Structural Ramsey theory and topological dynamics II

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Reminder from first lecture

- Extremely amenable group G:
  Every continuous action of G on a compact space has a fixed point.
- Ultrahomogeneous structure $\mathbb{F}$:
  Every isomorphism between finite substructures of $\mathbb{F}$ extends to an automorphism of $\mathbb{F}$.
- Fraïssé class $\mathcal{K}$:
  Countable class of finite structures with hereditarity, amalgamation and joint embedding property.
- Some Polish, non locally compact groups G are extremely amenable.
- When G closed subgroup of S_∞, then
  $G = \mathrm{Aut}(\mathbb{F})$, $\mathbb{F}$ countable ultrahomogeneous structure.
- The class $\mathcal{K}$ of finite substructures of $\mathbb{F}$ is a Fraïssé class.
- G is extremely amenable iff some combinatorial phenomenon takes place at the level of $\mathcal{K}$ (Ramsey type properties).

Second lecture: finite Ramsey theory, extreme amenability and universal minimal flows

- Ramsey theory on Fraïssé classes.
- The first main theorem: extreme amenability.
- The second main theorem: universal minimal flows.
- Examples of universal minimal flows.

Part I

Ramsey theory on Fraïssé classes

Example: complete graphs

- Color vertices of K_ω with finitely many colors.
  Fix $Y \subset K_\omega$ finite.
  Then there is $\tilde{Y} \cong Y$ where all vertices have same color.
  Idem when coloring the edges of K_ω.
  Idem when coloring the copies of any finite substructure $X \subset K_\omega$.
- $\mathcal{CG}$ has the Ramsey property.
- K_ω may be replaced by a finite large enough $Z \in \mathcal{CG}$,
  whose size depends on X, Y, and the number of colors.

A famous example

Proposition

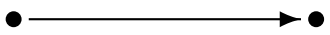
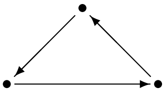
Any 2-coloring of the edges of K_6 has a triangle where all edges have same color.

Question

Does that happen for every Fraïssé class?

Finite oriented graphs

Proposition

Let $X =$  $Y =$ 

Z any finite oriented graph. Then:
There is a 2-coloring of the arcs of Z such that no copy of Y has all arcs with same color.

Proof.

Let $<$ be a linear ordering on Z.
Color an arc $x \leftarrow y$ blue if $x < y$, red otherwise.
Then every cycle has the two colors appearing. □

This problem disappears when working with ordered oriented graphs instead of oriented graphs.

Ramsey property

$\mathcal{K}$ a Fraïssé class.

Definition

$\mathcal{K}$ has the Ramsey property when
For any:

- $X \in \mathcal{K}$ *(small structure, to be colored),*
- $Y \in \mathcal{K}$ *(medium structure, to be reconstituted),*
- $k \in \mathbb{N}$ *(number of colors),*

There exists $Z \in \mathcal{K}$ (very large structure) such that:

$$Z \longrightarrow (Y)^X_k.$$

Whenever copies of X in Z are colored with k colors,
there is $\tilde{Y} \cong Y$ where all copies of X have same color.

Part II

The first main theorem: extreme amenability

The first main theorem

Definition

Let $L^{<}$ be a language with a distinguished binary symbol $<$.
$\mathcal{K}^{<}$ is a Fraïssé order class when it is a Fraïssé class with $<$ always interpreted as a total linear order.

Theorem (Kechris-Pestov-Todorcevic, 05)

Let $\mathcal{K}^{<}$ be a Fraïssé order class.
Let $\mathbb{F}^{<}$ be its Fraïssé limit.
Then TFAE:

i) $\mathrm{Aut}(\mathbb{F}^{<})$ *is extremely amenable.*
ii) $\mathcal{K}^{<}$ *has the Ramsey property.*

The very first example

- Finite linear orders:

Theorem (Ramsey, 30)

$\mathcal{LO}$ has the Ramsey property.

Corollary (Pestov, 98)

$\mathrm{Aut}(\mathbb{Q}, <)$ *extremely amenable.*

Corollary (Pestov, 98)

Homeo$_+(\mathbb{R})$ *(pointwise convergence topology) extremely amenable.*

Proof.

$\mathrm{Aut}(\mathbb{Q}, <) \hookrightarrow$ *Homeo*$_+(\mathbb{R})$ densely. $\square$

Example: Metric spaces

- Finite ordered metric spaces with rational distances: $\mathbb{U}_{\mathbb{Q}}^{<} = (\mathbb{U}_{\mathbb{Q}}, <^{\mathbb{U}_{\mathbb{Q}}})$.

