Thin equivalence relations in scaled pointclasses

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Classical theorems

An equivalence relation on $\mathbb{R} (= \omega^\omega)$ is thin if there is no perfect set of pairwise inequivalent reals.

Theorem (Silver 1980)

*Every thin coanalytic equivalence relation is Borel.*

Theorem (Harrington, Sami 1979)

*Projective determinacy implies that every thin $\Sigma^1_{2n}$ equivalence relation is $\Delta^1_{2n}$.*

Harrington and Sami showed the analogous results for many more pointclasses.
Examples

**Example**

Let $xEy$ if $x, y$ code wellorders with equal ranks or they don’t code wellorders.

$E$ is $\Sigma_1^1 - \Pi_1^1$ and thin.

**Example**

Consider a $\Sigma_{2n}^1$ norm on some $\Sigma_{2n}^1$ complete set.
Let $xEy$ if $x, y$ have the same rank or they are both not in the set.

$E$ is $\Pi_{2n}^1 - \Sigma_{2n}^1$ and thin assuming PD.
Let $J_1(\mathbb{R}) = V_{\omega+1}$ and $J_{\alpha+1}(\mathbb{R}) = \text{rud}(J_\alpha(\mathbb{R}) \cup \{J_\alpha(\mathbb{R})\})$.

**Definition**

A $\Sigma_1$ gap $[\alpha, \beta]$ is a maximal interval so that the same $\Sigma_1$ statements with parameters in $\mathbb{R} \cup \{\mathbb{R}\}$ hold in $J_\alpha(\mathbb{R})$ and $J_\beta(\mathbb{R})$.

The $\Sigma_1$ gaps partition the ordinals.

**Theorem (Steel)**

$\text{AD}^{J_\alpha(\mathbb{R})}$ implies that $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$ is scaled for every $\alpha$, i.e. $\Gamma$ sets have definable tree representations.
Main theorem

Theorem

Assume $\text{AD}^{L(\mathbb{R})}$. Suppose $\alpha$ begins a $\Sigma_1$ gap in $L(\mathbb{R})$ and $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$. Then every thin $\Gamma$ equivalence relation is $\check{\Gamma}$. 
\(\omega\)-cofinal pointclasses

Suppose \(\Gamma = \Sigma_1^J(\mathbb{R})\) is not closed under number quantification - i.e. under \(\forall n\) - then

- \(\alpha\) is a successor or
- \(cf(\alpha) = \omega\) in \(L(\mathbb{R})\).

\(\Gamma\) sets are countable unions of sets in \(J_\alpha(\mathbb{R})\).

**Lemma (Jackson)**

Assume \(AD\). Suppose \(\Gamma\) is a non-selfdual scaled pointclass closed under \(\exists^R\) but not under \(\forall n\). Then \(\Gamma\) is closed under wellordered unions.
Theorem (Harrington, Shelah 1980)

Assume ZF. Suppose $E = \mathbb{R}^2 - p[T]$ is a thin equivalence relation where $T$ is a tree on $\omega \times \omega \times \kappa$. Suppose that $\mathbb{R}^2 - p[T]$ is an equivalence relation in any Cohen generic extension of $L[T]$. Then there is an enumeration $(E_\alpha : \alpha < \gamma)$ of the equivalence classes with $\gamma \leq \kappa$. 
Theorem 1

**Theorem (Schindler, S.)**

Assume $\text{AD}^{L(\mathbb{R})}$. Suppose $\alpha$ begins a $\Sigma_1$ gap in $L(\mathbb{R})$ and $\Gamma = \Sigma^J_\alpha(\mathbb{R})$. If $\Gamma$ is not closed under number quantification, then every thin $\Gamma$ equivalence relation is $\tilde{\Gamma}$.

- $\Gamma$ sets are Suslin
- the class of Suslin sets is closed under countable intersections
- so $E = \mathbb{R}^2 - p[T]$ for some tree $T$
- let $(E_\alpha : \alpha < \gamma)$ enumerate the equivalence classes
- $\mathbb{R}^2 - E = \bigcup_{\alpha \neq \beta < \gamma} (E_\alpha \times E_\beta)$ is $\Gamma$
Mice

**Definition**

$M_n^#(x)$ is a minimal $\omega_1$-iterable $x$-premouse with $n$ Woodin cardinals and an extender above them.

$M_n^#(x)$ exists for every real $x$ if and only if $\Pi^1_{n+1}$ determinacy holds.

