Some Consequences of Martin’s Conjecture

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26th June 2009
Definition

The Borel equivalence relation $E$ on the standard Borel space $X$ is said to be countable iff every $E$-class is countable.

Standard Example

Let $G$ be a countable (discrete) group and let $X$ be a standard Borel $G$-space. Then the corresponding orbit equivalence relation $E^X_G$ is a countable Borel equivalence relation.

Theorem (Feldman-Moore)

If $E$ is a countable Borel equivalence relation on the standard Borel space $X$, then there exists a countable group $G$ and a Borel action of $G$ on $X$ such that $E = E^X_G$. 
Borel reductions

**Definition**

Let $E$, $F$ be Borel equivalence relations on the standard Borel spaces $X$, $Y$ respectively.

- $E \leq_B F$ iff there exists a Borel map $f : X \to Y$ such that
  \[ x E y \iff f(x) F f(y). \]

  In this case, $f$ is called a **Borel reduction** from $E$ to $F$.

- $E \sim_B F$ iff both $E \leq_B F$ and $F \leq_B E$.

- $E <_B F$ iff both $E \leq_B F$ and $E \not\sim_B F$.

**Definition**

More generally, $f : X \to Y$ is a **Borel homomorphism** from $E$ to $F$ iff

\[ x E y \implies f(x) F f(y). \]
Countable Borel equivalence relations

$E_\infty = \text{universal}$

Uncountably many relations

$E_0 = \text{hyperfinite}$

$id_{2^\mathbb{N}} = \text{smooth}$

**Definition**

The Borel equivalence relation $E$ is **smooth** iff $E \leq_B id_{2^\mathbb{N}}$.

**Definition**

$E_0$ is the equivalence relation of **eventual equality** on $2^\mathbb{N}$.

**Theorem (Adams-Kechris 2000)**

There exist $2^{\aleph_0}$ many countable Borel equivalence relations up to Borel bireducibility.
Countable Borel equivalence relations

- \( E_\infty = \text{universal} \)
- \( E_0 = \text{hyperfinite} \)
- \( \text{id}_{2^\mathbb{N}} = \text{smooth} \)

**Definition**
A countable Borel equivalence relation \( E \) is *universal* iff \( F \leq_B E \) for every countable Borel equivalence relation \( F \).

**Theorem (JKL)**
The orbit equivalence relation \( E_\infty \) of the shift action of the free group \( \mathbb{F}_2 \) on \( 2^{\mathbb{F}_2} \) is universal.
The Borel vs. measurable settings

Let $G$ be a countable group and let $X$ be a standard Borel $G$-space.

The Fundamental Question in the Borel setting

To what extent does the data $(X, E^X_G)$ "remember" the group $G$ and its action on $X$?

Dirty Little Secret

We cannot possibly recover the group $G$ from the data $(X, E^X_G)$ unless we add the hypotheses that:

- $G$ acts freely on $X$; and
- there exists a $G$-invariant probability measure $\mu$ on $X$. 
Essentially free relations

**Definition**

- The countable Borel equivalence relation $E$ on $X$ is **free** iff there exists a countable group $G$ with a free Borel action on $X$ such that $E^X_G = E$.

- The countable Borel equivalence relation $E$ is **essentially free** iff there exists a free countable Borel equivalence relation $F$ such that $E \sim_B F$.

**Theorem (Thomas 2006)**

The universal countable Borel equivalence relation $E_\infty$ is **not essentially free**.
Strongly universal relations

Question (Thomas 2006)

Does there exist a countable Borel equivalence relation $E$ on a standard Borel space $X$ such that:

- there exists an $E$-invariant probability measure $\mu$ on $X$;
- whenever $Y \subseteq X$ is a Borel subset with $\mu(Y) = 1$, then $E \upharpoonright Y$ is countable universal?

