On forcing with σ-ideals of closed sets

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Idealized forcing

Many classical forcing notions can be represented in the form $\mathbf{P}_I = \mathrm{Bor}(X) \setminus I$, where X is a Polish space and I is a σ-ideal on X.

Examples

The examples are: the Cohen forcing (σ-ideal of meager sets), the Sacks forcing (σ-ideal of countable sets), or the Miller forcing (K_σ sets in ω^ω)

Another way

Note that the forcing $\mathbf{P}_I = \mathrm{Bor}(X) \setminus I$ is equivalent to the quotient Boolean algebra $\mathrm{Bor}(X)/I$ (which is the separative quotient of $\mathbf{P}_I$).

The generic real

A forcing notion of the form $\mathrm{Bor}(\omega^\omega)/I$ adds the *generic real*, denoted $\dot{g}$ and defined in the following way:

$$[\![\dot{g}(n) = m]\!] = [(n,m)]_I$$

where $[(n,m)]$ is the basic clopen in ω^ω).

Genericity

Of course, the generic ultrafilter can be recovered from the generic real in the following way:

$$G = \{B \in \mathrm{Bor}(X) : g \in B\}$$

where g denotes the generic real.

The σ-ideal

We say that a σ-ideal I is *generated by closed sets*, if for each $A \in I$ there is a sequence of closed sets $F_n \in I$ such that $A \subseteq \bigcup_{n<\omega} F_n$.

Theorem (Solecki)

Let I be a σ-ideal generated by closed sets. If $A \subseteq X$ is analytic, then either $A \in I$, or else A contains a G_δ set G such that $G \notin I$.

Corollary

From the above theorem of Solecki we get that if I is generated by closed sets, then $\mathbf{P}_I$ is forcing equivalent to $\mathbf{Q}_I = \mathbf{\Sigma}^1_1 \setminus I$ ($\mathbf{P}_I$ is dense in $\mathbf{Q}_I$).

Theorem (Zapletal)

If I is a σ-ideal generated by closed sets, then the forcing $\mathbf{P}_I$ is proper.

Axiom A

Recall that a forcing notion $\mathbf{P}$ satisties Baumgartner's *Axiom A* if there is a sequence of partial orders $\leq_n$ on $\mathbf{P}$ such that $\leq_0 = \leq$, $\leq_{n+1} \subseteq \leq_n$ and

- if $\langle p_n \in \mathbf{P}, n < \omega \rangle$ is such that $p_{n+1} \leq_n p_n$, then there is $q \in \mathbf{P}$ such that $q \leq_n p_n$ for all n,
- for every $p \in \mathbf{P}$, for every n and for every name $\dot\alpha$ for an ordinal there exist $q \in \mathbf{P}$ and a countable set of ordinals A such that $q \leq_n p_n$ for each $n < \omega$, and $q \Vdash \dot\alpha \in A$.

Proposition (MS)

If I is a σ-ideal generated by closed sets, then the forcing $\mathbf{P}_I$ is equivalent to a forcing with trees, which satisfies Axiom A.

Sketch of the proof

Assume $X = \omega^\omega$ and fix I. Let $A \subseteq \omega^\omega$ be an analytic set and let T be a tree on $\omega \times \omega$ projecting to A.

Game $G_I(T)$

Consider the following game (between Adam and Eve).

- in his n-th move, Adam picks $\tau_n \in T$ such that τ_{n+1} extends τ_n.
- in her n-th move, Eve picks a clopen set O_n in ω^ω such that

$$\mathrm{proj}[T_{\tau_n}] \notin \mathcal{I} \;\Rightarrow\; O_n \cap \mathrm{proj}[T_{\tau_n}] \notin \mathcal{I}.$$

Winning condition

By the end of a play, Adam and Eve have a sequence of closed sets E_k in ω^ω defined as follows:

$$E_k = 2^\omega \setminus \bigcup_{i<\omega} O_{\rho^{-1}(i,k)}.$$

(ρ is some fixed bijection between ω and ω^2). Define $x = \pi(\bigcup_{n<\omega} \tau_n) \in \omega^\omega$. **Adam wins** if and only if

$$x \notin \bigcup_{k<\omega} E_k.$$

Lemma

Eve has a winning strategy in $G_I(T)$ if and only if $A = \mathrm{proj}[T] \in I$.

Strategy

If S is a strategy for Adam in $G_I(T)$, then by $\mathrm{proj}[S]$ we denote the set of points $x \in \omega^\omega$ which arise at the end of some game obeying S.

Lemma

If S is a winning strategy in $G_I(T)$, then $\mathrm{proj}[S]$ is an analytic subset of A and $\mathrm{proj}[S] \notin I$.

