# On properties of families of sets

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# The beginning

**Theorem (Bernstein)**

*There is $X \subset \mathbb{R}$ such that neither $X$ nor $\mathbb{R} \setminus X$ contain a perfect subset.*
***The family of perfect subsets of the reals has property B.***

**Definition**

A family $\mathcal{A}$ has **property B** iff there is a set $X$ such that $X \cap A \neq \emptyset$ and $A \setminus X \neq \emptyset$ for each $A \in \mathcal{A}$.

**Theorem (Bernstein)**

*If $|\mathcal{A}| = \kappa$ and $|A| \geq \kappa$ for each $A \in \mathcal{A}$ then $\mathcal{A}$ has **property** $B$.*

# Miller's results

**Theorem (E. W. Miller, 1937)**

*Let $n \in \omega$. If $\mathcal{A}$ is a family of **infinite countable sets**, and $|A \cap A'| < n$ for each $A \neq A' \in \mathcal{A}$ then $\mathcal{A}$ has **property** $B$.*

$\mathcal{A}$ is **$\mu$-almost-disjoint** iff $|A \cap A'| < \mu$ for $A \neq A' \in \mathcal{A}$

# Miller's results

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## Theorem (E. W. Miller)

*If a family $\mathcal{A}$ of infinite countable sets is $n$-almost disjoint for some $n \in \omega$ then $\mathcal{A}$ has **property** $B$.*

Proof:

- $\mathcal{A} = \{A_k : k < \omega\}$. Let $A'_k = A_k \setminus \cup\{A_\ell : \ell < k\}$.
- Pick $x_k \in A'_k$ and let $X = \{x_k : k < \omega\}$.
- $x_k \in X \cap A_k$, $|X \cap A_k| \leq k + 1 < \omega$.
- $\exists X \ (\forall A \in \mathcal{A}) \ 0 < |A \cap X| < \omega$

# Miller's results

## Theorem (E. W. Miller)

*If a family $\mathcal{A}$ of infinite countable sets is n-almost disjoint for some $n \in \omega$ then $\mathcal{A}$ has* ***property*** *$B$.*

- $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\} \subset \left[\omega_1\right]^\omega$
- $\mathcal{A} \in M_1 \prec M_2 \cdots \prec M_\alpha \prec M_{\alpha+1} \prec \cdots \prec \langle H(\kappa), \in, \lhd \rangle$,
- continuous chain of elementary submodels $(M_0 = \emptyset)$
- For $\alpha < \omega_1$ consider the family $\mathcal{A} \cap (M_{\alpha+1} \setminus M_\alpha)$ $=\{A_{\alpha,k} : k < \omega\}$
- **if** $A \in M_{\alpha+1} \setminus M_\alpha$ **then** $A \subset M_{\alpha+1}$ **and** $|A \cap M_\alpha| < n$
- Let $A'_{\alpha,k} = (A_{\alpha,k} \setminus (\cup\{A_{\alpha,\ell} : \ell < k\})) \setminus M_\alpha$.
- Pick $x_{\alpha,k} \in A'_{\alpha,k}$. Let $X = \{x_{\alpha,k} : \alpha < \omega_1, k < \omega\}$.
- $x_{\alpha,k} \in X$, $|X \cap A_{\alpha,k}| \le |M_\alpha \cap A_{\alpha,k}| + |\{x_{\alpha,0}, \ldots, x_{\alpha,k}\}| \le n + k$
- $\exists X \ (\forall A \in \mathcal{A})\ 0 < |X \cap A| < \omega$
- $A'_{\alpha,k} \subset A_{\alpha,k}$ $A_{\alpha,k} \setminus A'_{\alpha,k}$ is finite
- $A'_{\alpha,k} \cap A'_{\beta,m} = \emptyset$ if $\alpha < \beta$ then $A_{\alpha,k} \subset M_\beta$ and $A'_{\beta,m} \cap M_\beta = \emptyset$.

