Thin equivalence relations in scaled pointclasses

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Classical theorems

An equivalence relation on $\mathbb{R}$ $(= \omega^\omega)$ is thin if there is no perfect set of pairwise inequivalent reals.

Theorem (Silver 1980)

Every thin coanalytic equivalence relation is Borel.

Theorem (Harrington, Sami 1979)

Projective determinacy implies that every thin Σ^1_{2n} equivalence relation is Δ^1_{2n}.

Harrington and Sami showed the analogous results for many more pointclasses.

Examples

Example

Let xEy if x, y code wellorders with equal ranks or they don't code wellorders.

E is $\Sigma^1_1 - \Pi^1_1$ and thin.

Example

Consider a Σ^1_{2n} norm on some Σ^1_{2n} complete set.
Let xEy if x, y have the same rank or they are both not in the set.

E is $\Pi^1_{2n} - \Sigma^1_{2n}$ and thin assuming PD.

$L(\mathbb{R})$

Let $J_1(\mathbb{R}) = V_{\omega+1}$ and $J_{\alpha+1}(\mathbb{R}) = rud(J_\alpha(\mathbb{R}) \cup \{J_\alpha(\mathbb{R})\})$.

Definition

A Σ_1 gap $[\alpha, \beta]$ is a maximal interval so that the same Σ_1 statements with parameters in $\mathbb{R} \cup \{\mathbb{R}\}$ hold in $J_\alpha(\mathbb{R})$ and $J_\beta(\mathbb{R})$.

The Σ_1 gaps partition the ordinals.

Theorem (Steel)

$\mathrm{AD}^{J_\alpha(\mathbb{R})}$ *implies that $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$ is scaled for every α, i.e. Γ sets have definable tree representations.*

Main theorem

Theorem

Assume $\mathsf{AD}^{L(\mathbb{R})}$. *Suppose* α *begins a* Σ_1 *gap in* $L(\mathbb{R})$ *and* $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$. *Then every thin* Γ *equivalence relation is* $\check{\Gamma}$.

ω-cofinal pointclasses

Suppose $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$ is not closed under number quantification - i.e. under $\forall n$ - then

- α is a successor or
- $cf(\alpha) = \omega$ in $L(\mathbb{R})$.

Γ sets are countable unions of sets in $J_\alpha(\mathbb{R})$.

Lemma (Jackson)

Assume AD. *Suppose* Γ *is a non-selfdual scaled pointclass closed under* $\exists^{\mathbb{R}}$ *but not under* $\forall n$. *Then* Γ *is closed under wellordered unions.*

Harrington-Shelah

Theorem (Harrington, Shelah 1980)

Assume ZF. *Suppose* $E = \mathbb{R}^2 - p[T]$ *is a thin equivalence relation where* T *is a tree on* $\omega \times \omega \times \kappa$. *Suppose that* $\mathbb{R}^2 - p[T]$ *is an equivalence relation in any Cohen generic extension of* $L[T]$. *Then there is an enumeration* $(E_\alpha : \alpha < \gamma)$ *of the equivalence classes with* $\gamma \leq \kappa$.

Theorem 1

Theorem (Schindler, S.)

Assume $AD^{L(\mathbb{R})}$*. Suppose* α *begins a* Σ_1 *gap in* $L(\mathbb{R})$ *and* $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$*. If* Γ *is not closed under number quantification, then every thin* Γ *equivalence relation is* $\check{\Gamma}$*.*

- Γ sets are Suslin
- the class of Suslin sets is closed under countable intersections
- so $E = \mathbb{R}^2 - p[T]$ for some tree T
- let $(E_\alpha : \alpha < \gamma)$ enumerate the equivalence classes
- $\mathbb{R}^2 - E = \bigcup_{\alpha \neq \beta < \gamma}(E_\alpha \times E_\beta)$ is Γ

Mice

Definition

$M_n^\#(x)$ is a minimal ω_1-iterable x-premouse with n Woodin cardinals and an extender above them.

$M_n^\#(x)$ exists for every real x if and only if Π^1_{n+1} determinacy holds.

Theorem (Woodin's genericity iteration)

Suppose M is an $\omega_1 + 1$-iterable premouse with a Woodin cardinal. There is a forcing $\mathbb{Q}$ and for every real x an iteration map $\pi : M \to N$ such that x is $\mathbb{Q}^N$-generic over N.

Lemma

Suppose τ is a $\mathbb{P}$-name for a real. Let τ_i for $i = 0, 1$ be $\mathbb{P} \times \mathbb{P}$-names with $\tau_i^g = \tau^{g_i}$ for any $\mathbb{P} \times \mathbb{P}$-generic $g_0 \times g_1$.