Main Theorem (MC)

- Let $E$ be a countable Borel equivalence relation on the standard Borel space $X$ and let $\mu$ be a (not necessarily $E$-invariant) Borel probability measure on $X$.
- Then there exists a Borel subset $Y \subseteq X$ with $\mu(Y) = 1$ such that $E \upharpoonright Y$ is not universal.
Countable Borel Equivalence Relations

- Essentially Free
- $E_\infty$
- New methods needed
- Measure theoretic methods
- $E_0$
- Baire category methods

Simon Thomas (Rutgers)  Erwin Schrödinger Institute Workshop  26th June 2009
**Convention**

Throughout the powerset $\mathcal{P}(\mathbb{N})$ will be identified with $2^{\mathbb{N}}$ by identifying subsets of $\mathbb{N}$ with their characteristic functions.

**Definition**

If $x, y \in 2^{\mathbb{N}}$, then $x$ is Turing reducible to $y$, written $x \leq_T y$, iff there exists a $y$-oracle Turing machine which computes $x$.

**Remark**

In other words, there is an algorithm which computes $x$ modulo an oracle which correctly answers questions of the form “Is $n \in y$?”
A Notion of Largeness

Definition

For each $z \in 2^\mathbb{N}$, the corresponding cone is $C_z = \{ x \in 2^\mathbb{N} | z \leq_T x \}$.

- Suppose $z_n = \{ a_n, \ell | \ell \in \mathbb{N} \} \in 2^\mathbb{N}$ for each $n \in \mathbb{N}$ and define
  \[ \oplus z_n = \{ p_n^{a_n, \ell} | n, \ell \in \mathbb{N} \} \in 2^\mathbb{N}, \]
  where $p_n$ is the $n$th prime.
- Then $z_m \leq_T \oplus z_n$ for each $m \in \mathbb{N}$ and so $C_\oplus z_n \subseteq \bigcap_n C_{z_n}$.

Remark

It is well-known that if $C \subsetneq 2^\mathbb{N}$ is a proper cone, then $C$ is both null and meager.
The Turing equivalence relation

**Definition**

The *Turing equivalence relation* \( \equiv_T \) on \( 2^\mathbb{N} \) is defined by

\[
x \equiv_T y \text{ iff } x \leq_T y \text{ & } y \leq_T x,
\]

where \( \leq_T \) denotes Turing reducibility.

**Remark**

- Clearly \( \equiv_T \) is a countable Borel equivalence relation on \( 2^\mathbb{N} \).
- However, \( \equiv_T \) is *not* essentially free and is *not* induced by the action of any countable subgroup of \( \text{Sym}(\mathbb{N}) \) with its natural action on \( 2^\mathbb{N} \).
Martin’s Theorem

**Theorem (Martin)**

If $X \subseteq 2^\mathbb{N}$ is a $\equiv_T$-invariant Borel subset, then either $X$ or $2^\mathbb{N} \setminus X$ contains a cone.

**Remark**

For later use, notice that if $X \subseteq 2^\mathbb{N}$ is a $\equiv_T$-invariant Borel subset, then the following are equivalent:

(i) $X$ contains a cone.

(ii) For all $z \in 2^\mathbb{N}$, there exists $x \in X$ with $z \leq_T x$.
Ergodicity

Definition

Let $G$ be a countable group and let $X$ be a standard Borel $G$-space. Then the $G$-invariant probability measure $\mu$ is said to be ergodic iff $\mu(A) = 0, 1$ for every $G$-invariant Borel subset $A \subseteq X$.

Theorem

If $\mu$ is a $G$-invariant probability measure on the standard Borel $G$-space $X$, then the following statements are equivalent.

- The action of $G$ on $(X, \mu)$ is ergodic.
- If $Y$ is a standard Borel space and $f : X \to Y$ is a $G$-invariant Borel function, then there exists a $G$-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $f \mid M$ is a constant function.
Theorem (Folklore)

If $\varphi : 2^\mathbb{N} \to 2^\mathbb{N}$ is a $\equiv_T$-invariant Borel map, then there exists a cone $C$ such that $\varphi \upharpoonright C$ is a constant map.

Proof.

For each $n \in \mathbb{N}$, there exists $\varepsilon_n \in \{0, 1\}$ such that $X_n = \{ x \in 2^\mathbb{N} \mid \varphi(x)(n) = \varepsilon_n \}$ contains a cone.

