A simple proof of Gowers' dichotomy theorem

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Ramsey Theorems

Let us first recall some classical Ramsey type theorems.

The first result hardly deserves to be called a theorem, but can be extremely useful when used correctly.

Theorem (Dirichlet's principle)

Suppose

$$c\colon \mathbb{N} \to \{\textit{verde}, \textit{amarelo}\}$$

is a colouring of the natural numbers with two colours verde *and* amarelo*. Then there is an infinite subset $A \subseteq \mathbb{N}$ which is* monochromatic*.*

On the other hand, though not too hard to prove, Ramsey's Theorem is much deeper.

Theorem (Ramsey's Theorem - infinite version)

Suppose

$$c\colon [\mathbb{N}]^2 \to \{\textit{verde}, \textit{amarelo}\}$$

is a colouring with 2 colours. Then there is an infinite subset $A \subseteq \mathbb{N}$ *such that* $[A]^2$ *is monochromatic.*

And finally, the pride of infinite-dimensional Ramsey theory:

Theorem (Galvin - Prikry)

Suppose

$$c\colon [\mathbb{N}]^\infty \to \{\mathit{verde}, \mathit{amarelo}\}$$

is a Borel colouring with 2 colours. Then there is an infinite subset $A \subseteq \mathbb{N}$ such that $[A]^\infty$ is monochromatic.

Ramsey theory for Banach spaces?

For the geometric theory of Banach spaces, it would be extremely useful to have a similar principle for stabilising colourings.

So let us try to set up an ideal correspondence of the objects of classical Ramsey theory with Banach spaces:

$$\text{the base set } \mathbb{N} \sim \text{an infinite-dimensional Banach space } X$$
$$\text{infinite subsets } A \subseteq \mathbb{N} \sim \text{infinite-dimensional subspaces } Y \subseteq X$$
$$\text{numbers } n \in \mathbb{N} \sim \text{vectors } x \in X$$

Dirichlet's principle for Banach spaces?

So here is the analogue of Dirichlet's principle for Banach spaces.

First of all, since we can colour vectors according to their norm, we should restrict the consideration to vectors of norm 1.

Secondly, since we are dealing with continuous objects, our colours and colouring should also be continuous.

Question

Suppose X is a separable infinite-dimensional Banach space and

$$c : S_X \to [0,1]$$

is a Lipschitz function defined on the unit sphere S_X of X.
For any $\epsilon > 0$, is there an infinite-dimensional subspace $Y \subseteq X$ such that

$$\mathrm{diam}(c[S_Y]) < \epsilon \ ?$$

Gowers' c_0-theorem

In the early 1990s, W. T. Gowers proved that at least for c_0, i.e., the space

$$c_0 = \{(x_n) \in \mathbb{R}^{\mathbb{N}} \mid x_n \xrightarrow[n\to\infty]{} 0\},$$

there is a valid Dirichlet principle.

Theorem (Gowers)

Suppose

$$c\colon S_{c_0} \to [0,1]$$

is a Lipschitz function and $\epsilon > 0$.
Then there is an infinite-dimensional subspace $Y \subseteq c_0$ *such that*

$$\mathrm{diam}(c[S_Y]) < \epsilon.$$

The distortion theorem

Unfortunately, c_0 is the only space for which this Dirichlet principle is true.

Theorem (Odell-Schlumprecht)

Suppose X is a separable infinite-dimensional Banach space not containing an isomorphic copy of c_0.
Then there is an infinite-dimensional subspace $Y \subseteq X$ and a Lipschitz function

$$c \colon S_Y \to [0,1]$$

such that for any infinite-dimensional subspace $Z \subseteq Y$, we have

$$c[S_Z] = [0,1].$$

So this shows that even the most basic Ramsey principle fails in general Banach spaces.

We shall instead try to replace the stabilisation of colourings with a more dynamical principle that can hold in more general cases.

Vector spaces over countable fields

Suppose $\mathfrak{F}$ is a countable field (think of $\mathbb{Q}$ or $\mathbb{Q} + i\mathbb{Q}$) and E is a countable-dimensional $\mathfrak{F}$-vector space with basis $(e_n)_{n=1}^{\infty}$.