# Miller's results

**Theorem (E. W. Miller)**

*If a family $\mathcal{A}$ of infinite countable sets is n-almost disjoint for some $n \in \omega$ then $\mathcal{A}$ has **property** $B$.*

- $\mathcal{A} = \{A_{\alpha,k} : \alpha < \omega_1, K < \omega\} \subset [\omega_1]^{\omega}$
- $A'_{\alpha,k} \subset A_{\alpha,k}$, $A_{\alpha,k} \setminus A'_{\alpha,k}$ is finite , $A'_{\alpha,k} \cap A'_{\beta,m} = \emptyset$
- $\mathcal{A}$ **is $\omega$-essentially disjoint**

A family $\mathcal{A}$ is **$\mu$-essentially disjoint ($\mu$-ED)** iff for each $A \in \mathcal{A}$ there is $F(A) \in [A]^{<\mu}$ such that $\{A \setminus F(A) : A \in \mathcal{A}\}$ is disjoint

**Theorem (Erdős-Hajnal,1961)**

*If a family $\mathcal{A}$ of infinite countable sets is n-almost disjoint for some $n \in \omega$ then $\mathcal{A}$ is $\omega$-ED.*

# Miller's results

**Theorem (E. W. Miller)**

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**Theorem (Erdős-Hajnal,1961)**

*If $\mathcal{A}$ is an n-almost disjoint family of infinite countable countable sets for some $n \in \omega$, then $\mathcal{A}$ is $\omega$-ED.*

A family $\mathcal{A}$ of infinite sets has property $B(\mu)$ iff there is a set $X$ such that $0 < |X \cap A| < \mu$ for each $A \in \mathcal{A}$.

**Theorem (Erdős-Hajnal,1961)**

*If $\mathcal{A}$ is an n-almost disjoint family of infinite countable sets for some $n \in \omega$, then $\mathcal{A}$ has property $B(\omega)$.*

# Miller's result

$\mathcal{A}$ is $\mu$-**ED** iff $\forall A \in \mathcal{A}\ \exists F(A) \in [A]^{<\mu}$ s. t. $\{A \setminus F(A) : A \in \mathcal{A}\}$ is disjoint

$\mathcal{A}$ has property $B(\mu)$ iff $\exists X\ \forall A \in \mathcal{A}\ 0 < |X \cap A| < \mu$.

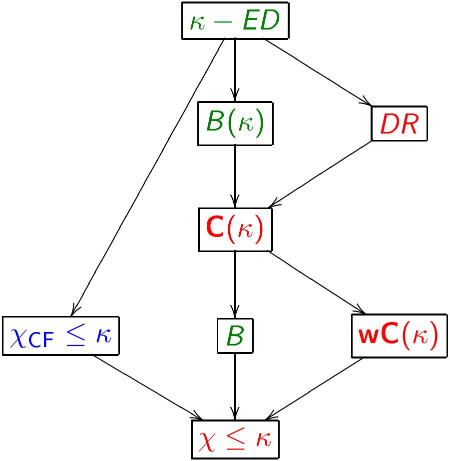
$\mathcal{A}$ **is** $\omega$-**ED** implies $\mathcal{A}$ **has property** $B(\omega)$ implies $\mathcal{A}$ **has property** $B$.

- **We consider only subfamilies of $[\lambda]^\kappa$ for some $\omega \leq \kappa \leq \lambda$.**

Notation: $M(\lambda, \kappa, \mu) \to \Phi$ means that
**every $\mu$-almost disjoint family $\mathcal{A} \subset [\lambda]^\kappa$ has property $\Phi$**

Miller, Erdös-Hajnal: $M(\lambda, \omega, n) \to B$, $M(\lambda, \omega, n) \to B(\omega)$,
$M(\lambda, \omega, n) \to \omega$-**ED**,

## $\mu \leq \kappa \leq \lambda$
## $\Psi \to \Phi$ iff $\mathbf{M}(\lambda, \kappa, \mu) \to \Psi$ implies $\mathbf{M}(\lambda, \kappa, \mu) \to \Phi$.




**DR:** $\forall A \in \mathcal{A}\ \exists F(A) \in [A]^{|A|}$ s t $\{F(A) : A \in \mathcal{A}\}$ **is disjoint**

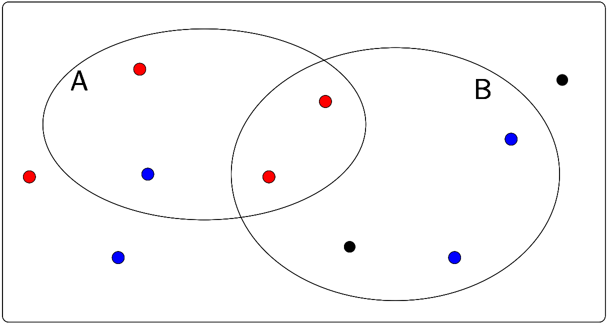
$\mathbf{C}(\kappa)$ : $\exists f\ \forall A \in \mathcal{A}\ f''A = \kappa$