**Theorem (Woodin’s genericity iteration)**

Suppose $M$ is an $\omega_1 + 1$-iterable premouse with a Woodin cardinal. There is a forcing $\mathbb{Q}$ and for every real $x$ an iteration map $\pi : M \to N$ such that $x$ is $\mathbb{Q}^N$-generic over $N$. 
Lemma

Suppose $\tau$ is a $\mathbb{P}$-name for a real. Let $\tau_i$ for $i = 0, 1$ be $\mathbb{P} \times \mathbb{P}$-names with $\tau^g_i = \tau^{g_i}$ for any $\mathbb{P} \times \mathbb{P}$-generic $g_0 \times g_1$.

Lemma (Hjorth)

Let $E$ be a thin $\Sigma^1_{2n}$ equivalence relation and $\mathbb{P} \in M^\#_{2n-1}$. For densely many $p \in \mathbb{P}$:

$$(p, p) \models_{\mathbb{P} \times \mathbb{P}} \tau_0 E \tau_1$$

- build a binary tree of conditions
- associate a filter $g_x$ to each $x \in 2^\omega$ so that
  - $M^\#_{2n-1} [g_x, g_y] \models \neg \tau^{g_x} E \tau^{g_y}$
  - $g_x \times g_y$ is $\mathbb{P} \times \mathbb{P}$-generic for $x \neq y$
- the set of $\tau^{g_x}$ is a perfect set of inequivalent reals, contradiction
Projective case

We first sketch the proof of

**Theorem**

Suppose $M_{2n-1}^\#(x)$ exists for every real $x$. Then every thin $\Sigma_{2n}^1$ equivalence relation is $\Pi_{2n}^1$.

**Idea:** with reals $a, b$ we associate

- reals $c, d$ with $aEc$ and $bEd$
- $M_{2n-1}^\# \rightarrow N$ with $N[c, d] \models \neg cEd$.

**Fact:** the condition that $N$ is an iterate of $M_{2n-1}^\#$ via an iteration tree bounded below the least Woodin cardinal is $\Sigma_{2n}^1$ (in every real coding $M_{2n-1}^\#$ as a parameter).
Proof

- Let $\kappa$ be the critical point of the top measure of $M_{2n-1}^#$.
- Let $\pi : M \to M_{2n-1}^#|\kappa$ elementary with
  - $M \triangleleft M_{2n-1}^#$
  - $\gamma < \pi(\gamma) = \delta$
  - $V_M^{\gamma} = V_{M_{2n-1}^{\gamma}}$
Proof

\( \neg aEb \) if and only if there are reals \( c, d \) and an iteration map \( \rho : M_{2n-1}^\# \rightarrow N \) living on \( M_{2n-1}^\# | \gamma \) with

- \( aEc \) and \( bEd \)
- \( (c, d) \) is \( Q^\rho(M) \)-generic over \( N \)
- \( N[c, d] \models \neg cEd \)
Capturing terms

Suppose $\Gamma = \Sigma_1^J(\mathbb{R})$ is closed under number quantification.

**Theorem (Woodin)**

Suppose $\text{AD}^J(\mathbb{R})$ holds and $A \in \Gamma$. There is a mouse $N$ with a Woodin cardinal $\delta$ and the property:

for every $\lambda \geq \delta$ there is a $\text{Col}(\omega, \lambda)$-name $\dot{A}$ so that $A \cap P[g] = \dot{A}^g$

for any iteration map $\pi : N \to P$ and $\text{Col}(\omega, \pi(\lambda))$-generic $g$ over $P$. 
Suppose $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$ is closed under number quantification. Let $N$ be a mouse as above with a capturing term for the complete $\Gamma$ set.

**Theorem (Schindler, S.)**

Assume $\text{AD}^{J_\alpha(\mathbb{R})}$. Then every thin $\Gamma$ equivalence relation is $\tilde{\Gamma}$ in any real coding $N$ as a parameter.
Let $E$ be a thin $\Gamma = \sum J_\alpha^\infty(\mathbb{R})$ equivalence relation. Let $N$ be a mouse as above with $Col(\omega, \delta)$-capturing terms $\dot{E}, \dot{S}$ for $E$ and its $\Gamma$ scale.

Lemma (Schindler, S.)

Assume $AD^{J_\alpha(\mathbb{R})}$. Let $n \geq 1$ and $\pi : M \rightarrow N\upharpoonright(\delta^n)^N$ sufficiently elementary with $\dot{E}, \dot{S} \in \text{rng}(\pi)$ and $\bar{E} = \pi^{-1}(\dot{E})$. Then $\bar{E}g \subseteq E$ for every $Col(\omega, \pi^{-1}(\delta))$-generic filter $g$ over $M$.

The analogous result holds for iterates of $M$. 