Hence there exists a cone $C \subseteq \bigcap X_n$ and clearly $\varphi \upharpoonright C$ is a constant map.
Proof of Martin’s Theorem

- Suppose that \( X \subseteq 2^\mathbb{N} \) is a \( \equiv^T \)-invariant Borel subset.

- Consider the two player Borel game \( G(X) \)

\[
    s(0) \ s(1) \ s(2) \ s(3) \ \cdots
\]

where \( I \) wins iff \( s = ( s(0) \ s(1) \ s(2) \ \cdots ) \in X \).

- Then the Borel game \( G(X) \) is determined. Suppose, for example, that \( \sigma : 2^{<\mathbb{N}} \rightarrow 2 \) is a winning strategy for \( I \).

- Let \( \sigma \leq_T t \in 2^\mathbb{N} \) and consider the run of \( G(X) \) where
  - \( II \) plays \( t = ( s(1) \ s(3) \ s(5) \ \cdots ) \)
  - \( I \) responds with \( \sigma \) and plays \( ( s(0) \ s(2) \ s(4) \ \cdots ) \).

- Then \( s \in X \) and \( s \equiv_T t \). Hence \( t \in X \) and so \( C_\sigma \subseteq X \).
Strong Ergodicity

Definition

- Suppose that $E$, $F$ are countable Borel equivalence relations on the standard Borel spaces $X$, $Y$ and that $\mu$ is an $E$-invariant Borel probability measure on $X$.
- Then $E$ is said to be $F$-ergodic iff for every Borel homomorphism $\varphi : X \to Y$ from $E$ to $F$, there exists a Borel subset $Z \subseteq X$ with $\mu(Z) = 1$ such that $\varphi$ maps $Z$ into a single $F$-class.

Example (Jones-Schmidt)

$E_\infty$ is $E_0$-ergodic.
**Definition**

Let $E$ be a countable Borel equivalence relation on the standard Borel space $X$. Then $\equiv_T$ is said to be $E$-m-ergodic iff for every Borel homomorphism $\varphi : 2^\mathbb{N} \to X$ from $\equiv_T$ to $E$, there exists a cone $C \subseteq 2^\mathbb{N}$ such that $\varphi$ maps $C$ into a single $E$-class.

**Target**

Classify the countable Borel equivalence relations $E$ such that $\equiv_T$ is $E$-m-ergodic.

**Question**

*When is it “obvious” that $\equiv_T$ is not $E$-m-ergodic?*
Definition

- The Borel homomorphism $\varphi : X' \to X$ from $E'$ to $E$ is said to be a weak Borel reduction iff $\varphi$ is countable-to-one. In this case, we write $E' \leq^w_B E$.

- A countable Borel equivalence relation $E$ is said to be weakly universal iff $F \leq^w_B E$ for every countable Borel equivalence relation $F$.

Some Examples

- If $E$ is universal, then $E$ is weakly universal.
- The Turing equivalence relation $\equiv_T$ is weakly universal.

Observation

If $E$ is weakly universal, then $\equiv_T$ is not $E$-m-ergodic.
Strong Ergodicity for Turing equivalence

Strong Ergodicity Theorem (MC)

If $E$ is any countable Borel equivalence relation, then exactly one of the following conditions holds:

(a) $E$ is weakly universal.
(b) $\equiv_T$ is $E$-m-ergodic.

Remark

- There are currently no nonsmooth countable Borel equivalence relations $E$ for which it has been proved that $\equiv_T$ is $E$-m-ergodic.
- In particular, it is not known whether $\equiv_T$ is $E_0$-m-ergodic, where $E_0$ denotes the eventual equality equivalence relation on $2^\mathbb{N}$. 
The Kechris-Miller Theorem

**Observation**

Let $E, F$ be countable Borel equivalence relations.
- If $E \leq_B F$, then $E \leq^w_B F$.
- If $E \subseteq F$, then $E \leq^w_B F$.