$\mathbf{wC}(\kappa)$ : $\exists f\ \forall A \in \mathcal{A}\ f''A \in [\kappa]^{\kappa}$

$\chi \leq \kappa$: $\exists f\ (\mathrm{ran}(f) \subset \kappa$ and $\forall A \in \mathcal{A}\ |f''A| \geq 2)$

# Colorings

- $\mathcal{A} \subset \mathcal{P}(X)$ a family of sets, $f : X \to \rho$ function
- $f$ is a **proper coloring** of $\mathcal{A}$ iff $|f''A| \geq 2$ for each $A \in \mathcal{A}$.
- $f$ is called a **conflict free coloring** iff $\forall A \in \mathcal{A}\ \exists \xi_A \in \rho\ \exists! a \in A$ $f(a) = \xi_A$.
- $\chi_{\mathrm{CF}}(\mathcal{A})$= $\min\{\rho : \exists f : X \to \rho$ conflict free coloring$\}$



# $\omega$-EDness and coloring

$f : \cup\mathcal{A} \to \rho$ is a **conflict free coloring** for $\mathcal{A}$ iff

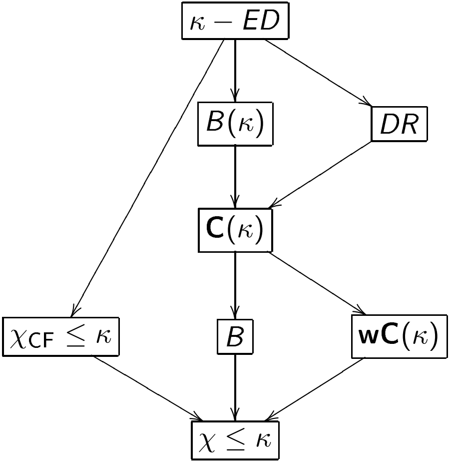
$$\forall A \in A\ \exists \xi_A \in \rho\ \exists! a \in A\ f(a) = \xi_A.$$

$\chi_{CF}(\mathcal{A}) = \min\{\rho : \exists f : X \to \rho \text{ conflict free coloring}\}$

- Let $\mathcal{A} \subset [\lambda]^\omega$
- **If $\mathcal{A}$ is $\omega$-ED, then $\chi_{CF}(\mathcal{A}) \le \omega$.**
- Proof: For $A \in \mathcal{A}$ let $F(A) \in [A]^{<\omega}$ s.t. $\{A \setminus F(A) : A \in \mathcal{A}\}$ is disjoint
- Let $f : \lambda \to \omega$ s.t. $f \restriction A \setminus F(A)$ is injective
- If $A \in \mathcal{A}$ then $F(A)$ is finite and $f \restriction A \setminus F(A)$ is injective so there is $a \in A \setminus F(A)$ s.t. $f(a) \notin f''F(A)$.
- Miller: If $\mathcal{A} \subset [\lambda]^\omega$ is $n$-a.d. for some $n \in \omega$ then $\chi_{CF}(\mathcal{A}) \le \omega$.
- $\chi(\mathcal{A}) \le \chi_{CF}(\mathcal{A})$.

## $\mu \leq \kappa \leq \lambda$

## $\Psi \to \Phi$ iff **M**$(\lambda, \kappa, \mu) \to \Psi$ **implies M**$(\lambda, \kappa, \mu) \to \Phi$.




**DR:** $\forall A \in \mathcal{A}\ \exists F(A) \in [A]^{|A|}$ s t $\{F(A) : A \in \mathcal{A}\}$ **is disjoint**

**C**$(\kappa)$ : $\exists f\ \forall A \in \mathcal{A}\ f''A = \kappa$

**wC**$(\kappa)$ : $\exists f\ \forall A \in \mathcal{A}\ f''A \in [\kappa]^{\kappa}$

$\chi \leq \kappa$: $\exists f$ (ran$(f) \subset \kappa$ and $\forall A \in \mathcal{A}\ |f''A| \geq 2$)

$\chi_{\text{CF}} \leq \kappa$: $\exists f$ (ran$(f) \subset \kappa$ and $\forall A \in \mathcal{A}\ \exists \xi \in \kappa\ \exists! a \in A$ $f(a) = \xi$)

# $n$-almost disjoint families

- If $\mathcal{A} \subset [\lambda]^{\omega}$ is $n$-a.d. for some $n \in \omega$ then $\mathcal{A}$ is $\omega$-ED, and so $\chi_{\mathsf{CF}}(\mathcal{A}) \leq \omega$.
- There is a 2-ad. family $\mathcal{A} \subset [\omega_1]^{\omega_1}$ which is not $\omega$-ED.
- Remark: $\mathcal{A}$ above is $\omega_1$-ED, and so it has property $B$, but $\omega_1$-ED does not implies $\chi_{\mathsf{CF}}(\mathcal{A}) \leq \omega$.