Then E is a countable set, which we give the discrete topology, and we equip the infinite product $E^{\mathbb{N}}$ with the product topology. So $E^{\mathbb{N}}$ is Polish.

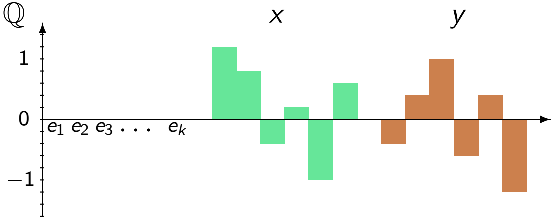
Now if $x = \sum a_n e_n \in E$, we let $\mathrm{supp}(x) = \{n \in \mathbb{N} \mid a_n \neq 0\}$, which is a finite set of integers.

Also, for non-zero vectors $x, y \in E$ and an integer k, set

$$x < y \quad \Leftrightarrow \quad \max \mathrm{supp}(x) < \min \mathrm{supp}(y)$$

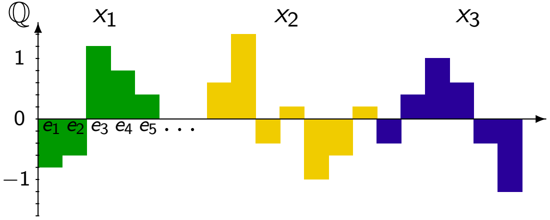
and

$$k < x \quad \Leftrightarrow \quad k < \min \mathrm{supp}(x).$$



A block sequence of (e_n) is an infinite sequence of non-zero vectors $(x_1, x_2, x_3, \ldots)$ such that

$$x_1 < x_2 < x_3 < x_4 < \ldots.$$



A block subspace of E is an infinite-dimensional subspace $X \subseteq E$ spanned by an infinite block sequence (x_n) (written $X = [x_n]$).

By elementary linear algebra, any infinite-dimensional subspace of E contains an infinite-dimensional block subspace $X = [x_n]$.

Therefore, from the point of view of Ramsey theory, we only have to worry about block subspaces.

Notation

In the following,

$$X, Y, Z, \ldots$$

denote (infinite-dimensional) block subspaces of E,

$$x, y, z, \ldots$$

denote non-zero vectors in E, and

$$\vec{x}, \vec{y}, \vec{z}, \ldots$$

denote finite block sequences $\vec{x} = (x_0, x_1, \ldots, x_n)$ in E.

The Gowers game

Let $X \subseteq E$ be any block subspace and define the Gowers game G_X played below X as follows:

$$\begin{array}{lccccccc} \textbf{I} & Y_0 & & Y_1 & & Y_2 & & \dots \\ \textbf{II} & & z_0 \in Y_0 & & z_1 \in Y_1 & & z_2 \in Y_2 & \dots \end{array}$$

Here the Y_n are block subspaces of X and the $z_n \in Y_n$ are nonzero vectors such that $z_n < z_{n+1}$.

The outcome of an infinite run of the game is the infinite block sequence (z_n) in X.

If also $\vec{x}$ is a finite block sequence, $G_X(\vec{x})$ is defined as above, except that the outcome is now $\vec{x}^\frown(z_n)$.

The Infinite Asymptotic game

Fix a block subspace $X \subseteq E$ and define the Infinite Asymptotic game F_X played below X as follows:

$$
\begin{array}{lcccccccc}
\mathbf{I} & k_0 & & k_1 & & k_2 & & \dots \\
\mathbf{II} & & k_0 < z_0 & & k_1 < z_1 & & k_2 < z_2 & \dots
\end{array}
$$

Here I and II alternate in choosing respectively natural numbers k_n and non-zero vectors $k_n < z_n \in X$ according to the constraint $z_n < z_{n+1}$.

The outcome of an infinite run of the game is again the infinite block sequence (z_n) in X.

Again, if $\vec{x}$ is a finite block sequence, the game $F_X(\vec{x})$ is played as above, except that the outcome is now $\vec{x}^\frown(z_n)$.

One easily sees that the infinite asymptotic game is equivalent to the Gowers game in which we moreover demand that player I plays block subspaces Y_n of finite codimension in X.