**Theorem (Kechris-Miller)**

If $E, F$ are countable Borel equivalence relations on the uncountable standard Borel spaces $X, Y$ respectively, then the following conditions are equivalent:

(i) $E \leq^w_B F$.

(ii) There exists a countable Borel equivalence relation $S \subseteq F$ on $Y$ such that $S \sim_B E$.
**The weak universality of Turing equivalence**

**Proposition (Kechris)**
\[
\equiv_T \text{ is weakly universal.}
\]

**Proof.**
Identifying the free group \( F_2 \) with a suitably chosen group of recursive permutations of \( \mathbb{N} \), we have that \( E_\infty \subseteq \equiv_T \).

**Important Remark**
If \( C = \{ x \in 2^\mathbb{N} \mid z \leq_T x \} \) is a cone, then the map \( y \mapsto y \oplus z \) is a weak Borel reduction from \( \equiv_T \) to \( \equiv_T \upharpoonright C \) and hence \( \equiv_T \upharpoonright C \) is also weakly universal.
Martin’s Conjecture (MC)

If \( \varphi : 2^\mathbb{N} \rightarrow 2^\mathbb{N} \) is a Borel homomorphism from \( \equiv_T \) to \( \equiv_T \), then exactly one of the following conditions holds:

(i) There exists a cone \( C \subseteq 2^\mathbb{N} \) such that \( \varphi \) maps \( C \) into a single \( \equiv_T \)-class.

(ii) There exists a cone \( C \subseteq 2^\mathbb{N} \) such that \( x \leq_T \varphi(x) \) for all \( x \in C \).

Theorem (Slaman-Steel)

Suppose that \( \varphi : 2^\mathbb{N} \rightarrow 2^\mathbb{N} \) is a Borel homomorphism from \( \equiv_T \) to \( \equiv_T \). If \( \varphi(x) <_T x \) on a cone, then there exists a cone \( C \subseteq 2^\mathbb{N} \) such that \( \varphi \) maps \( C \) into a single \( \equiv_T \)-class.
Some easy consequences of Martin’s Conjecture

Theorem \((MC)\)

If \(\varphi : 2^\mathbb{N} \to 2^\mathbb{N}\) is a Borel homomorphism from \(\equiv_T\) to \(\equiv_T\), then exactly one of the following conditions holds:

(i) There exists a cone \(C \subseteq 2^\mathbb{N}\) such that \(\varphi\) maps \(C\) into a single \(\equiv_T\)-class.

(ii) There exists a cone \(C \subseteq 2^\mathbb{N}\) such that \(\varphi \upharpoonright C\) is a weak Borel reduction from \(\equiv_T \upharpoonright C\) to \(\equiv_T\).

Furthermore, in case (ii), if \(D \subseteq 2^\mathbb{N}\) is any cone, then \([\varphi(D)]_{\equiv_T}\) contains a cone.
Some easy consequences of Martin’s Conjecture

Corollary (MC)

- $\equiv_T <_B (\equiv_T \sqcup \equiv_T)$.
- *In particular, $\equiv_T$ is not countable universal.*

Corollary (MC)

*If $A \subseteq 2^\mathbb{N}$ is a $\equiv_T$-invariant Borel subset, then $\equiv_T \upharpoonright A$ is weakly universal iff $A$ contains a cone.*

Remark

There are currently no naturally occurring classes $D \subseteq 2^\mathbb{N}$ for which it is known that $\equiv_T \upharpoonright D$ is not weakly universal.
Proof of the Strong Ergodicity Theorem (MC)

Let $E$ be any countable Borel equivalence relation.

Since $E \leq_B \equiv_T$, we can suppose that $E \subseteq \equiv_T$.

Suppose that $\varphi : 2^\mathbb{N} \to 2^\mathbb{N}$ is a Borel homomorphism from $\equiv_T$ to $E$ and that $\varphi$ does not map any cone to a single $E$-class.

Then $\varphi$ is also a Borel homomorphism from $\equiv_T$ to $\equiv_T$ and clearly $\varphi$ does not map any cone to a single $\equiv_T$-class.