**Theorem (HJSSs)**

*For each infinite cardinals $\kappa \leq \lambda$ and $n \in \omega$ if $\mathcal{A} \subset [\lambda]^{\kappa}$ is n-almost disjoint then $\chi_{\mathsf{CF}}(\mathcal{A}) \leq \omega$.*

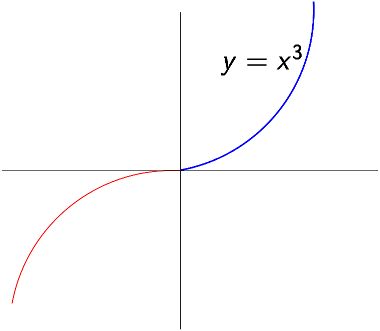
- $\chi_{\mathrm{CF}}([\lambda]^{\kappa}, \mu\text{-a.d.}) = \sup\{\chi_{\mathsf{CF}}(\mathcal{A}) : \mathcal{A} \subset [\lambda]^{\kappa}, |\mathcal{A}| = \lambda, \mathcal{A} \text{ is } \mu\text{-ad}\}$.

**Theorem (HJSSs)**

$\chi_{\mathrm{CF}}([\lambda]^{\kappa}, n\text{-a.d.}) \leq \omega$ *for each infinite cardinals $\kappa \leq \lambda$ and $n \in \omega$.*

Def: $\chi_{\mathrm{CF}}([\lambda]^{\kappa}, \mu\text{-}\mathbf{a.d.}) \le \rho$ iff every $\mu$-**almost disjoint** family $\mathcal{A} \subset [\lambda]^{\kappa}$ of size $\lambda$ has a conflict free coloring with $\rho$ **colors.**
Thm: $\chi_{\mathrm{CF}}([\lambda]^{\kappa}, n\text{-}\mathbf{a.d.}) \le \omega$.

- **Do we really need $\omega$ colors?**
- **Special case**: Let $\mathcal{E}$ be the family of lines of the plane $\mathbb{R}^2$. $\mathcal{E}$ is 2-almost disjoint.
- Every line contains exactly one blue point or exactly one red point
- $\chi_{\mathrm{CF}}(\mathcal{E}) \le 3$ $\quad \chi_{\mathrm{CF}}(\mathcal{E}) = 3$



# $n$-almost disjoint families: sharper theorems

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- **Special case**: Let $\mathcal{E}$ be the family of lines of the plane $\mathbb{R}^2$. $\mathcal{E}$ is 2-almost disjoint.
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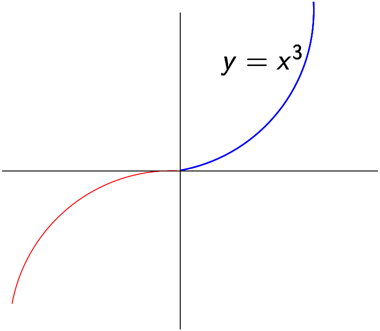



- $f$ is a **weak conflict free coloring** for $\mathcal{A}$ iff $\forall A \in \mathcal{A}\ \exists \xi\ \exists! a \in A$ $f(a) = \xi$.
- $\mathrm{w}\chi_{\mathrm{CF}}(\mathcal{A})$ is the minimal $\rho$ s.t. $\mathcal{A}$ has a weak conflict free coloring with $\rho$ colors.
- $\mathrm{w}\chi_{\mathrm{CF}}(\mathcal{A}) \leq \chi_{\mathrm{CF}}(\mathcal{A}) \leq \mathrm{w}\chi_{\mathrm{CF}}(\mathcal{A}) + 1$

## Sharper theorems

- $f : \cup\mathcal{A} \to \rho$ is a **CF-coloring** iff $\forall A \in \mathcal{A}\ (\exists \zeta < \rho)\ |A \cap f^{-1}\{\zeta\}| = 1$.
- $f$ is **wCF-coloring** if $\operatorname{dom}(f) \subset \cup\mathcal{A}$
- $\chi_{\mathrm{CF}}([\lambda]^{\kappa}, \mu\text{-}\mathbf{a.d.}) = \sup\{\chi_{\mathrm{CF}}(\mathcal{A}) : \mathcal{A} \subset [\lambda]^{\kappa}, |\mathcal{A}| = \lambda, \mathcal{A} \textbf{ is } \mu\textbf{-ad}\}$.

