Playing with ctbl support iteration

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Forcing has two aspects.
When we add a Cohen real, we

- get an object not covered by ground model - a real that avoids all ground model meager sets
- cover every object by ground model - every real is in a ground model null set.

MA addresses the first aspect, CPA - the second.

CPA makes possible constructions of length ω_1 in which one meets ω_2 requirements.
The space of requirements is covered by ω_1 sets s.t. each set requirements are met in one step of the construction.

CPA(*null*)

Adam and Eve meet ω_1 times. At round α Adam chooses a non-null Borel set $A_\alpha \subseteq 2^\omega$ and a Borel function $f_\alpha : A_\alpha \to 2^\omega$, Eve responds with non-null Borel $E_\alpha \subseteq A_\alpha$.
Adam wins if $\bigcup_{\alpha<\omega_1} f_\alpha[E_\alpha] = 2^\omega$.
$\mathtt{CPA}(null) \equiv$ Eve has no winning strategy.

random reals

$\mathcal{R}_{\omega_2}$ - a poset for adding ω_2 random reals,
Baire subsets of 2^{ω_2} of positive measure.

Theorem

Suppose $V \models CH$. Then $\mathcal{R}_{\omega_2} \Vdash \mathtt{CPA}(null)$.

Theorem

$\mathtt{CPA}(\mathit{null}) \Rightarrow$ universally measurable sets are unions of ω_1 Borel sets.

proof

Fix universally measurable $U \subseteq 2^\omega$. Talk Eve into this tactic:

- Adam: $f_\alpha : A_\alpha \to 2^\omega$,
- Eve: E_α contained either in $A_\alpha \cap f_\alpha^{-1}[U]$ or in $A_\alpha \setminus f_\alpha^{-1}[U]$.

Find Adam's winning counter-play $\langle A_\alpha, f_\alpha \rangle_{\alpha<\omega_1}$.
The sets $f_\alpha[E_\alpha]$ cover the space 2^ω, each set is either in or out of U, and, being analytic, is a union of ω_1 Borel sets.

So we get:

Theorem (Larson-Shelah)

Suppose $V \models GCH$. Then $\mathcal{R}_{\omega_2}$ forces that universally measurable sets are unions of ω_1 Borel sets, hence that there are $\mathfrak{c}$ many of them.

- *ctbl* - the σ-ideal of ctbl subsets of 2^ω,
- $ctbl^\alpha$ - the αth Fubini power of *ctbl*, $\alpha < \omega_1$.

Fubini power

$X \in ctbl^\alpha$ iff $X \subseteq (2^\omega)^\alpha$ and there is a tree $S \subseteq (2^\omega)^{<\alpha}$ s.t.

- non-terminal nodes splits into co-ctbly many successors,
- at limit levels all branches are taken,
- $lim\, S = \{s \in (2^\omega)^\alpha : \forall \beta < \alpha\ s \restriction \beta \in S\}$ is disjoint with X.

Sacks

Positive sets ordered by inclusion

$$\mathcal{P}_\alpha = Borel((2^\omega)^\alpha) \setminus ctbl^\alpha.$$

$\mathcal{P}_\alpha \equiv$ the αth stage of ctbl support iteration of the Sacks forcing.

Adam and Eve meet ω_1 times. At round α Adam chooses $\alpha' < \omega_1$, $p_\alpha \in \mathcal{P}_{\alpha'}$, and a Borel function

$$f_\alpha : p_\alpha \to 2^\omega.$$

Eve responds with $q_\alpha \in \mathcal{P}_{\alpha'}$, $q_\alpha \subseteq p_\alpha$.
Adam wins if

$$\bigcup_{\alpha<\omega_1} f_\alpha[q_\alpha] = 2^\omega.$$

CPA - Covering Property Axiom

CPA $\equiv$ Eve has no winning strategy.

Theorem CPA in the Sacks model

CPA holds in the Sacks model.

Sacks model $= \omega_2$ Sacks reals added to CH via ctbl support.

Adam and Eve meet ω_1 times. At round α Adam chooses $\alpha' < \omega_1$, $p_\alpha \in \mathcal{P}_{\alpha'}$, and a Borel function

$$f_\alpha : p_\alpha \to 2^\alpha.$$

Eve responds with $q_\alpha \in \mathcal{P}_{\alpha'}$, $q_\alpha \subseteq p_\alpha$.
Adam wins if for every $t \in 2^{\omega_1}$ for stationary many α

$$t \restriction \alpha \in f_\alpha[q_\alpha].$$

$\Diamond$CPA

$\Diamond$CPA $\equiv$ Eve has no winning strategy.

Theorem $\Diamond$CPA in the Sacks model

$\Diamond$CPA holds in the Sacks model.