Hence there exists a cone $C$ such that $\varphi \upharpoonright C$ is countable-to-one.

Since $\equiv_T \upharpoonright C$ is weakly universal and $(\equiv_T \upharpoonright C) \leq_B^w E$, it follows that $E$ is weakly universal.
Some applications of the Strong Ergodicity Theorem

Theorem (MC)

There exist uncountably many weakly universal countable Borel equivalence relations up to Borel bireducibility.

Definition

The countable group $G$ is (weakly) action universal iff there exists a standard Borel $G$-space $X$ such that $E^X_G$ is (weakly) universal.

Theorem (MC)

If $G$ is a countable group, then the following are equivalent.

(a) $G$ is weakly action universal.

(b) The conjugacy relation on the space of subgroups of $G$ is weakly universal.
Borel Boundedness

**Definition**

If \( c, d \in \mathbb{N}^\mathbb{N} \), then:

- \( c \leq^* d \) iff \( c(n) \leq d(n) \) for all but finitely many \( n \in \mathbb{N} \).
- \( c =^* d \) iff both \( c \leq^* d \) and \( d \leq^* c \).

**Easy Observation**

Suppose that \( E \) is a countable Borel equivalence relation on the standard Borel space \( X \) and that \( \sigma : X \to \mathbb{N}^\mathbb{N} \) is any map. Then there exists a map \( \psi : X/E \to \mathbb{N}^\mathbb{N} \) such that \( \sigma(x) \leq^* \psi([x]_E) \) for all \( x \in X \).
Suppose that $E$ is a countable Borel equivalence relation on the standard Borel space $X$ and that $\sigma : X \to \mathbb{N}^\mathbb{N}$ is a Borel map. Then there exists a Borel map $\psi : X \to \mathbb{N}^\mathbb{N}$ such that for all $x \in X$,

$$\sigma(y) \leq^* \psi(x) \quad \text{for all } y \in [x]_E$$

**Proof.**

By Feldman-Moore, we can realize $E$ by a Borel action of a countable group $G = \{ \gamma_m \mid m \in \mathbb{N} \}$.

Define $\psi(x)(n) = \max\{ \sigma(\gamma_m \cdot x)(n) \mid m \leq n \}$. 
**Borel Boundedness**

**Definition (Boykin-Jackson)**

The countable Borel equivalence relation $E$ on the standard Borel space $X$ is said to be **Borel-Bounded** iff for every Borel map $\theta : X \to \mathbb{N}^\mathbb{N}$, there exists a Borel homomorphism $\varphi : X \to \mathbb{N}^\mathbb{N}$ from $E$ to $\equiv^*$ such that $\theta(x) \leq^* \varphi(x)$ for all $x \in X$.

**Theorem (Boykin-Jackson)**

If $E$ is hyperfinite, then $E$ is Borel-Bounded.

**Question (Boykin-Jackson)**

Is Borel-Boundedness equivalent to hyperfiniteness?

**Problem (Boykin-Jackson)**

Find an example of a countable Borel equivalence relation which is **not** Borel-Bounded.
Proposition

If $(X, \mu)$ is a standard Borel probability space and $\theta : X \to \mathbb{N}^\mathbb{N}$ is a Borel map, then there exists a function $h \in \mathbb{N}^\mathbb{N}$ such that

\[ \mu(\{ x \in X \mid \theta(x) \leq^* h \}) = 1. \]

Proof.

For each $n \in \mathbb{N}$, there exists $h(n) \in \mathbb{N}$ such that

\[ \mu(\{ x \in X \mid \theta(x)(n) > h(n) \}) \leq (1/2)^{n+1}. \]

By the Borel-Cantelli Lemma, we have that

\[ \mu(\{ x \in X \mid \theta(x)(n) > h(n) \text{ for infinitely many } n \}) = 0. \]
An application of Martin’s Conjecture

Theorem (MC)

The Turing equivalence relation $\equiv_T$ is not Borel-Bounded.