- $w\chi_{\mathrm{CF}}([\lambda]^{\kappa}, \mu\text{-}\mathbf{a.d.}) = \sup\{w\chi_{\mathrm{CF}}(\mathcal{A}) : \mathcal{A} \subset [\lambda]^{\kappa}, |\mathcal{A}| = \lambda, \textbf{ is } \mu\textbf{-ad}\}$.
- $w\chi_{\mathrm{CF}}([\lambda]^{\kappa}, \mu\text{-}\mathbf{a.d.}) \le \rho$ iff every $\mu$-almost disjoint family $\mathcal{A} \subset [\lambda]^{\kappa}$ of size $\lambda$ has a weak conflict free coloring with $\rho$ colors.

- $w\chi_{\mathrm{CF}}([\omega]^{\omega}, 2\text{-a.d.}) \le 2$.
- $w\chi_{\mathrm{CF}}([\omega_1]^{\omega}, 2\text{-a.d.}) \le 2$.
- $w\chi_{\mathrm{CF}}([\omega_2]^{\omega}, 2\text{-a.d.}) \le 3$ and $w\chi_{\mathrm{CF}}([\omega_3]^{\omega}, 2\text{-a.d.}) \le 3$

# Sharper theorems

**Theorem (HJSSz)**

*If $\kappa$ is an infinite cardinal, $m, d$ are natural numbers, then*

$$w\chi_{\mathrm{CF}}\left(\left[\kappa^{+m}\right]^{\kappa}, d\text{-}\mathbf{a.d.}\right) \leq \left\lfloor \frac{(m+1)(d-1)+1}{2} \right\rfloor + 1.$$

# Sharper theorems

**Theorem**

*If $\kappa$ is an infinite cardinal, $m, d$ are natural numbers, then*

$$w\chi_{\mathrm{CF}}\left(\left[\kappa^{+m}\right]^{\kappa}, d\text{-a.d.}\right) \leq \left\lfloor \frac{(m+1)(d-1)+1}{2} \right\rfloor + 1.$$

**Theorem**

*If GCH holds, and if $d = 2$ or $d$ is odd then we have equality in the result above for each $\kappa$ and $m$.*

- **Do we really need $\omega$ colors?**

**Theorem**

***Yes, we need:*** $\chi_{\mathrm{CF}}\left(\left[\beth_{\omega}\right]^{\omega}, 2\text{-}\mathbf{a.d.}\right) = \omega$.

# The exact value of $\chi_{\mathrm{CF}}$

- $\mathrm{w}\chi_{\mathrm{CF}}(\mathcal{A}) \leq \chi_{\mathrm{CF}}(\mathcal{A}) \leq \mathrm{w}\chi_{\mathrm{CF}}(\mathcal{A}) + 1$
- **GCH:** $w\chi_{\mathrm{CF}}([\omega_m]^\omega, 2\text{-}\mathbf{a.d.}) = \lfloor m/2 \rfloor + 2$;
- $\lfloor m/2 \rfloor + 2 \leq \chi_{\mathrm{CF}}([\omega_m]^\omega, 2\text{-a.d.}) \leq \lfloor m/2 \rfloor + 3.$
- Easy $\chi_{\mathrm{CF}}([\omega]^\omega, 2\text{-a.d.}) = \chi_{\mathrm{CF}}([\omega_1]^\omega, 2\text{-a.d.}) = 3$
- **Open: GCH** $\vdash \chi_{\mathrm{CF}}([\omega_2]^\omega, 2\text{-}\mathbf{a.d.}) = 4$
- **GCH implies** $\chi_{\mathrm{CF}}([\omega_3]^\omega, 2\text{-}\mathbf{a.d.}) = 4$
- **Question:** Assume that $f$ is a function, $\mathrm{dom}(f) \subset \mathbb{Q}^2$, $\mathrm{ran}(f) \subset 3$, $\mathrm{dom}(f)$ does not contain 3 collinear points. Is there a function $g : \mathbb{Q}^2 \to 3$ such that $g \supset f$ and $g$ is a CF-coloring for the lines?