Easily, $\Diamond \Rightarrow \Diamond$CPA $\Rightarrow$ CPA, CH $\Rightarrow$ CPA.

proof - sketch

For $\alpha \geq \omega_1$ let $\mathcal{P}_\alpha$ consist of all cylinders in $(2^\omega)^\alpha$ of the form

$$\{s \in (2^\omega)^\alpha : s \circ e \in p\},$$

where $p \in \mathcal{P}_\beta$ for some $\beta < \omega_1$ and $e : \beta \to \alpha$ is an order preserving embedding.
$\mathcal{P}_\alpha \equiv$ the αth stage of ctbl support iteration of the Sacks forcing.
Let $\dot{r}_\alpha$ and $\dot{G}_\alpha$ be the canonical terms for the αth Sacks real and the generic subset of $\mathcal{P}_\alpha$.
Let $\dot{\sigma}$ be a $\mathcal{P}_{\omega_2}$ term for Eve's strategy.

- $\{\alpha < \omega_2 : \mathcal{P}_{\omega_2} \Vdash \dot{\sigma} \restriction V[\dot{G}_\alpha] \in V[\dot{G}_\alpha]\}$ is unbounded and ω_1-closed in ω_2;
- $\mathrm{cf}\alpha = \omega_1 \Rightarrow \mathcal{P}_\alpha \Vdash \diamondsuit$;
- $\forall\, \alpha < \omega_2\ \mathcal{P}_\alpha \Vdash$ "$\mathcal{P}_{\omega_2}/\mathcal{P}_\alpha \equiv$ cs iteration of ω_2 Sacks reals".

WLOG

- $\sigma \restriction V \in V$ is Eve's strategy in V;
- $V \models \Diamond$.

We will find in V a counter-play for Adam s.t. in $V[G_{\omega_2}]$ makes him a winner.
We work in V.

Fix a $\Diamond$ sequence

$$\langle \varepsilon_\alpha \subseteq \alpha \times \alpha : \alpha < \omega_1 \rangle$$

that predicts subsets of $\omega_1 \times \omega_1$.
Fix large enough regular $\varkappa$.

Adam's counter-play.

At round α Adam attempts to find an elementary embedding

$$\langle \alpha, \varepsilon_\alpha \rangle \to \langle H_\varkappa, \in\restriction H_\varkappa \rangle \, .$$

Let M_α be the range of the embedding, M^*_α - the transitive collapse of M_α, and $e_\alpha : M^*_\alpha \to M_\alpha$ - the inverse of the collapsing map. Adam wants

- $M_\alpha \cap \omega_1 = \alpha$,
- $e_\alpha(0)$ is a condition in $\mathcal{P}_{\omega_2}$, call it p,
- $e_\alpha(1)$ is a sequence $\langle \dot{t}_\beta \rangle_{\beta<\omega_1}$ of $\mathcal{P}_{\omega_2}$ terms s.t.

$$\forall \beta \, \mathcal{P}_{\omega_2} \Vdash \dot{t}_\beta \in \{0, 1\} \, .$$

If Adam fails, then he playes anything legal.

If he succeeds, then he looks at

- $\alpha' = \mathrm{ot}(M_\alpha \cap \omega_2) = (\omega_2)^{M^*_\alpha}$;
- the set S_α of all $s \in (2^\omega)^{M_\alpha \cap \omega_2}$ for which there is (in V) an M_α-generic filter $H_s \subseteq M_\alpha \cap \mathcal{P}_{\omega_2}$ containing p and s.t. $s(\xi) = \dot{r}_\xi / H_s$ for all $\xi \in M_\alpha \cap \omega_2$;

and plays

- $p_\alpha = \{s \circ e_\alpha : s \in S_\alpha\} \subseteq (2^\omega)^{\alpha'}$;
- $f_\alpha : p_\alpha \to 2^\alpha$ defined by $s \circ e_\alpha \longmapsto \langle \dot{t}_\beta / H_s \rangle_{\beta < \alpha}$.

Properness implies that

- $p_\alpha \in \mathcal{P}_{\alpha'}$;
- H_s is uniquely determined by s;
- the function f_α is Borel.
- p_α and f_α don't depend on the particular choice of M_α, they depend on M^*_α, ultimately on ε_α.

Suppose Adam follows the above counter-play while Eve plays according to her strategy σ.
Let $\{q_\alpha\}_{\alpha<\omega_1}$ list the sets played by Eve.