Theorem (Nešetřil, 05)

$\mathcal{M}_{\mathbb{Q}}^{<}$ has the Ramsey property.

Corollary

$\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}}, <^{\mathbb{U}_{\mathbb{Q}}})$ *extremely amenable.*

Corollary

$\mathrm{iso}(\mathbb{U})$ *extremely amenable.*

Proof.

$\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}}, <^{\mathbb{U}_{\mathbb{Q}}}) \hookrightarrow \mathrm{iso}(\mathbb{U})$ densely. $\square$

Part III

The second main theorem: universal minimal flows

Compact G-flows

Recall: A G-flow is a continuous action of G on a topological space X.
Notation: $G \curvearrowright X$.

Remark

In what follows, G will be Hausdorff and X will be compact Hausdorff.

Example

$\mathbb{F}$: a countable ultrahomogeneous structure.
$LO(\mathbb{F})$: the set of linear orderings on $\mathbb{F}$.
$LO(\mathbb{F}) \subset 2^{\mathbb{F}\times\mathbb{F}}$ is compact.
If $<\in LO(\mathbb{F})$, $g \in \mathrm{Aut}(\mathbb{F})$, define $g\cdot <$:

$$x\ (g\cdot <)\ y \quad \leftrightarrow \quad g^{-1}(x) < g^{-1}(y).$$

Then $\mathrm{Aut}(\mathbb{F}) \curvearrowright \overline{\mathrm{Aut}(\mathbb{F})\cdot <}$ is a compact $\mathrm{Aut}(\mathbb{F})$-flow.

Minimal flows

Definition

Let $G \curvearrowright X$ be a G-flow.
$G \curvearrowright X$ is minimal when every $x \in X$ has dense orbit in X:

$$\overline{G \cdot x} = X$$

Theorem

Let G be a topological group.
Then there is a unique universal minimal flow $G \curvearrowright M(G)$:
$\forall$ $G \curvearrowright X$ minimal,
$\exists \pi : M(G) \longrightarrow X$ continuous, onto, equivariant:

$$\forall g \in G \ \ \forall x \in X \ \ \pi(g \cdot x) = g \cdot \pi(x).$$

The work of Kechris, Pestov and Todorcevic, II

- Finding $G \curvearrowright M(G)$ is hard in general.
- $M(G)$ may not be metrizable (e.g. G countable discrete).
- G is extremely amenable iff $|M(G)| = 1$.

Theorem (Kechris - Pestov - Todorcevic, 05)

Combinatorics on Fraïssé classes gives access to an explicit description of $G \curvearrowright M(G)$ when G closed subgroup of S_∞.

Extensions

Definition

Let L be a language, $<$ new symbol for a binary relation.
$L^* = L \cup \{<\}$.
$\mathcal{K}$ *class of finite L-structures,* $\mathcal{K}^*$ *class of finite* L^**-structures.*

$\mathcal{K}^*$ *is an* *extension* *of* $\mathcal{K}$ *when:*

$$\forall (X, <^X), \ \ (X, <^X) \in \mathcal{K}^* \to X \in \mathcal{K}.$$

Example

$\mathcal{G}^<$ *finite ordered graphs,* $\mathcal{G}$ *finite graphs.*

The extension property

Definition

Let $\mathcal{K}$ be a Fraïssé class, $\mathcal{K}^<$ a Fraïssé order class.
Assume $\mathcal{K}^<$ is an extension of $\mathcal{K}$.
$\mathcal{K}^<$ has the extension property with respect to $\mathcal{K}$ when:

For any $(X, <^X) \in \mathcal{K}^<$
There exists $Y \in \mathcal{K}$ such that

$$(X, <^X) \textit{ embeds in } (Y, \prec) \textit{ whenever } (Y, \prec) \in \mathcal{K}^<.$$

Flow minimality and extension property

Theorem (Kechris - Pestov - Todorcevic, 05)

Assume

- $\mathcal{K}$ *a Fraïssé class, with limit* $\mathbb{F}$.
- $\mathcal{K}^<$ *an extension of* $\mathcal{K}$, *Fraïssé order class, with limit* $\mathbb{F}^<$.
- $\mathbb{F}^<$ *is of the form* $(\mathbb{F}, <^{\mathbb{F}})$.
- $\mathcal{K}^<$ *has the extension property with respect to* $\mathcal{K}$.