- If $n < \omega \le \lambda$ then $\chi_{\rm CF}([\lambda]^{\omega}, n\text{-}\mathbf{a.d.}) \le \omega$
- One can conjecture: If $\omega_1 \le \lambda$ then $\chi_{\rm CF}([\lambda]^{\omega_1}, \omega\text{-}\mathbf{a.d.}) \le \omega_1$

**Theorem**

*Let $\omega_2 \le \lambda$ be an infinite cardinal. Assume that $\mu^\omega = \mu^+$ for each $\mu < \lambda$ with $\mathrm{cf}(\mu) = \omega$. Then $\chi_{\rm CF}([\lambda]^{\kappa}, \omega\text{-}\mathbf{a.d.}) \le \omega_2$ for each $\omega_2 \le \kappa \le \lambda$.*

- $\chi_{\rm CF}([\lambda]^{\omega_2}, \omega\text{-}\mathbf{a.d.}) \le \omega_2$
- If $\omega_2 \le \kappa \le \lambda$ and $\mathcal{A} \subset [\lambda]^{\kappa}$ is $\omega$-ad then there is $X \subset \lambda$ s.t. $\{A \cap X : A \in \mathcal{A}\} \subset [X]^{\omega_2}$

# $\mathcal{A}$ is $\omega$-almost disjoint

## $\omega_1$ colors may not be enough

$\bigstar(\lambda)$: *there is a* ***stationary set*** $S \subset E^{\lambda}_{\omega_1}$ *and an $\omega$-almost disjoint family* $\{A_\alpha : \alpha \in S\}$ *such that* $A_\alpha \subset \alpha$ *is cofinal for* $\alpha \in S$.

### Theorem

*Assume that GCH holds and we have* $\bigstar(\lambda)$ *for some regular cardinal* $\lambda > \omega_1$. *Then there is a stationary set* $S^* \subset E^{\lambda}_{\omega_1}$ *and there is an $\omega$-almost disjoint family* $\{E_\alpha : \alpha \in S^*\}$ *such that*

(1) $E_\alpha \subset \alpha$ *is cofinal in* $\alpha$ *for each* $\alpha \in S^*$,

(2) *for each* $B \in [\lambda]^{\lambda}$ *there is* $\alpha \in S^*$ *with* $E_\alpha \subset B$.

### Corollary

*Assume that GCH. If* $\bigstar(\lambda)$ *holds for some regular cardinal* $\lambda > \omega_1$, *then there is an $\omega$-almost disjoint family* $\mathcal{A} \subset [\lambda]^{\omega_1}$ *with* $\chi(\mathcal{A}) = \lambda$. *Especially* $\chi_{\mathrm{CF}}([\lambda]^{\omega_1}, \omega\text{-}\mathbf{a.d.}) = \lambda$.

$\bigstar(\lambda)$: *there is a stationary set $S \subset E^{\lambda}_{\omega_1}$ and an $\omega$-almost disjoint family $\{A_\alpha : \alpha \in S\}$ such that $\cup A_\alpha = \alpha$ for each $\alpha \in S$.*

GCH + $\bigstar(\lambda) \Longrightarrow \exists$ $\omega$-A.D. $\mathcal{A} \subset [\lambda]^{\omega_1}$ s.t. $\chi(\mathcal{A}) = \lambda$.

## Theorem

*Assume GCH. If $\bigstar(\lambda)$ holds for some regular cardinal $\lambda > \omega_1$, then for each $\omega_1 < \kappa < \lambda$ there is an $\omega$-almost disjoint family $\mathcal{F} \subset [\lambda]^{\kappa}$ with $\chi_{\mathrm{CF}}(\mathcal{F}) = \omega_2$.*

## Corollary

*Con(ZFC + $\exists$supercompact) $\Longrightarrow$ Con(ZFC+ GCH +*

(1) $\chi_{\mathrm{CF}}([\omega_{\omega+1}]^{\omega_1}, \omega\text{-a.d.}) = \omega_{\omega+1}$,

(2) $\chi_{\mathrm{CF}}([\omega_{\omega+1}]^{\omega_n}, \omega\text{-a.d.}) = \omega_2$ *for* $2 \le n \le \omega$.

