Structural Ramsey theory and topological dynamics III

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Reminder from second lecture

$\mathcal{K}$ a Fraïssé class, $\mathbb{F}$ its Fraïssé limit.

- $\mathcal{K}$ has the Ramsey property when
  For any:
  - $X \in \mathcal{K}$ (small structure, to be colored),
  - $Y \in \mathcal{K}$ (medium structure, to be reconstituted),
  - $k \in \mathbb{N}$ (number of colors),

  $$\mathbb{F} \longrightarrow (Y)^X_k.$$

  Whenever copies of X in $\mathbb{F}$ are colored with k colors,
  there is $\tilde{Y} \cong Y$ where all copies of X have same color.
- Continuous actions of $\mathrm{Aut}(\mathbb{F})$ on compact spaces capture Ramsey properties of $\mathcal{K}$ and its extensions $\mathcal{K}^*$ (and vice-versa).
- We are now interested in the case where Y is replaced by $\mathbb{F}$.

Third lecture: oscillation stability and infinite Ramsey theory

- Oscillation stability for topological groups.
- Coloring vertices.
- Higher dimension.

Part I

Oscillation stability for topological groups

A typical question

Let $\mathbb{F}$ be a countable ultrahomogeneous structure. Color vertices of $\mathbb{F}$ with finitely many colors:

$$\mathbb{F} = A_1 \cup \ldots \cup A_k.$$

- Is there a copy of $\mathbb{F}$ in $\mathbb{F}$ where only one color appears?
- If so, can we see it at the level of $\mathrm{Aut}(\mathbb{F})$?

Left completion of topological groups

Let G be a Polish group. Then:

- G admits a left-invariant metric. Call $\widehat{G}$ the completion.
- Multiplication extends to $\widehat{G}$.
- Inversion does not extend to $\widehat{G}$ in general.

Therefore, $\widehat{G}$ is a topological semigroup.

Examples

- *If $\mathbb{F}$ countable ultrahomogeneous,*

$$\widehat{\mathrm{Aut}(\mathbb{F})} = Emb(\mathbb{F}).$$

- *If X complete separable ultrahomogeneous metric space,*

$$\widehat{\mathrm{iso}(X)} = Emb(X).$$

Oscillation stability, definitions

Definition

Let G be a topological group.
G is oscillation stable when

- *for every $f : G \longrightarrow [0, 1]$ uniformly continuous,*
- *for every $\varepsilon > 0$,*

there is a right ideal $I \subset \widehat{G}$ such that

$$\mathrm{osc}(\hat{f}, I) := \sup_{x,y \in I} |\hat{f}(y) - \hat{f}(x)| < \varepsilon.$$

If H a subgroup of G,
(G, H) is oscillation stable if same conclusion holds when f compatible with G/H.

The work of Kechris, Pestov and Todorcevic, III

Theorem (Kechris-Pestov-Todorcevic, 05)

Let $\mathcal{K}$ be a Fraïssé class, $\mathbb{F}$ its Fraïssé limit.
Let $X \in \mathcal{K}$ with no non trivial automorphism. Then TFAE:

i) $\forall k \in \mathbb{N}, \quad \mathbb{F} \longrightarrow (\mathbb{F})^X_k$.

ii) $(\mathrm{Aut}(\mathbb{F}), \mathit{Stab}(X))$ *is oscillation stable.*

Corollary

Let $\mathcal{K}^<$ be a Fraïssé order class, $\mathbb{F}^<$ its Fraïssé limit. Then TFAE:

i) $\forall X \in \mathcal{K}^<, \quad \forall k \in \mathbb{N}, \quad \mathbb{F} \longrightarrow (\mathbb{F})^X_k$.

ii) $\forall H \leq \mathrm{Aut}(\mathbb{F})$ *open,* $(\mathrm{Aut}(\mathbb{F}), H)$ *is oscillation stable.*

Hjorth's theorems

Theorem (Hjorth, 08)

Let $\mathcal{K}^<$ be a Fraïssé order class, $\mathbb{F}^<$ its Fraïssé limit. Then there is $X \in \mathcal{K}^<$, $|X| = 2$, such that:

$$\mathbb{F}^< \nrightarrow (\mathbb{F}^<)^X_2 .$$

Equivalently, $(Aut(\mathbb{F}^<), Stab(X))$ is not oscillation stable.

Theorem (Hjorth, 08)

Let G be a non trivial Polish group. Then G is not oscillation stable.

Remark

Recently, new proof of this Theorem by Melleray via continuous logic.

Part II

Coloring vertices

Big Ramsey degree for vertices

Let $\mathbb{F}$ be countable ultrahomogeneous structure.