Then TFAE:

i) $\mathrm{Aut}(\mathbb{F}) \curvearrowright \overline{\mathrm{Aut}(\mathbb{F}) \cdot <^{\mathbb{F}}}$ *is minimal.*

ii) $\mathcal{K}^<$ *has the extension property with respect to* $\mathcal{K}$.

The second main theorem

Theorem (Kechris - Pestov - Todorcevic, 05)

Assume

- $\mathcal{K}$ *a Fraïssé class, with limit* $\mathbb{F}$.
- $\mathcal{K}^<$ *an extension of* $\mathcal{K}$*, Fraïssé order class, with limit* $\mathbb{F}^<$.
- $\mathbb{F}^<$ *is of the form* $(\mathbb{F}, <^{\mathbb{F}})$.
- $\mathcal{K}^<$ *has the Ramsey and the extension property with respect to* $\mathcal{K}$.

Then:
The universal minimal flow of $\mathrm{Aut}(\mathbb{F})$ *is*

$$\mathrm{Aut}(\mathbb{F}) \curvearrowright \overline{\mathrm{Aut}(\mathbb{F}) \cdot <^{\mathbb{F}}}.$$

In particular, $M(\mathrm{Aut}(\mathbb{F}))$ *is metrizable.*

Strategy to find universal minimal flows

- Choose your favorite countable ultrahomogeneous structure $\mathbb{F}$.
- Consider its class $\mathcal{K}$ of finite substructures.
- Try to enrich $\mathcal{K}$ with linear orderings to obtain $\mathcal{K}^<$ such that
  - $\mathcal{K}^<$ is a Fraïssé class with the Ramsey property.
  - $\mathcal{K}^<$ has the extension property with respect to $\mathcal{K}$.
- Express the limit of $\mathcal{K}^<$ as $(\mathbb{F}, <^{\mathbb{F}})$.
- Describe the action $\mathrm{Aut}(\mathbb{F}) \curvearrowright \overline{\mathrm{Aut}(\mathbb{F})\cdot <^{\mathbb{F}}}$.

Part IV

Examples of universal minimal flows

Graphs

• $\mathcal{G}$ finite graphs:

Theorem (Nešetřil-Rödl, 77)

Let $\mathcal{G}^{<}$ be the class of all finite ordered graphs. Then $\mathcal{G}^{<}$ has the Ramsey and the extension property.

Corollary

$\mathrm{Aut}(\mathcal{R}) \curvearrowright M(\mathrm{Aut}(\mathcal{R}))$ *is* $\mathrm{Aut}(\mathcal{R}) \curvearrowright LO(\mathcal{R})$.

• $\mathcal{G}_n$ finite K_n-free graphs:

Theorem (Nešetřil-Rödl, 77)

Let $\mathcal{G}_n^{<}$ be the class of all finite ordered K_n-free graphs. Then $\mathcal{G}_n^{<}$ has the Ramsey and the extension property.

Corollary

$\mathrm{Aut}(H_n) \curvearrowright M(\mathrm{Aut}(H_n))$ *is* $\mathrm{Aut}(H_n) \curvearrowright LO(H_n)$.

Partial orders

• $\mathcal{P}$ finite partial orders:

Definition

Let $P \in \mathcal{P}$. A linear order on P is compatible when it extends $<^P$.

Theorem (Nešetřil, 05)

Let $\mathcal{P}^{e<}$ be the class of all finite compatibly ordered partial orders. Then $\mathcal{P}^{e<}$ has the Ramsey and the extension property.

Corollary

Let $eLO(\mathbb{P})$ be the class of all compatible linear orders on $\mathbb{P}$. Then $\mathrm{Aut}(\mathbb{P}) \curvearrowright M(\mathrm{Aut}(\mathbb{P}))$ is $\mathrm{Aut}(\mathbb{P}) \curvearrowright eLO(\mathbb{P})$.

Examples: ultrametric spaces

• $\mathcal{U}$ finite ultrametric spaces, distances in $\{1/2^n : n \in \mathbb{N}\}$:
Equivalently: finite metric subspaces of the Baire space $\mathbb{N}^{\mathbb{N}}$.

Theorem (NVT, 08)

Let $\mathcal{U}^<$ be the class of all finite ordered metric subspaces of $\mathbb{N}^{\mathbb{N}}$.
Then $\mathcal{U}^<$ has neither the Ramsey property nor the extension property.