## Definition

Let $\mu$ be a singular cardinal with $\mathrm{cf}(\omega) = \omega$. For a sufficiently large $\vartheta$ and $x \in \mathcal{H}(\vartheta)$, a **$(\omega_1, \mu)$-dominating sequence over $x$** is a **continuous, strictly increasing sequence $\langle M_\alpha : \alpha < \mu^+ \rangle$ of elementary submodels of $\mathcal{H}(\vartheta)$** such that

(s1) $(\omega_1 + 1) \cup \{x\} \subset M_0$, $|M_\alpha| \leq \mu$ and $\mu^+ \subseteq \bigcup_{\alpha<\mu^+} M_\alpha$,

(s2) for each $\alpha < \mu^+$ the set $M_\alpha$ is the union of sets $\{M'_{\alpha,n} : n < \omega\}$ such that $\left[M'_{\alpha,n}\right]^\omega \subset M_\alpha$.

## Theorem (Fuchino-S.)

*Assume $\nu^\omega = \nu^+$ for each cardinal $\nu$ with $\mathrm{cf}(\nu) = \omega$. Let $\mu$ be a singular cardinal with $\mathrm{cf}(\mu) = \omega$. If $\square^{***}_{\omega_1,\mu}$ **holds**, then, for any sufficiently large $\chi$ and $x \in \mathcal{H}(\chi)$, there is a **$(\omega_1, \mu)$-dominating sequence** over $x$.*

# $\mathcal{A}$ is $\omega$-almost disjoint

## $\omega_1$ colors may be enough

$\langle M_\alpha : \alpha < \mu^+ \rangle \prec \mathcal{H}(\vartheta)$ is a a **$(\omega_1, \mu)$-dominating sequence over $x$** iff

(s1) $(\omega_1 + 1) \cup \{x\} \subset M_0$, $|M_\alpha| \le \mu$ and $\mu^+ \subseteq \bigcup_{\alpha<\mu^+} M_\alpha$,

(s2) for each $\alpha < \mu^+$ the set $M_\alpha$ is the union of sets $\{M'_{\alpha,n} : n < \omega\}$ such that $\left[M'_{\alpha,n}\right]^\omega \subset M_\alpha$.

### Theorem

*Let $\lambda$ be an infinite cardinal. Assume that*

(i) $\mu^\omega = \mu^+$ *for each cardinal* $\mu < \lambda$ *with* $\mathrm{cf}(\mu) = \omega$,

(ii) *for each singular cardinal* $\mu < \lambda$ *with* $\mathrm{cf}(\mu) = \omega$ *if* $\vartheta$ *is sufficiently large and* $x \in \mathcal{H}(\vartheta)$ *then there is a* $(\omega_1, \mu)$*-dominating sequence over* $x$.

*Then* $\chi_{\mathrm{CF}}(\left[\lambda\right]^\kappa, \omega\text{-}\mathbf{a.d.}) \le \omega_1$ *for each* $\omega_1 \le \kappa \le \lambda$.

$CH \vdash \chi_{\mathrm{CF}}(\left[\omega_m\right]^{\omega_k}, \omega\text{-}\mathbf{a.d.}) \le \omega_1$ for $2 \le k \le m < \omega$.

# $\mathcal{A}$ is $\omega$-almost disjoint

## $\omega$ colors are not enough

**Theorem (Komjáth)**

*There is an $\omega$-almost disjoint family $\mathcal{A} \subset \left[2^\omega\right]^\omega$ with $\chi(\mathcal{A}) = 2^\omega$. Hence $\chi_{\mathrm{CF}}(\left[2^\omega\right]^\omega, \omega\text{-a.d.}) = 2^\omega$.*

Komjáth proved that there is an $\omega$-almost disjoint family $\mathcal{A} \subset \left[2^\omega\right]^\omega$ such that for each $X \in \left[2^\omega\right]^{\omega_1}$ there is $A \in \mathcal{A}$ with $A \subset X$.

Komjáth: $\chi_{\mathrm{CF}}([2^\omega]^\omega, \omega\text{-a.d.}) = 2^\omega$

**Theorem**

*If CH holds then* $\chi_{\mathrm{CF}}([\omega_1]^{\omega_1}, \omega\text{-}\mathbf{a.d.}) = \omega_1$.