We want to see that

$$\mathcal{P}_{\omega_2} \Vdash \forall\, t \in 2^{\omega_1}\ \ \{\alpha < \omega_1 : t \restriction \alpha \in f_\alpha[q_\alpha]\} \text{ is stationary}\,.$$

Fix a condition $p \in \mathcal{P}_{\omega_2}$, a $\mathcal{P}_{\omega_2}$ term $\dot{D}$ for a club subset of ω_1, and a sequence $\langle \dot{t}_\alpha \rangle_{\alpha<\omega_1}$ of $\mathcal{P}_{\omega_2}$ terms for members of $\{0,1\}$.
Find an elementary submodel

$$\langle N, \in\restriction N\rangle \prec \langle H_\varkappa, \in\restriction H_\varkappa\rangle$$

of size ω_1 s.t.

$$\omega_1 \cup \{p,\ \dot{D},\ \langle \dot{t}_\alpha \rangle_{\alpha<\omega_1}\} \ \subseteq N\,.$$

Fix a bijection $\psi : \omega_1 \to N$ s.t.

$$\psi(0) = p,\ \psi(1) = \langle \dot{t}_\alpha \rangle_{\alpha<\omega_1},\ \psi(2) = \dot{D}\,.$$

Note that for club many $\alpha < \omega_1$,

$$\psi[\alpha] \cap \omega_1 = \alpha \quad \wedge \quad \langle \psi[\alpha], \in \restriction \psi[\alpha] \rangle \prec \langle H_\varkappa, \in \restriction H_\varkappa \rangle\,.$$

Look at

$$\varepsilon = \{\langle \xi, \zeta \rangle \in \omega_1 \times \omega_1 : \psi(\xi) \in \psi(\zeta)\}\,,$$

and use the predictive capabilities of $\langle \varepsilon_\alpha \rangle_{\alpha<\omega_1}$ to get α s.t. $\varepsilon_\alpha = \varepsilon \cap (\alpha \times \alpha)$.
Note that ε_α and $M_\alpha = \psi[\alpha]$ pass Adam's search criteria.

Now

$$q = \{s \in (2^\omega)^{\omega_2} : s \circ e_\alpha \in q_\alpha\}$$

is $(M, \mathcal{P}_{\omega_2})$ generic, $q \leq p$, and

$$q \Vdash f_\alpha(\langle \dot{r}_\xi : \xi \in M_\alpha \cap \omega_2 \rangle) = \langle \dot{t}_\beta \rangle_{\beta < \alpha}\,,$$

which gives

$$q \Vdash \langle \dot{t}_\beta \rangle_{\beta < \alpha} \in f_\alpha[q_\alpha].$$

Also, $\dot{D} \in M_\alpha$ and $\alpha = M_\alpha \cap \omega_1$ imply by genericity of q that $q \Vdash \alpha \in \dot{D}$.

other forcings

Analogous axioms can be formulated for other forcings.
E.g., we can change everywhere above 2^ω to ω^ω,
Sacks forcing to Miller forcing, and
ctbl to $\mathcal{K}_\sigma$, the σ-ideal generated by compact subsets of ω^ω.
We get CPA for Miller forcing.

abstracting

$\widetilde{\mathcal{I}}$ is a family of σ-ideals;
each $\mathcal{I} \in \widetilde{\mathcal{I}}$ is a σ-ideal of Borel subsets of a Polish space $\mathbb{X}_\mathcal{I}$,
$\mathcal{P}_\mathcal{I} = \mathit{Borel}(\mathbb{X}_\mathcal{I}) \setminus \mathcal{I}$.

ideal game

Consider two games, regular and diamond.
In both games there are ω_1 rounds.
At round α Adam chooses $\mathcal{I}_\alpha \in \widetilde{\mathcal{I}}$, $p_\alpha \in \mathcal{P}_{\mathcal{I}_\alpha}$, and a Borel function

- $f_\alpha : p_\alpha \to 2^\omega$, in the regular game, or
- $f_\alpha : p_\alpha \to 2^\alpha$, in the diamond game.

Eve responds with $q_\alpha \subseteq p_\alpha$, $q_\alpha \in \mathcal{P}_{\mathcal{I}_\alpha}$.
Adam wins

- the regular game if $\bigcup_{\alpha<\omega_1} f_\alpha[q_\alpha] = 2^\omega$,
- the diamond game if for every $t \in 2^{\omega_1}$ for stationary many α, $t \restriction \alpha \in f_\alpha[q_\alpha]$.

$\mathtt{CPA}(\widetilde{\mathcal{I}})$ and $\diamondsuit\mathtt{CPA}(\widetilde{\mathcal{I}})$

Axioms $\mathtt{CPA}(\widetilde{\mathcal{I}})$ and $\diamondsuit\mathtt{CPA}(\widetilde{\mathcal{I}})$ both say that

- Eve has no winning strategy in the respective game.

Theorem CPA for Sacks Miller Cohen Solovay .....