So from this follows that if $\mathbb{A} \subseteq E^{\mathbb{N}}$, then

If II has a strategy in $G_X(\vec{x})$ to play in $\mathbb{A}$, then II has a strategy in $F_X(\vec{x})$ to play in $\mathbb{A}$.

Conversely,

If I has a strategy in $F_X(\vec{x})$ to play in $\mathbb{A}$, then I has a strategy in $G_X(\vec{x})$ to play in $\mathbb{A}$.

So the infinite asymptotic game is easier to play for II than the Gowers game, while the converse is true for player I.

Strategically Ramsey sets

Definition

We say that a set $\mathbb{A} \subseteq E^{\mathbb{N}}$ is strategically Ramsey if for any $V \subseteq E$ and finite block sequence $\vec{v}$, there is a block subspace $Y \subseteq V$ such that one of the following two mutually exclusive conditions hold:

- *II has a strategy in $G_Y(\vec{v})$ to play in $\mathbb{A}$,*
- *I has a strategy in $F_Y(\vec{v})$ to play in the complement $\sim\mathbb{A}$.*

Since the game $G_Y(\vec{v})$ is easier to play for I than the game $F_Y(\vec{v})$, we see that the conclusion is stronger than just stating that the game $G_Y(\vec{v})$ to play in or out of $\mathbb{A}$ is determined.

So being strategically Ramsey is more than determinacy of a certain game. But on the other hand, the conclusion is only obtained on a subspace $X \subseteq E$ and not on E itself, so the definition also involves Ramsey theory.

The proof of Galvin–Prikry's Theorem for open sets requires relatively subtle Ramsey theoretical techniques.

On the other hand, using hardly any Ramsey theory at all, but only Gale–Stewart's proof of determinacy of open games, one can show that open sets are strategically Ramsey.

Open sets are strategically Ramsey

Suppose $Y = [y_n]$. We write $Y \subseteq^* X$ to denote that $y_n \in X$ for all but finitely many n, i.e., that $Y \cap X$ has finite codimension in Y.

So, as for infinite subsets of $\mathbb{N}$, if

$$X_0 \supseteq^* X_1 \supseteq^* X_2 \supseteq^* \ldots$$

is a decreasing sequence of subspaces, there is some X_∞ such that $X_\infty \subseteq^* X_n$ for all n.

Recall the game $G_X(\vec{x})$:

$$\begin{array}{lcccccccc} \textbf{I} & Y_0 & & Y_1 & & Y_2 & & & \dots \\ \textbf{II} & & z_0 \in Y_0 & & z_1 \in Y_1 & & z_2 \in Y_2 & & \dots \end{array}$$

where $Y_0, Y_1, \ldots \subseteq X$.

By the asymptotic nature of $G_X(\vec{x})$, we have for $Z \subseteq^* X$

If II has a strategy in $G_X(\vec{x})$ to play in $\mathbb{A}$,
then II has a strategy in $G_Z(\vec{x})$ to play in $\mathbb{A}$.

Similarly,

If I has a strategy in $F_X(\vec{x})$ to play in $\mathbb{A}$,
then I has a strategy in $F_Z(\vec{x})$ to play in $\mathbb{A}$.

Proposition

Open sets are strategically Ramsey.

Proof

Suppose $\mathbb{U} \subseteq E^{\mathbb{N}}$ is open and suppose $V \subseteq E$ and $\vec{v}$ are given.

By a simple diagonalisation over all finite block sequences $\vec{x}$, we can find some $X \subseteq V$ such that for all $Y \subseteq^* X$ and $\vec{x}$,

II has a strategy in $G_Y(\vec{x})$ to play in $\mathbb{U}$

if and only if

II has a strategy in $G_X(\vec{x})$ to play in $\mathbb{U}$.

By a further diagonalisation, we can find some $Y \subseteq X$ such that for all $\vec{x}$,

> *if there is some $Z \subseteq Y$ such that for all $y \in Z$, II has no strategy in $G_Y(\vec{x}\hat{\ }y)$ to play in $\mathbb{U}$, then there is some n such that for all $y \in Y$, if $n < y$, then II has no strategy in $G_Y(\vec{x}\hat{\ }y)$ to play in $\mathbb{U}$.*

We will show that either

- II has a strategy in $G_Y(\vec{v})$ to play in $\mathbb{A}$, or
- I has a strategy in $F_Y(\vec{v})$ to play in the complement $\sim\mathbb{A}$.