**Theorem**

*Assume* $MA_{\aleph_1}$. *Then* $\chi_{\mathrm{CF}}([\omega_1]^{\omega_1}, \omega\text{-}\mathbf{a.d.}) = \omega$ *and* $\chi_{\mathrm{CF}}([\omega_1]^{\omega}, \omega\text{-}\mathbf{a.d.}) = \omega$.

## More ZFC result

- Closure operation: Find $\langle N_\alpha : \alpha < \kappa \rangle$ s.t **closed enough**
- $\rho^{[\nu]} = \rho$ iff there is a family $\mathcal{B} \subset [\rho]^{\le\nu}$ of size $\rho$ such that for all $u \in [\rho]^{\nu}$ there is $\mathcal{P} \in [\mathcal{B}]^{<\nu}$ such that $u \subset \cup\mathcal{P}$.
- **Shelah's Revised GCH theorem: If $\rho \ge \beth_\omega$, then $\rho^{[\nu]} = \rho$ for each large enough regular $\nu < \beth_\omega$.**
- Let $\mu \le \kappa \le \lambda$ be cardinals. $\mathbf{M}(\lambda, \kappa, \mu) \to ED(\kappa)$ holds iff every $\mu$-**almost disjoint** family $\mathcal{A} \subset [\lambda]^{\kappa}$ is $\kappa$-**ED**

### Theorem

*If $\mu < \beth_\omega \le \lambda$ then $\mathbf{M}(\lambda, \beth_\omega, \mu) \to ED(\beth_\omega)$, and so $\chi_{\mathrm{CF}}([\lambda]^{\beth_\omega}, \mu\text{-}\mathbf{a.d.}) \le \beth_\omega$.*

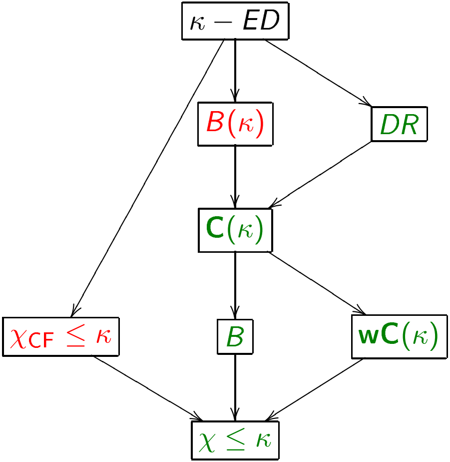
- $V = L \vdash \chi_{\mathrm{CF}}([\lambda]^{\kappa}, \omega\text{-}\mathbf{a.d.}) \le \omega_1$ for $\omega_1 \le \kappa \le \lambda$.
- $GCH \vdash \chi_{\mathrm{CF}}([\lambda]^{\kappa}, \omega\text{-}\mathbf{a.d.}) \le \omega_2$ for $\omega_2 \le \kappa \le \lambda$.
- $ZFC \vdash \chi_{\mathrm{CF}}([\lambda]^{\kappa}, \omega\text{-}\mathbf{a.d.}) \le \beth_\omega$ for $\beth_\omega \le \kappa \le \lambda$.

# Without GCH

- $CH \vdash \chi_{\mathrm{CF}}\left(\left[\omega_m\right]^{\omega_k}, \omega\text{-}\mathbf{a.d.}\right) \leq \omega_1$ for $1 \leq k \leq m < \omega$.
- Con( $\chi_{\mathrm{CF}}\left(\left[\omega_2\right]^{\omega_1}, \omega\text{-a.d.}\right) = \omega_2$. )

**Separate the properties in the diagram!**

The assumption that **every $\mu$-a.d. family $\mathcal{A} \subset [\lambda]^\kappa$ has property $\Phi$** does not imply that **every $\mu$-a.d. family $\mathcal{A} \subset [\lambda]^\kappa$ has property $\Psi$.**




- Stepping up:
  Does $\mathbf{M}(\lambda, \kappa, \rho) \to \Phi$ imply $\mathbf{M}(\lambda, \kappa', \rho) \to \Phi$ for $\kappa < \kappa'$?
- $\mathcal{A} \ll \mathcal{A}'$ iff $\forall A' \in \mathcal{A}' \ \exists A \in \mathcal{A} \ A \subset A'$.
- $\Phi(\mathcal{A}) \to \Phi(\mathcal{A}')$
- $\Phi(\mathcal{A}) \not\to \Phi(\mathcal{A}')$
- $\neg\Phi(\mathcal{A}')$