$\diamondsuit\texttt{CPA}(\widetilde{\mathcal{I}})$ is forced by ctbl support iteration of length ω_2 of

- Sacks real: $\widetilde{\mathcal{I}} = \widetilde{ctbl} = \{ctbl^\alpha\}_{\alpha<\omega_1}$;
- Miller real: $\widetilde{\mathcal{I}} = \widetilde{\mathcal{K}}_\sigma = \{\mathcal{K}_\sigma^\alpha\}_{\alpha<\omega_1}$;
- Cohen real: $\widetilde{\mathcal{I}} = \widetilde{\mathcal{M}} = \{\mathcal{M}^\alpha\}_{\alpha<\omega_1}$, $\mathcal{M} = meager(2^\omega)$;
- Solovay real: $\widetilde{\mathcal{I}} = \widetilde{\mathcal{N}} = \{\mathcal{N}^\alpha\}_{\alpha<\omega_1}$, $\mathcal{N} = null(2^\omega)$;
- .....

Cohen reals

$\mathcal{C}_{\omega_2}$ - poset for adding ω_2 Cohen reals,
non-meager Baire subsets of 2^{ω_2}.
Equivalent to the finite support iteration of length ω_2 of $\mathcal{C} = \mathcal{P}_{\mathcal{M}}$.

random reals

$\mathcal{R}_{\omega_2}$ - a poset for adding ω_2 random reals,
Baire subsets of 2^{ω_2} of positive measure.
$\mathcal{R}_{\omega_2}$ is not an iteration of $\mathcal{R} = \mathcal{P}_{\mathcal{N}}$.

$\mathcal{N}^2 = \mathcal{N}((2^\omega)^2)$ but $\mathcal{N}^\omega$ is a proper subideal of $\mathcal{N}((2^\omega)^\omega)$.
Eg., $\{\langle x_n\rangle_{n\in\omega} \in (2^\omega)^\omega : \forall n\, x_n(0) = 0\} \in \mathcal{N}((2^\omega)^\omega) \setminus \mathcal{N}^\omega$.

Likewise for $\mathcal{M}$.

Theorem CPA for *meager* and *null*

- Assume $V \models CH$. Then $\mathcal{C}_{\omega_2} \Vdash \texttt{CPA}(meager)$.
- Assume $V \models \diamondsuit$. Then $\mathcal{C}_{\omega_2} \Vdash \diamondsuit\texttt{CPA}(meager)$.
- Likewise for *null* and $\mathcal{R}_{\omega_2}$.

$\texttt{CPA}(meager)$

At round α Adam chooses a non-meager Borel set A_α and a Borel function $f_\alpha : A_\alpha \to 2^\omega$;
Eve responds with non-meager Borel set $E_\alpha \subseteq A_\alpha$.
Adam wins if $\bigcup_{\alpha<\omega_1} f_\alpha[E_\alpha] = 2^\omega$.
$\texttt{CPA}(meager) \equiv$ Eve has no winning strategy.

Kunen - small MAD

$\texttt{CPA}(meager) \Rightarrow \exists$ MAD of size ω_1.

$\mathbb{S}$ - a Polish space; $\mathbb{X}_\mathcal{I}$ ($\mathcal{I} \in \widetilde{\mathcal{I}}$) more Polish spaces;
each $\mathcal{I}$ from $\widetilde{\mathcal{I}}$ is a σ-ideal in $Borel(\mathbb{S} \times \mathbb{X}_\mathcal{I})$;
$\mathcal{P}_\mathcal{I} = Borel(\mathbb{S} \times \mathbb{X}_\mathcal{I}) \setminus \mathcal{I}$.
As before, at round α
Adam chooses $\mathcal{I}_\alpha \in \widetilde{\mathcal{I}}$, $p_\alpha \in \mathcal{P}_{\mathcal{I}_\alpha}$, and a Borel function

- $f_\alpha : p_\alpha \to 2^\omega$, in the regular game, or
- $f_\alpha : p_\alpha \to 2^\alpha$, in the diamond game.

Eve responds with $q_\alpha \subseteq p_\alpha$, $q_\alpha \in \mathcal{P}_{\mathcal{I}_\alpha}$.

Adam wins

- the regular game if $\exists s \in \mathbb{S} \bigcup_{\alpha<\omega_1} f_\alpha[q_\alpha|s] = 2^\omega$
- the diamond game if $\exists s \in \mathbb{S}$ for every $t \in 2^{\omega_1}$ for stationary many α, $t \restriction \alpha \in f_\alpha[q_\alpha|s]$;

Here $q_\alpha|s = q_\alpha \cap (\{s\} \times \mathbb{X}_{\mathcal{I}_\alpha})$.

$\mathtt{CPA}'(\widetilde{\mathcal{I}})$ and $\diamondsuit\mathtt{CPA}'(\widetilde{\mathcal{I}})$

Axioms $\mathtt{CPA}'(\widetilde{\mathcal{I}})$ and $\diamondsuit\mathtt{CPA}'(\widetilde{\mathcal{I}})$ both say, as before, that

- Eve has no winning strategy in the respective game.

The games are harder for Adam. The axioms are stronger.