Let T be the set of all $\vec{x}$ such that II has no strategy in $G_Y(\vec{x})$ to play in $\mathbb{U}$.

Since $\mathbb{U}$ is open, we have $[T] \cap \mathbb{U} = \emptyset$.

Also, suppose that $\vec{x} \in T$.

Then, as II has no strategy in $G_Y(\vec{x})$ to play in $\mathbb{U}$, I can play some $Z \subseteq Y$ such that for all $y \in Z$, II has no strategy in $G_Y(\vec{x}\hat{\ }y)$ to play in $\mathbb{U}$.

So for some n and all $y \in Y$, if $n < y$, then II has no strategy in $G_Y(\vec{x}\hat{\ }y)$ to play in $\mathbb{U}$, i.e., $\vec{x}\hat{\ }y \in T$.

Thus, in the game $F_Y(\vec{x})$, we can let I play this n, and so for any response $y \in Y$ by II such that $n < y$, we still have $\vec{x}\hat{\ }y \in T$.

This shows that if we have $\vec{v} \in T$, then T provides a quasi-strategy for I in $F_Y(\vec{v})$ to play in $[T] \subseteq \sim\mathbb{U}$.

On the other hand, if $\vec{v} \notin T$, then, by definition, II has a strategy in $G_Y(\vec{v})$ to play in $\mathbb{U}$. **Q.E.D.**

The extent of strategically Ramsey sets

Theorem

Analytic sets strategically Ramsey.

This result was originally proved by Gowers for so called weakly Ramsey sets in Banach spaces, whereas the above result speaks about sets of block sequences in countable-dimensional vector spaces over countable fields.

Secondly, as a byproduct of the proof of our theorem, we have that

Theorem

The class of strategically Ramsey sets is closed under countable unions.

Theorem (MA_κ)

The class of strategically Ramsey sets is closed under κ-unions.

Again, J. Bagaria and J. López-Abad, proved the same conclusion for the class of weakly Ramsey sets in Banach spaces under an assumption relatively consistent with the existence of a weakly compact cardinal.

Moreover, as $\mathbf{\Sigma}^1_2$ sets are unions of $\aleph_1$ Borel sets, we have

Corollary (MA_{ω_1})

$\mathbf{\Sigma}^1_2$*-sets are strategically Ramsey.*

Strategically Ramsey sets in normed vector spaces

The full power of the notion of strategically Ramsey sets really becomes apparent once we work within an ambient normed vector space.

So suppose that $\mathfrak{F}$ is a countable subfield of $\mathbb{R}$ or of $\mathbb{C}$ and $\|\cdot\|$ is a norm on the vector space E, i.e., $\|\cdot\|: E \to \mathbb{R}$.

For $X \subseteq E$, let $B_X = \{x \in X \mid \|x\| \leqslant 1\}$ denote the unit ball of X.

Also, if $\Delta = (\delta_n)$ is a sequence of positive real numbers $\delta_n > 0$, which we denote by $\Delta > 0$, we define for any set $\mathbb{A} \subseteq E^{\mathbb{N}}$,

$$\mathbb{A}_\Delta = \{(z_n) \in E^{\mathbb{N}} \mid \exists (y_n) \in \mathbb{A} \quad \forall n \quad \|z_n - y_n\| < \delta_n\}$$

and

$$\begin{aligned} \mathrm{Int}_\Delta(\mathbb{A}) &= \{(z_n) \in E^{\mathbb{N}} \mid \forall (y_n) \ \big(\forall n \ \ \|z_n - y_n\| < \delta_n \to (y_n) \in \mathbb{A}\big)\} \\ &= \sim (\sim \mathbb{A})_\Delta. \end{aligned}$$