Definition

- *Vertices have big Ramsey degree l in $\mathbb{F}$ when $l \leq \omega$ is least such that: For every $k \in \mathbb{N}$ and every*

$$\mathbb{F} = A_1 \cup \ldots \cup A_k$$

  there is a copy of $\mathbb{F}$ in $\mathbb{F}$ where $\leq l$ colors appear.
- *When $l = 1$, $\mathbb{F}$ is indivisible.*

Example: Graphs

Proposition

The Rado graph $\mathcal{R}$ is indivisible.

Proof.

Let $\mathcal{R} = R \cup B$. Write $\mathcal{R} = \{x_n : n \in \omega\}$.
If all vertices are red, done.
Otherwise, pick a blue $\tilde{x}_0$.
In general, if $\tilde{x}_0, \ldots, \tilde{x}_n$ are blue and $\{\tilde{x}_0, \ldots, \tilde{x}_n\} \cong \{x_0, \ldots, x_n\}$, consider

$$E = \{x \in \mathcal{R} : \{\tilde{x}_0, \ldots, \tilde{x}_n, x\} \cong \{x_0, \ldots, x_n, x_{n+1}\}\},$$

$E \subset \mathcal{R}$ ultrahomogeneous, universal for finite graphs, hence $\cong \mathcal{R}$.
If $E \subset R$, done.
Otherwise, pick $\tilde{x}_{n+1} \in E$ blue.
Carry on. □

Examples: K_n-free graphs

Recall: $\mathcal{H}_n$ the countable ultrahomogeneous K_n-free graph.

Theorem (Komjáth-Rödl, 86)

$\mathcal{H}_3$ is indivisible.

Theorem (El-Zahar - Sauer, 87)

$\mathcal{H}_n$ is indivisible for every $n \in \mathbb{N}$.

Remark

Those are particular cases where amalgamation is free.

Free amalgams

Definition

Let $f_i : A \longrightarrow B_i$, $i = 0, 1$ be embeddings of finite structures. Their free amalgam is the structure obtained by

- *Considering disjoint copies of B_0, B_1.*
- *Gluing them by identifying $f_0''A$ and $f_1''A$.*

Definition

A Fraïssé class has free amalgamation property when it is closed under free amalgams.

Examples

- *Finite graphs.*
- *Finite K_n-free graphs.*
- *Finite graphs with red edges, blue edges, no monochromatic triangle.*

Orbits

Definition

Let $\mathbb{F}, \mathbb{G}$ be countable structures.
$\mathbb{F} \preccurlyeq \mathbb{G}$ *means* $\mathbb{F} = \bigcup_{i=1}^{n} F_i$ *with each* F_i *embeddable in* $\mathbb{G}$.

Definition

Let $X \subset \mathbb{F}$ *finite.*
Then $Stab(X)$ *acts on* $\mathbb{F} \setminus X$.
Corresponding orbits are orbits of $\mathbb{F}$.
Orb := *set of all orbits of* $\mathbb{F}$.

Sauer's theorem

Theorem (Sauer, 03)

Let $\mathcal{K}$ be a Fraïssé class in a finite binary language, $\mathbb{F}$ its Fraïssé limit. Assume $\mathcal{K}$ has free amalgamation property. Then the big Ramsey degree vertices in $\mathbb{F}$ is

$$\mathsf{sup}\{|\mathcal{A}| : \mathcal{A} \subset (Orb, \preccurlyeq) \ \text{ antichain}\}.$$

Examples

- *$\mathcal{R}$ Rado graph: 1.*
- *H_n countable ultrahomogeneous K_n-free graph: 1.*
- *Countable ultrahomogeneous graph with red edges, blue edges, no monochromatic triangle: 2.*

Examples: metric spaces

Recall: $\mathbb{U}_m$ countable ultrahomogeneous metric space, distances in $\{1, \ldots, m\}$.

Theorem (NVT-Sauer, 09)

$\mathbb{U}_m$ is indivisible for every $m \in \mathbb{N}$.

Remark

- *Proof works with $\{1, \ldots, m\}$ replaced by $\{s_1, \ldots, s_m\}$ and $\forall i < m \;\; s_{i+1} \leq 2s_i$.*
- *Proof fails if $\exists i < m \;\; 2s_i < s_{i+1}$.*

More on metric spaces: Hilbert and Urysohn spheres

$\mathbb{U}$: the unique complete separable ultrahomogeneous metric space into which any separable metric space embeds.

Theorem (López-Abad - NVT - Sauer, 09)

Let $\mathbb{S}$ *be a sphere in* $\mathbb{U}$*,*
$\varepsilon > 0$, $\mathbb{S} = R \cup B$.
Then $\mathbb{S} \hookrightarrow (R)_\varepsilon$ *or* $\mathbb{S} \hookrightarrow (B)_\varepsilon$.