Forcing with strategies

Consider the following forcing $\mathbf{T}_I$:
$\{S : S \text{ is a winning strategy for Adam in } G_I(T) \text{ for some tree } T\}$
ordered as follows:

$$S_0 \leq S_1 \text{ iff } \mathrm{proj}[S_0] \subseteq \mathrm{proj}[S_1].$$

Dense embedding

Notice that $\mathbf{T}_I \ni S \mapsto \mathrm{proj}[S] \in \mathbf{Q}_I$ is a dense embedding, hence the three forcing notions $\mathbf{P}_I$, $\mathbf{Q}_I$ and $\mathbf{T}_I$ are forcing equivalent. Let us show that $\mathbf{T}_I$ satisfies Axiom A.

Winning condition revised

Recall that the winning condition for Adam in $G_I(T)$ says that

$$x \notin \bigcup_k E_k.$$

Fix k. For each play in $G_I(T)$ both x and E_k are built "step-by-step" (E_k from basic clopen sets which sum up to $\omega^\omega \setminus E_k$). Hence, if π is a play and $x \notin E_k$, then there is $m < \omega$ such that the partial play $\pi \restriction m$ already determines that "$x \notin E_k$".

Fusion

Let $S \in \mathbf{T}_I$ be a winning strategy for Adam. For each play π in S there is the least $m < \omega$ such that $\pi \restriction m$ determines that "$x \notin E_i$" for $i \leq k$. Therefore, we can define the k-th front of the tree S, denoted by $F_k(S)$ so that each play determines "$x \notin E_i$" before passing through $F_k(S)$.

Axiom A

We define the inequalities $\leq_k$ as follows: $S_0 \leq S_1$ if and only if

- $S_0 \leq S_1$,
- $F_k(S_0) = F_k(S_1)$.

Fubini power

Recall the classical definition of the *Fubini product* of two σ-ideals I and J (on X and Y, resp.). For $A \subseteq X \times Y$

$$A \in I * J \quad \text{iff} \quad \{x \in X : A_x \notin J\} \in I.$$

This allows to define I^n for each $n < \omega$. We can also define I^α (for $\alpha < \omega_1$) as follows

Definition

Call a set $D \subseteq X^\alpha$ an (I,α) *tree* if

(i) for each $\beta < \alpha$ and for each $x \in \pi_{\alpha,\beta}[D]$ we have

$$(\pi_{\alpha,\beta+1}[D])_x \notin \mathcal{I}$$

(ii) for each limit $\beta < \alpha$ and $x \in X^\beta$,

$$x \in \pi_{\alpha,\beta}[D] \quad \Leftrightarrow \quad \forall\gamma < \beta \quad x{\restriction}\gamma \in \pi_{\alpha,\gamma}[D].$$

Say that D is an (I,α) *full tree* if in (i): $X \setminus (\pi_{\alpha,\beta+1}[D])_x \in \mathcal{I}$.

Definable trees

If $\mathbf{\Gamma}$ is a projective class, then we say that D is an (I,α) $\mathbf{\Gamma}$ *tree* if D is a tree and

(iii) $\pi_{\alpha,\beta}[D] \in \mathbf{\Gamma}$ for each $\beta \leq \alpha$ (so in particular $D \in \mathbf{\Gamma}$).

Definition

We say that $A \subseteq X^\alpha$ is in I^α, if there is a (I, α) full tree D disjoint from A.

Theorem (Zapletal)

If I is an *iterable* σ-ideal and $\alpha < \omega_1$, then the countable-support iteration $(\mathbf{P}_I)^{*\alpha}$ is equivalent to the forcing $\mathbf{P}_{I^\alpha}$. Moreover, each I^α-positive Borel set contains an (I, α) Bor tree.

Definition

We say that I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ if for each $\mathbf{\Sigma}^1_1$ set $A \subseteq X \times X$ the set

$$\{x \in X : A_x \in I\}$$

is $\mathbf{\Pi}^1_1$.

Theorem (Kanovei, Zapletal)

Suppose I is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ iterable σ-ideal and let $\alpha < \omega_1$. Then for any analytic set $A \subseteq X^\alpha$

- either $A \in I^\alpha$,
- or A contains an (I, α) Bor tree.

Lemma

Let I be a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal generated by closed sets. If $V \subseteq W$ is a forcing extension, then I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ and generated by closed sets also in W.

Iterability

The above lemma in particular implies that $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideals generated by closed sets, are iterable.

Proposition (MS)

Let I be a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal generated by closed sets. If $A \subseteq X^\omega$ is analytic, then

- either $A \in I^\omega$,
- or A contains a (I, ω) $\mathbf{G}_\delta$ tree.

The end

Thank You.