Lemma (Hjorth)

Let E be a thin Σ^1_{2n} equivalence relation and $\mathbb{P} \in M^\#_{2n-1}$. For densely many $p \in \mathbb{P}$:

$$(p, p) \vDash^{M^\#_{2n-1}}_{\mathbb{P}\times\mathbb{P}} \tau_0 E \tau_1$$

- build a binary tree of conditions
- associate a filter g_x to each $x \in 2^\omega$ so that
  - $M^\#_{2n-1}[g_x, g_y] \vDash \neg\tau^{g_x} E \tau^{g_y}$
  - $g_x \times g_y$ is $\mathbb{P} \times \mathbb{P}$-generic for $x \neq y$
- the set of τ^{g_x} is a perfect set of inequivalent reals, contradiction

Projective case

We first sketch the proof of

Theorem

Suppose $M^{\#}_{2n-1}(x)$ exists for every real x. Then every thin Σ^1_{2n} equivalence relation is Π^1_{2n}.

Idea: with reals a, b we associate

- reals c, d with aEc and bEd
- $M^{\#}_{2n-1} \to N$ with $N[c, d] \models \neg cEd$.

Fact: the condition that N is an iterate of $M^{\#}_{2n-1}$ via an iteration tree bounded below the least Woodin cardinal is Σ^1_{2n} (in every real coding $M^{\#}_{2n-1}$ as a parameter).

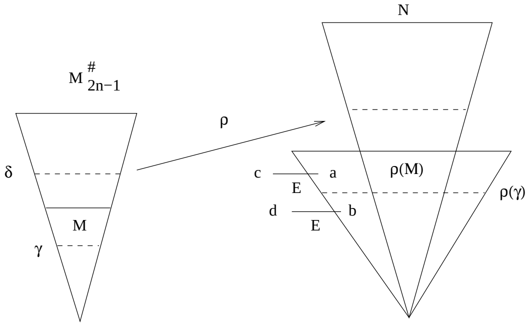
Proof

- let κ be the critical point of the top measure of $M^{\#}_{2n-1}$
- let $\pi : M \to M^{\#}_{2n-1}|\kappa$ elementary with
  - $M \lhd M^{\#}_{2n-1}$
  - $\gamma < \pi(\gamma) = \delta$
  - $V_\gamma^M = V_\gamma^{M^{\#}_{2n-1}}$

Proof

$\neg aEb$ if and only if there are reals c, d and and an iteration map $\rho : M^{\#}_{2n-1} \to N$ living on $M^{\#}_{2n-1}|\gamma$ with

- aEc and bEd
- (c, d) is $\mathbb{Q}^{\rho(M)}$-generic over N
- $N[c, d] \vDash \neg cEd$



Capturing terms

Suppose $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$ is closed under number quantification.

Theorem (Woodin)

Suppose $\mathsf{AD}^{J_\alpha(\mathbb{R})}$ *holds and* $A \in \Gamma$. *There is a mouse* N *with a Woodin cardinal* δ *and the property:*
for every $\lambda \geq \delta$ *there is a* $Col(\omega, \lambda)$*-name* $\dot{A}$ *so that* $A \cap P[g] = \dot{A}^g$ *for any iteration map* $\pi : N \to P$ *and* $Col(\omega, \pi(\lambda))$*-generic* g *over* P.

Theorem 2

Suppose $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$ is closed under number quantification. Let N be a mouse as above with a capturing term for the complete Γ set.

Theorem (Schindler, S.)

Assume $\mathsf{AD}^{J_\alpha(\mathbb{R})}$. Then every thin Γ equivalence relation is $\check{\Gamma}$ in any real coding N as a parameter.

Upwards absoluteness

Let E be a thin $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$ equivalence relation. Let N be a mouse as above with $Col(\omega, \delta)$-capturing terms $\dot{E}, \dot{S}$ for E and its Γ scale.

Lemma (Schindler, S.)

Assume $\mathsf{AD}^{J_\alpha(\mathbb{R})}$. *Let* $n \geq 1$ *and* $\pi : M \to N|(\delta^{+n})^N$ *sufficiently elementary with* $\dot{E}, \dot{S} \in rng(\pi)$ *and* $\bar{E} = \pi^{-1}(\dot{E})$.
Then $\bar{E}^g \subseteq E$ *for every* $Col(\omega, \pi^{-1}(\delta))$*-generic filter* g *over* M.

The analogous result holds for iterates of M.