The fundamental theorem of infinite asymptotic games

Theorem

Suppose I has a strategy in F_X to play in some set $\mathbb{A}$ and $\Delta > 0$. Then there is a sequence of finite intervals

$$I_0 < I_1 < I_2 < I_3 < \ldots \subseteq \mathbb{N}$$

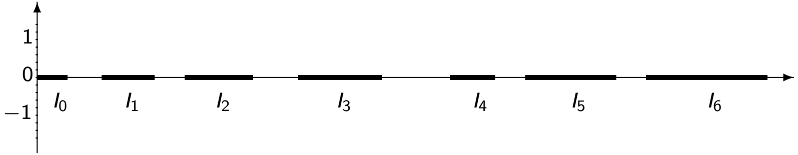
such that for any block sequence $(z_n) \subseteq B_X$, if

$$(*) \qquad \forall n\, \exists m \qquad I_0 < z_n < I_m < z_{n+1},$$

then $(z_n) \in \mathbb{A}_\Delta$.

This result has been through some iterations. Originally, some version of this was proved for closed sets by E. Odell and Th. Schlumprecht, while another result was proved for coanalytic sets by C. Rosendal. The above result is essentially from a paper by V. Ferenczi and C. Rosendal.

To understand the condition $(*)$, note that it just requires that any two successive vectors z_n and z_{n+1} are separated by an interval I_m and the support of the first vector z_1 begins after I_0.



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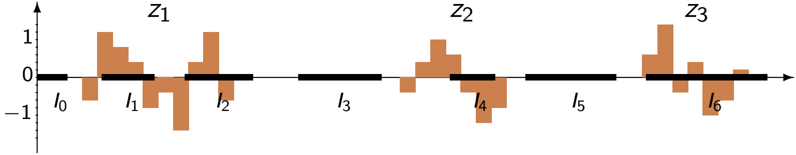

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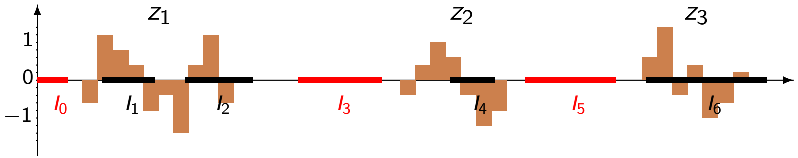

This really says that if ever I has a strategy in F_X to play in A, then for any $\Delta > 0$, I has a trivial strategy in F_X to play in the slightly bigger $\mathbb{A}_\Delta$.

For in this case, I does not need to look at what II is playing, but can say in advance of the game that if the block sequence is sufficiently separated, as given by the intervals (I_n), then $(z_n) \in \mathbb{A}_\Delta$.

So if we combine the fundamental theorem for infinite asymptotic games with the definition of strategically Ramsey sets, we obtain

Corollary (Gowers' Selection Theorem)

Suppose $\mathbb{A} \subseteq E^{\mathbb{N}}$ is strategically Ramsey (e.g., analytic) such that for some $\Delta > 0$ and all block subspaces $X \subseteq E$ there is a block sequence $(z_n) \subseteq B_X$ lying in $\mathrm{Int}_\Delta(\mathbb{A})$.
Then there is a block subspace $X \subseteq E$ such that II has a strategy in G_X to ensure that the outcome lies in $\mathbb{A}$.

Adversarial Ramsey principles

Note that, as opposed to most other games between two players, in both the Gowers game and the infinite asymptotic game, only player II directly contributes to the outcome.

So are there similar games in which both players contribute?

A. M. Pełczar and later V. Ferenczi considered such a game that we will refine here.

The A_X-game

Let $X \subseteq E$ and define the game A_X as follows:

$$\begin{array}{llllll} \textbf{I} & & n_0 < x_0, Z_0 & & n_1 < x_1, Z_1 & & \dots \\ \textbf{II} & n_0 & & y_0 \in Z_0, n_1 & & y_1 \in Z_1, n_2 & \dots \end{array}$$

where $Z_0, Z_1, \ldots \subseteq X$ and the vectors are subject to the condition $x_0 < y_0 < x_1 < y_1 < \ldots$.

The *outcome* is the infinite block sequence $(x_0, y_0, x_1, y_1, \ldots)$.

Note that here, I simultaneously acts a player I in G_X and as player II in F_X and vice versa for II.