Theorem

$\diamondsuit\mathtt{CPA}'(\widetilde{\mathtt{ctbl}})$ holds in the Sacks model.
Likewise for other forcings and ideals.

Laver and BC

Let $\widetilde{\mathcal{L}} = \{\mathcal{L}_\alpha\}_{\alpha<\omega_1}$, where $\mathcal{L}_\alpha$ is the σ-ideal in *Borel*$((\omega^\omega)^\alpha)$ associated with ctbl support iteration of Laver forcing.
Then $\diamondsuit\mathtt{CPA}'(\widetilde{\mathcal{L}})$ holds in the Laver model.

$\mathtt{CPA}'(\widetilde{\mathcal{L}}) \Rightarrow$ every strong measure zero subset of 2^ω is ctbl.

Carlson and Dual BC

$\mathtt{CPA}'(\textit{meager}) \Rightarrow$ every strongly meager subset of 2^ω is ctbl.

$\mathbb{S}, \mathbb{T}, \mathbb{Z}$ - Polish spaces, $\Delta \subseteq \mathbb{S} \times \mathbb{T} \times \mathbb{Z}$ - a Borel set.
A set $Z \subseteq \mathbb{Z}$ is s-small, $s \in \mathbb{S}$, if $\exists\, t \in \mathbb{T}\ Z \subseteq \Delta_{st}$.
Let Δ^* collect sets that are s-small for every $s \in \mathbb{S}$.

Fubini

Let $\mathcal{I}$ be a σ-ideal in $Borel(\mathbb{S} \times \mathbb{X})$, $\mathbb{X}$ - a Polish space, and let $\mathcal{H}$ be a hereditary family of subsets of $\mathbb{Z}$.
For every Borel function $g : \mathbb{S} \times \mathbb{X} \to \mathbb{T}$ form the set

$$\Delta(g) = \{\langle\langle s, x\rangle, z\rangle \in \mathbb{S} \times \mathbb{X} \times \mathbb{Z} : \langle s, g(s,x), z\rangle \in \Delta\} .$$

$Fubini(\Delta, \mathcal{I}, \mathcal{H})$ iff always

$$p \in \mathcal{P}_{\mathcal{I}} \quad \Rightarrow \quad \text{for co}\mathcal{H} \text{ many } z \quad p \setminus \Delta(g)^z \in \mathcal{P}_{\mathcal{I}}.$$

Lemma

$\mathtt{CPA}'(\widetilde{\mathcal{I}}) \wedge \mathit{Fubini}(\Delta, \widetilde{\mathcal{I}}, \mathcal{H}) \Rightarrow \Delta^* \subseteq \mathcal{H}$.

proof

Suppose $Z \in \Delta^* \setminus \mathcal{H}$. Talk Eve into the following tactic:

- Adam $f_\alpha : p_\alpha \to \mathbb{T}$,
- Eve $q_\alpha = A \setminus \Delta(f_\alpha)^{z_\alpha}$, $z_\alpha \in Z$ is from $\mathit{Fubini}(\Delta, \mathcal{I}_\alpha, \mathcal{H})$.

Then $f_\alpha[q_\alpha|s] \cap \Delta_s^{z_\alpha} = \varnothing$ for all s.
Find Adam's counter-play $\langle p_\alpha, f_\alpha \rangle_{\alpha<\omega_1}$ and $s \in \mathbb{S}$ s.t.

$$\bigcup_\alpha f_\alpha[q_\alpha|s] = \mathbb{T}.$$

Then the corresponding set $\{z_\alpha\}_{\alpha<\omega_1}$ fails the s-test:
$t \in f_\alpha[q_\alpha|s] \Rightarrow z_\alpha \notin \Delta_{st}$.

Laver and BC cont.

Set $\mathbb{S} = \omega^\omega$ and $\mathbb{T} = \mathbb{Z} = 2^\omega$. For $s \in \mathbb{S}$ let $\mathbb{T}_s = \prod_n [2^{s(n)}]^{\leq n}$.
For $t \in \mathbb{T}_s$ let $\overline{t} = \{z \in 2^\omega : \exists^\infty n \; z \restriction s(n) \in t(n)\}$.
Claim: $Z \subseteq 2^\omega$ is stronly null iff $\forall s \; \exists t \in \mathbb{T}_s \; Z \subseteq \overline{t}$.
Endow $[2^{s(n)}]^{\leq n}$ with discrete topology. Choose homeomorphisms $\varphi_s : \mathbb{T} \to \mathbb{T}_s$ so that the following set is G_δ

$$\Delta = \left\{ \langle s, t, z \rangle \in \mathbb{S} \times \mathbb{T} \times \mathbb{Z} : z \in \overline{\varphi_s(t)} \right\}.$$

Then Δ^* is exactly the family of strong measure zero subsets of $\mathbb{Z}$, and $Fubini(\Delta, \widetilde{\mathcal{L}}, \mathcal{H})$ holds for $\mathcal{H} = ctbl$.