Corollary (MC)

If $E$ is a weakly universal countable Borel equivalence relation, then $E$ is not Borel-Bounded. In particular, $E_\infty$ is not Borel-Bounded.

Proof.

By Boykin-Jackson, if $E$ is Borel-Bounded and $F \leq_B^w E$, then $F$ is also Borel-Bounded.
Growth Rates

Definition

Identifying each \( r \in 2^\mathbb{N} \) with the corresponding subset of \( \mathbb{N} \), define the Borel map \( \theta : 2^\mathbb{N} \to \mathbb{N}^\mathbb{N} \) by:

- \( \theta(r) \) is the increasing enumeration of \( r \cap 2^\mathbb{N} \), if \( r \cap 2^\mathbb{N} \) is infinite;
- \( \theta(r) \) is the zero function, otherwise.

Observation

For each \( h \in \mathbb{N}^\mathbb{N} \), the \( \equiv_T \)-invariant Borel set

\[
D_h = \{ r \in 2^\mathbb{N} \mid (\exists s \in 2^\mathbb{N}) s \equiv_T r \text{ and } h < \theta(s) \}
\]

contains a cone.
Proof of Theorem (MC)

- Suppose that $\varphi : 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a Borel homomorphism from $\equiv_T$ to $\equiv^*$ such that $\theta(r) \leq^* \varphi(r)$ for all $r \in 2^\mathbb{N}$.

- Since $\equiv^*$ is hyperfinite, it follows that $\equiv_T$ is $\equiv^*$-$m$-ergodic.

- Hence there exists a cone $C$ such that $\varphi$ maps $C$ into a single $\equiv^*$-class; say, $[h]_{\equiv^*}$.

- But then $C \cap D_h = \emptyset$, which is a contradiction.
Strongly universal relations

**Question (Thomas 2006)**

Does there exist a countable Borel equivalence relation $E$ on a standard Borel space $X$ such that:

- there exists an *ergodic* $E$-invariant probability measure $\mu$ on $X$;
- whenever $Y \subseteq X$ is a Borel subset with $\mu(Y) = 1$, then $E \upharpoonright Y$ is countable universal?

**Theorem (MC)**

Let $E$ be a countable Borel equivalence relation on the standard Borel space $X$ and let $\mu$ be a (not necessarily $E$-invariant) Borel probability measure on $X$. Then there exists a Borel subset $Y \subseteq X$ with $\mu(Y) = 1$ such that $E \upharpoonright Y$ is not weakly universal.
Proof of Theorem (MC)

- Let $E$ be a countable Borel equivalence relation on the standard Borel space $X$ and let $\mu$ be a Borel probability measure on $X$.
- Let $\theta : 2^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$ be the Borel map defined earlier.
- By the Feldman-Moore Theorem, there exists a Borel map $\psi : 2^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$ such that if $r \equiv_T s$, then $\theta(s) \leq^* \psi(r)$.
- Let $\varphi : X \rightarrow 2^\mathbb{N}$ be a weak Borel reduction from $E$ to $\equiv_T$ and let $\pi : X \rightarrow \mathbb{N}^\mathbb{N}$ be the Borel map defined by $\pi = \psi \circ \varphi$.
- Then there exists a function $h \in \mathbb{N}^\mathbb{N}$ such that the Borel set $Y = \{ x \in X \mid \pi(x) \leq^* h \}$ satisfies $\mu(Y) = 1$.
- Since the Borel set $Z = [\varphi(Y)]_{\equiv_T}$ satisfies $Z \cap D_h = \emptyset$, it follows that $\equiv_T \upharpoonright Z$ is not weakly universal.
- Since $(E \upharpoonright Y) \leq^w_B (\equiv_T \upharpoonright Z)$, it follows that $E \upharpoonright Y$ is not weakly universal.
Some Open Problems

Problem

Prove that $\equiv_T$ is $E_0$-m-ergodic.

Problem

Find a naturally occurring classes of degree $D \subseteq 2^\mathbb{N}$ such that

$\equiv_T | D$ is not weakly universal.

For example, how about the classes of minimal degrees,
hyperimmune-free degrees, ... ?