In fact, we can separate this into the games G_X and F_X, which then have a joint outcome.

The B_X-game

In the game B_X, we change the roles of I and II, while preserving the order in which they are playing.

Thus, I is still the first player to play a vector, but is now the one having to choose vectors in arbitrary subspaces, while II only has to choose vectors beyond some integer.

I		$x_0 \in Z_0, n_0$		$x_1 \in Z_1, n_1$		$\dots$
II	Z_0		$n_0 < y_0, Z_1$		$n_1 < y_1, Z_2$	$\dots$

Again, the outcome is the block sequence $(x_0, y_0, x_1, y_1, \dots)$.

The adversarial Ramsey property

Definition

A set $\mathbb{A} \subseteq E^{\mathbb{N}}$ *is said to be* adversarially Ramsey *if for any* $X \subseteq E$ *there is some* $Y \subseteq X$ *such that one of the following two conditions hold:*

- *II has a strategy in* A_Y *to play in* $\mathbb{A}$*,*
- *I has a strategy in* B_Y *to play in* $\sim\mathbb{A}$*.*

Notice again that this is stronger than just demanding that the games A_Y and B_Y are determined.

For either **II** will have a strategy in the game A_Y to play in $\mathbb{A}$

$$\begin{array}{llllllll} \textbf{I} & & n_0 < x_0, Z_0 & & n_1 < x_1, Z_1 & & & \dots \\ \textbf{II} & n_0 & & y_0 \in Z_0, n_1 & & & y_1 \in Z_1, n_2 & \dots \end{array}$$

where **I** is playing the block subspaces.

Or **I** will have a strategy in the game B_Y to play in $\sim\mathbb{A}$

$$\begin{array}{llllllll} \textbf{I} & & x_0 \in Z_0, n_0 & & x_1 \in Z_1, n_1 & & & \dots \\ \textbf{II} & Z_0 & & n_0 < y_0, Z_1 & & & n_1 < y_1, Z_2 & \dots \end{array}$$

where now **II** is playing the block subspaces.

Adversarial Ramsey sets and determinacy

The connection between adversarial Ramsey sets and determinacy is very tight. In fact, we have the following direct implication.

Proposition

Let Γ be a class of subsets of Polish spaces closed under continuous preimages. Assume that any Γ set $\mathbb{A} \subseteq E^{\mathbb{N}}$ is adversarially Ramsey. Then Γ games on $\mathbb{N}$ are determined.

The reason for this is that we can code integers by vectors in any block subspace using, e.g., their first coordinate on the basis (e_n).

So, by the famous theorem of H. Friedman, any proof to the effect that Borel sets are adversarially Ramsey must use $\aleph_1$ iterations of the power set operation.

In particular, we cannot hope to adopt the techniques used to prove that analytic sets are strategically Ramsey sets.

Also, the best we can hope for in ZFC is that Borel sets should be adversarially Ramsey.

Though, the class of adversarially Ramsey sets is not formally closed under complementation, if Γ is a point class of adversarially Ramsey sets, which is closed under continuous preimages, any $\check{\Gamma}$ set is also adversarially Ramsey.

The techniques introduced by A. M. Pełczar and V. Ferenczi essentially give that closed sets are adversarially Ramsey, but the best current result is

Theorem

$\mathbf{\Pi}^0_3$ *and* $\mathbf{\Sigma}^0_3$ *sets are adversarially Ramsey.*

We do not know if Borel sets are adversarially Ramsey, and there are concrete examples to show that D. A. Martin's proof of Borel determinacy at least is not easily adaptable to our setting.

On the other hand, determinacy of $\mathbf{\Pi}^0_3$ games can be fully proved in second order number theory (this is due to Morton Davis), and the techniques of the proof commute sufficiently with Ramsey theory of block sequences to give the above result.

Rumours have it that A. Montalban and R. Shore have shown that $\mathbf{\Delta}^0_4$ determinacy for games on $\mathbb{N}$ cannot be proved in second order arithmetic, while it can be proven for a difference of finitely many $\mathbf{\Sigma}^0_3$ sets.

So are Borel sets adversarially Ramsey?