Theorem Martin's Axiom - generics

Assume $\diamondsuit\mathtt{CPA}'(\widetilde{\mathcal{I}})$ for a family $\widetilde{\mathcal{I}}$.
Let $\mathcal{J} \subseteq \mathcal{P}(\mathbb{S})$ be an ideal s.t. for every $\mathcal{I} \in \widetilde{\mathcal{I}}$ and $S \in \mathcal{J}$ every $p \in \mathcal{P}_{\mathcal{I}}$ has $q \in \mathcal{P}_{\mathcal{I}}$ s.t. $q \subseteq p \setminus (S \times \mathbb{X}_{\mathcal{I}})$.
Then $cov(\mathcal{J}) > \omega_1$.

proof

Let $\{S_\alpha\}_{\alpha<\omega_1} \subseteq \mathcal{J}$ is ascend. Talk Eve into the following tactic:

- Adam $f_\alpha : p_\alpha \to 2^\alpha$,
- Eve $q_\alpha \subseteq p_\alpha \setminus (S_\alpha \times \mathbb{X}_{\mathcal{I}_\alpha})$.

Find Adam's counter-play $\langle p_\alpha, f_\alpha \rangle_{\alpha<\omega_1}$ and $s \in \mathbb{S}$ s.t.
$\mathbf{0} \restriction \alpha \in f_\alpha[q_\alpha|s]$ for stationary many α.
As $f_\alpha[q_\alpha|s] \neq \varnothing \Rightarrow q_\alpha|s \neq \varnothing \Rightarrow s \notin S_\alpha$, s falls out of stationary many S_α.
So, $s \notin \bigcup_{\alpha<\omega_1} S_\alpha$ because S_α ascend.

Imagine a binary relation ρ , $\mathbb{C} = dom\,\rho$ and $\mathbb{A} = rng\,\rho$.
If $c\rho a$ say that the answer a covers the challenge c.
A set of answers is covering if it covers every challenge.
The norm $\|\rho\|$ is the least size of a covering set.
Consider only Borel invariants
($\mathbb{C}$ and $\mathbb{A}$ are Polish spaces and ρ is a Borel subset of $\mathbb{C} \times \mathbb{A}$).

Many cardinal characteristics of the continuum are norms:

- the dominating number $\mathfrak{d}$ is $\|\mathfrak{D}\|$,
  $\mathfrak{D} = \{\langle s, t\rangle \in \omega^\omega \times \omega^\omega : \forall^\infty n\, s(n) \leq t(n)\}$,
- the number $\mathfrak{b}$ is $\|\mathfrak{B}\|$,
  $\mathfrak{B} = \{\langle s, t\rangle \in \omega^\omega \times \omega^\omega : \exists^\infty n\, s(n) > t(n)\}$;
- *non* $\mathcal{M}$, the least size of a nonmeager subset of 2^ω, is $\|\mathfrak{Non}\,\mathcal{M}\|$, $\mathfrak{Non}\,\mathcal{M}$ is a G_δ subset of $2^\omega \times 2^\omega$ whose vertical sections constitute a basis of the filter of comeager sets in 2^ω.

reductions

Write $\rho \sqsubset \rho'$ if there are functions translating answers $\alpha : \mathbb{A} \to \mathbb{A}'$ and challenges $\gamma : \mathbb{C}' \to \mathbb{C}$ s.t. $\gamma\rho\alpha \subseteq \rho'$.

$$\begin{array}{ccc} \mathbb{A} & \longrightarrow \alpha \longrightarrow & \mathbb{A}' \\ \rho \uparrow & & \vdots\ \rho' \\ \mathbb{C} & \longleftarrow \gamma \longleftarrow & \mathbb{C}' \end{array}$$

Notation for binary relations - (functions are relations):

- forward composition: $R_0 R_1 = \{\langle x, z\rangle : \exists y\ xR_0 y\ \&\ yR_1 z\}$;
- functorial notation: $R : X \to Y$ if $dom\, R = X$ and $rng\, R \subseteq Y$;
- images: given R, let $xR = \{y : xRy\}$, likewise $Ry = \{x : xRy\}$.

Constructive proofs of inequalities often give rise to reductions. E.g., from a suitable proof of $non\, \mathcal{M} \geq \mathfrak{b}$ we can get $\mathfrak{Non}\, \mathcal{M} \sqsubset \mathfrak{B}$ (even with Borel translating functions)

$\mathtt{CPA}(\rho)$

Axiom $\mathtt{CPA}(\rho)$ says that Eve has no winning strategy in the ρ game:
Adam and Eve, meet ω_1 times.
Adam wants to cover a space of topics $\mathbb{T} = 2^\omega$.
At round α Eve reveals topics via a Borel relation $R_\alpha : \mathbb{T} \to \mathbb{C}$.
Adam responds with $a_\alpha \in \mathbb{A}$ - covers topics from the set $R_\alpha \rho a_\alpha$.
Adam wins if $\mathbb{T} = \bigcup_\alpha R_\alpha \rho a_\alpha$ (all topics are covered).