Definition

$<$, linear ordering on metric space, is convex when all balls are $<$-convex.

Theorem (NVT, 08)

Let $\mathcal{U}^{c<}$ be the class of finite convexly ordered metric subspaces of $\mathbb{N}^{\mathbb{N}}$.
Then $\mathcal{U}^{c<}$ has the Ramsey and the extension property.

Corollary

$\mathrm{iso}(\mathbb{U}^{ult}) \curvearrowright M(\mathrm{iso}(\mathbb{U}^{ult}))$ *is* $\mathrm{iso}(\mathbb{U}^{ult}) \curvearrowright LexO(\mathbb{U}^{ult})$.

Vector spaces

• $\mathcal{V}_F$ finite vector spaces, F finite field.

Definition

Let $V \in \mathcal{V}_F$. A natural linear ordering of V is obtained by

- *fixing B linearly ordered basis of V,*
- *fixing a linear ordering of F with least element 0_F,*
- *taking the resulting lexicographical ordering induced on V.*

$\mathcal{V}_F^{n<}$: the class of naturally ordered finite vector spaces over F.

Vector spaces, cont'd

Theorem (Thomas, 86)

- $\mathcal{V}_F^{n<}$ *is a Fraïssé order class with reduct* $\mathcal{V}_F$,
- $\mathcal{V}_F^{n<}$ *has the extension property.*

Theorem (Graham-Leeb-Rothschild, 72)

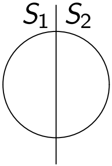
$\mathcal{V}_F^{n<}$ *has the Ramsey property.*

Corollary

Let $nLO(F^{<\omega})$ *be the set of all linear orderings on* $F^{<\omega}$ *with natural restrictions on finite-dimensional subspaces. Then:*
$GL(F^{<\omega}) \curvearrowright M(GL(F^{<\omega}))$ *is* $GL(F^{<\omega}) \curvearrowright nLO(F^{<\omega})$.

The case of $S(2)$

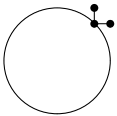
- Finite substructures of $(S(2), <)$ never have the Ramsey property: $\exists$2-coloring of the vertices with no monochromatic 3-cycle.
- Ramsey property holds if $S(2)$ is enriched differently:



- Key fact: $(S(2), S_1, S_2) \cong (\mathbb{Q}, Q_1, Q_2, <)$, Q_1, Q_2 dense subsets of $\mathbb{Q}$ (Reversing the arcs between points in different parts).
- Ramsey and extension property hold for the corresponding finite substructures.

The case of $S(2)$, cont'd

- The second main theorem holds in that case.
- $\mathrm{Aut}(S(2)) \curvearrowright M(\mathrm{Aut}(S(2)))$ is $\mathrm{Aut}(S(2)) \curvearrowright \overline{\mathrm{Aut}(S(2)) \cdot (S_1, S_2)}$.
- $\overline{\mathrm{Aut}(S(2)) \cdot (S_1, S_2)} \cong (\mathbb{S}^1$ with rational and corational points doubled).
- Thus, $\mathrm{Aut}(S(2)) \curvearrowright M(\mathrm{Aut}(S(2)))$ is $\mathrm{Aut}(S(2)) \curvearrowright (\mathbb{S}^1$ with rational and corational points doubled):



Summary

- A flow $G \curvearrowright X$ is minimal when every $x \in X$ has a dense orbit.
- Every Hausdorff topological group has a largest minimal flow: the universal minimal flow $G \curvearrowright M(G)$.
- G is extremely amenable iff $|M(G)| = 1$.
- If $\mathcal{K}$ Fraïssé class with Fraïssé limit $\mathbb{F}$, extensions of $\mathcal{K}$ with Ramsey and extension property give access to an explicit description of

$$\mathrm{Aut}(\mathbb{F}) \curvearrowright M(\mathrm{Aut}(\mathbb{F})).$$

Perspectives

- Towards a new proof of Gromov-Milman theorem (extreme amenability of $O(\ell_2)$):
  Is there a Ramsey theorem for finite ordered affinely independent Euclidean metric spaces, distances in $\mathbb{Q}$?
- Is there a unified approach to prove Ramsey property for classes of finite structures?
- Recent developments of the theory:
  - Projective version (Irwin-Solecki).
  - Dual version (Solecki).
- A possible development of the theory: continuous logic?