Corollary

Let $\varepsilon > 0$*,*
$\mathbb{S}_{\mathcal{C}([0,1])} = R \cup B$.
Then $\mathbb{S}_{\mathcal{C}([0,1])} \hookrightarrow (R)_\varepsilon$ *or* $\mathbb{S}_{\mathcal{C}([0,1])} \hookrightarrow (B)_\varepsilon$.

Theorem (Odell-Schlumprecht, 94)

This is false if $\mathcal{C}([0,1])$ *is replaced by* ℓ_2*.*

Examples: affine spaces

$\mathcal{A}_F$: finite affine spaces over F (finite field), Fraïssé limit $F^{<\omega}$.

Theorem (Hindman, 74)

Big Ramsey degree of vertices in $\mathbb{F}_2^{<\omega}$: 1.

Theorem (Folklore)

Assume $F \neq \mathbb{F}_2$.
Big Ramsey degree of vertices in $F^{<\omega}$: ω.

Proof.

Let $x \in F^{<\omega}$.
$c(x)$: number of times it changes value (ignoring value 0).
Then range of c contains arbitrarily long intervals on copies of $F^{<\omega}$. □

Part III

Higher dimension

Coloring more complex structures

Let $\mathbb{F}$ be countable ultrahomogeneous structure,
$X \subset \mathbb{F}$ finite,
$\binom{\mathbb{F}}{X}$ set of copies of X in $\mathbb{F}$.

Definition

X has big Ramsey degree l in $\mathbb{F}$ when $l \leq \omega$ is least such that:
For every $k \in \mathbb{N}$ and every

$$\binom{\mathbb{F}}{X} = A_1 \cup \ldots \cup A_k$$

there is a copy $\tilde{F}$ of $\mathbb{F}$ in $\mathbb{F}$ where $\leq l$ colors appear on $\binom{\tilde{F}}{X}$.

Example

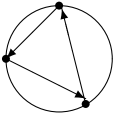
From Hjorth's theorem, there is always $X \subset \mathbb{F}$ of size 2 with big Ramsey degree ≥ 2.

Examples: Tournaments

Theorem (Lachlan-Woodrow, 84)

There are only three countable ultrahomogeneous tournaments:

- *The rationals* $(\mathbb{Q}, <)$: $x \leftarrow y$ *iff* $x < y$.
- *The countable random tournament* $\overrightarrow{\mathcal{R}}$.
- *The dense local order* $S(2)$.



Theorem (D. Devlin, 79)

In $(\mathbb{Q}, <)$, *every* X *has degree* $\tan^{(2|X|-1)}(0)$.

Theorem (Laflamme-Sauer-Vuksanovic, 04)

In $\overrightarrow{\mathcal{R}}$, *every* X *has degree...* $< \omega$, *equal to something*.

Theorem (Laflamme-NVT-Sauer, 09)

In $S(2)$, *every* X *has degree* $\tan^{(2|X|-1)}(0) \cdot 2|X|/|\mathrm{Aut}(X)|$.

One more example

Previous results use coding of structures in $2^{<\omega}$.
The following one does not:

Theorem (Sauer, 98)

In H_3 countable ultrahomogeneous triangle-free graph, edges have big Ramsey degree 2.

Summary

- $\mathbb{F} \longrightarrow (\mathbb{F})^X_k$ can be captured by a property of $\mathrm{Aut}(\mathbb{F})$: oscillation stability.
- Hjorth's theorem provides an obstruction for some $|X| \geq 2$.
- Still, can study big Ramsey degrees:
  - For vertices, least $l \leq \omega$ such that:
    For every
    $$\mathbb{F} = A_1 \cup \ldots \cup A_k$$
    there is a copy of $\mathbb{F}$ in $\mathbb{F}$ where $\leq l$ colors appear.
  - In general, least $l \leq \omega$ such that:
    For every
    $$\binom{\mathbb{F}}{X} = A_1 \cup \ldots \cup A_k$$
    there is a copy $\tilde{F}$ of $\mathbb{F}$ in $\mathbb{F}$ where $\leq l$ colors appear on $\binom{\tilde{F}}{X}$.
- El-Zahar - Sauer's theorem allows to compute big Ramsey degrees for vertices when language is finite and binary and amalgamation is free.
- In higher dimensions, most known results use combinatorics on trees.

Perspectives

- Can oscillation stability be better understood from the group theory side?
- For vertex partitions:
  - Is there a characterization of structures with finite big Ramsey degree (even in finite binary language)?
  - Can we characterize complete separable metric spaces X so that

$$\forall \varepsilon > 0, \quad \forall X = R \cup B, \quad X \hookrightarrow (R)_\varepsilon \ \text{ or } \ X \hookrightarrow (B)_\varepsilon .$$

  - Can Odell-Schlumprecht theorem be reproved combinatorially?
- For higher dimensions:
  - What about big Ramsey degrees in K_n-free graphs?
  - What about the countable ultrahomogeneous partial order?
  - What about countable ultrahomogeneous directed graphs?