$\diamondsuit\mathtt{CPA}(\rho)$

$\mathbb{T} = 2^{\omega_1}$ and $R_\alpha : 2^\alpha \to \mathbb{C}$.
Adam wins if for every topic $t \in 2^{\omega_1}$ for stationary many α,
$t \restriction \alpha \in R_\alpha \rho a_\alpha$.

Easily:
$\diamondsuit\mathtt{CPA}(\rho) \Rightarrow \mathtt{CPA}(\rho)$.
$\mathtt{CPA}(\rho) \Rightarrow \mathtt{CPA}(\rho')$ if $\rho \sqsubset \rho'$ with a Borel translation of challenges.

preserving answers in ρ

Say that $\mathcal{P}$ *preserves* answers if

$$\mathcal{P} \Vdash \text{"the ground model answers are covering"}\ .$$

This preservation property is well known. E.g., a poset preserves answers

- in $\mathfrak{D}$ iff it is ω^ω bounding;
- in $\mathfrak{B}$ iff it adds no dominating real;
- in $\mathfrak{Non}\,\mathcal{M}$ iff it forces that the ground model reals are non-meager.

Lemma

Suppose that $\mathcal{P}_{\mathcal{I}}$ is proper.The following are equivalent.

- $\mathcal{P}_{\mathcal{I}}$ preserves answers in ρ
- For any $p \in \mathcal{P}_{\mathcal{I}}$ and any Borel relation $F : p \to \mathbb{C}$ there exist $q \in \mathcal{P}_{\mathcal{I}}$ and $a \in \mathbb{A}$ s.t. $q \subseteq p \cap F\rho a$ $(q \Vdash F_{\dot{g}} \cap \rho^{\check{a}} \neq \emptyset)$.

Proposition

$\mathtt{CPA}(\widetilde{\mathcal{I}}) \Rightarrow \mathtt{CPA}(\underset{\sim}{\rho})$ and $\diamondsuit\mathtt{CPA}(\widetilde{\mathcal{I}}) \Rightarrow \diamondsuit\mathtt{CPA}(\rho)$
if each $\mathcal{P}_{\mathcal{I}}$ $(\mathcal{I} \in \widetilde{\mathcal{I}})$ preserves answers in ρ.

an application - Sacks, Miller, and $\diamondsuit\mathtt{CPA}(\mathfrak{B})$

$\widetilde{ctbl}$ and $\widetilde{\mathcal{K}_\sigma}$ preserve $\mathfrak{B}$, so
$\diamondsuit\mathtt{CPA}(\mathfrak{B})$ is true in the Sacks model and in the Miller model.

Proposition - proof

Assume $\mathtt{CPA}(\widetilde{\mathcal{I}})$. Fix a strategy of Eve in the ρ game and define her strategy in the $\widetilde{\mathcal{I}}$ game as follows. Consider round α. At the ρ board, following her strategy, Eve plays $R_\alpha : \mathbb{T} \to \mathbb{C}$. She notices Adam's move $f_\alpha : p_\alpha \to \mathbb{T}$ at the $\widetilde{\mathcal{I}}$ board, applies lemma to $F_\alpha = f_\alpha R_\alpha$, and plays q_α for which she can find $a_\alpha \in \mathbb{A}$ s.t.

$$q_\alpha \subseteq p_\alpha \cap F_\alpha \rho a_\alpha \,.$$

Let $f_\alpha : p_\alpha \to \mathbb{T}\,,\ \alpha < \omega_1$ be Adam's winning counter-play at the $\widetilde{\mathcal{I}}$ board.

We define a winning counter-play for him at the ρ board.

Following her strategies, Eve plays $R_\alpha : \mathbb{T} \to \mathbb{C}$ and finds q_α and a_α s.t. $q_\alpha \subseteq f_\alpha R_\alpha \rho a_\alpha$. Fortunately for Adam f_α is a function, so $q_\alpha f_\alpha \subseteq R_\alpha \rho a_\alpha$. Now $\mathbb{T} = \bigcup_\alpha q_\alpha f_\alpha \subseteq \bigcup_\alpha R_\alpha \rho a_\alpha$, so $\langle a_\alpha \rangle_{\alpha < \omega_1}$ is a winning counter-play for Adam.

Proposition

$\mathtt{CPA}(\rho) \Rightarrow \|\rho\| = \omega_1$.

proof

Let $\mathbb{T} = \mathbb{C}$. Let Eve play as $R_\alpha : \mathbb{T} \to \mathbb{C}$ the identity function.
If $\langle a_\alpha \rangle_{\alpha<\omega_1} \subseteq \mathbb{A}$ is Adam's winning counter-play, then
$\mathbb{C} = \mathbb{T} = \bigcup_\alpha R_\alpha \rho a_\alpha = \bigcup_\alpha \rho a_\alpha$.

Theorem

Let $\mathcal{P}_{\omega_2}$ be the limit of a ctbl support iteration of length ω_2 of a "nice" sequence of proper Borel posets.
Then $\mathcal{P}_{\omega_2} \Vdash \|\rho\| \leq \omega_1 \Leftrightarrow \mathcal{P}_{\omega_2} \Vdash \diamondsuit\mathtt{CPA}(\rho)$.

proof

A suitable adaptation of a similar result about parametrized $\diamondsuit$ from [DHM].

free lunch

Often $\mathfrak{c} = \omega_2$ and $\|\rho\| = \omega_1$ is forced by a ctbl support iteration of length ω_2 of proper Borel posets that add reals but preserve answers.
By one of Shelah's preservation theorems the limit $\mathcal{P}_{\omega_2}$ also preserves answers.
So, the ground model answers witness $\mathcal{P}_{\omega_2} \Vdash \|\rho\| \leq \omega_1$.
Bonus: by the above theorem $\mathcal{P}_{\omega_2} \Vdash \diamondsuit\mathtt{CPA}(\rho)$.

A more brutal method of forcing $\diamondsuit\mathtt{CPA}(\rho)$ is finite support iteration.
Say that a poset $\mathcal{P}$ kills challenges if it adds an answer that answers all ground model challenges,
i.e., there is a term $\dot{a}$ for a member of $\mathbb{A}$ s.t.
for every ground model $c \in \mathbb{C}$, $\mathcal{P} \Vdash c\rho\dot{a}$.

finite support iteration thm

Let $\mathcal{P}_{\omega_1}$ be a finite support iteration of length ω_1 of ccc posets.
If $\forall\, \alpha < \omega_1\ \mathcal{P}_\alpha \Vdash$ "$\mathcal{P}_{\omega_1}/\mathcal{P}_\alpha$ kills challenges in ρ",
then $\mathcal{P}_{\omega_1} \Vdash \diamondsuit\mathtt{CPA}(\rho)$.

Since $\mathcal{C} = \mathcal{P}_{\mathcal{M}}$ preserves answers in $\mathfrak{Non}\,\mathcal{M}$,
$\diamondsuit\mathtt{CPA}(\{\mathcal{M}\})$ implies $\diamondsuit\mathtt{CPA}(\mathfrak{Non}\,\mathcal{M})$.
Assuming $V \models \diamondsuit$, from $\mathcal{C}_{\omega_2} \Vdash \diamondsuit\mathtt{CPA}(\mathit{meager})$, and the fact that $\diamondsuit\mathtt{CPA}(\mathit{meager})$ is just $\diamondsuit\mathtt{CPA}(\{\mathcal{M}\})$, we get $\mathcal{C}_{\omega_2} \Vdash \diamondsuit\mathtt{CPA}(\mathfrak{Non}\,\mathcal{M})$.
Actually we can drop $V \models \diamondsuit$ here.

Proposition

$\mathcal{C}_{\omega_2} \Vdash \diamondsuit\mathtt{CPA}(\mathfrak{Non}\,\mathcal{M})$.

proof

$\mathcal{C}_{\omega_2} \approx \mathcal{C}_{\omega_2} * \mathcal{C}_{\omega_1}$, view $\mathcal{C}_{\omega_1}$ as $\mathcal{C}$ iterated ω_1 times with finite support. Every tail, adds a Cohen real, kills the intermediate model challenges in $\mathfrak{Non}\,\mathcal{M}$. The theorem applies.

in fact

Any finite support iteration of nontrivial ccc forcings of length ω_1 forces $\diamondsuit\mathtt{CPA}(\mathfrak{Non}\,\mathcal{M})$.

Likewise for random.

Proposition

$\mathcal{R}_{\omega_2} \Vdash \diamondsuit\texttt{CPA}(\mathfrak{Non}\,\mathcal{N})$

$\diamondsuit^*$

In fact, in both propositions, we have a club of good $\alpha < \omega_1$. $\diamondsuit^*\texttt{CPA}(\mathfrak{Non}\,\mathcal{M})$ and $\diamondsuit^*\texttt{CPA}(\mathfrak{Non}\,\mathcal{N})$ are forced.

Parametrized $\Diamond$ of [DHM]

$\Diamond(\rho) \equiv$ Eve has no winning tactic.

So,

$$\Diamond \mathtt{CPA}(\rho) \;\Rightarrow\; \Diamond(\rho).$$