The lifting problem for $\text{Aut}(X, \mu)$

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Measure preserving transformations

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We identify measure preserving Borel bijections that agree \(\mu\)-a.e. The equivalence class
\[ [T] = \{S : X \to X : S \text{ is m.p. and } T = S \text{ a.e.}\} \]

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\textbf{Warning:} We usually write \(T\) for \([T]\) if there is no danger of confusion.
The group of measure preserving transformations

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$\text{Aut}(X, \mu)$ is a Polish group in the topology induced by the sub-neighbourhood basis

$$N(T_0, \varepsilon, A) = \{ T \in \text{Aut}(X, \mu) : \mu(T(A) \triangle T_0(A)) < \varepsilon \}$$

where $A \subseteq X$ is Borel, $\varepsilon > 0$ and $T_0 \in \text{Aut}(X, \mu)$. 
Near actions vs. spatial actions, I

Measure preserving group actions is the central object of Ergodic Theory. One distinguishes between actions that are defined almost everywhere, and those that are really defined everywhere:

Definition.

(1) A m.p. near-action of a group $G$ on $(X, \mu)$ is a homomorphism $h: G \rightarrow \text{Aut}(X, \mu)$.

(2) A spatial model for a near action $h$ is an (actual, pointwise) action $\sigma: G \times X \rightarrow X$ such that for each $g \in G$ $x \mapsto \sigma(g, x)$ is a representative of $h(g)$.

Nb. If $G$ is a topological group, then in (1) it is natural to require $h$ to be continuous or Borel (i.e. continuous near-action, Borel near-action). Likewise in (2), we could require $\sigma$ to be continuous or Borel (i.e. continuous spatial model, Borel spatial model).
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This was generalized by Mackey (circa 1960), who showed the same for all locally compact 2nd countable groups.

Very recently Kwiatkowska and Solecki (2009) have generalized this to a new and much larger class of groups.
Running counter to this is the following:

Theorem. (Glasner-Tsirelson-Weiss, 2004). There are (many) Polish groups for which no "non-trivial" Borel near-action admits a Borel spatial model. In particular, the near-action of $\text{Aut}(X, \mu)$ on $(X, \mu)$ does not admit a Borel spatial model.

The purpose of this talk is to discuss the situation if we drop the assumption of the spatial model being Borel. Specifically:

Theorem 1. (T., 2009) If $\text{CH}$ holds, then the near-action of $\text{Aut}(X, \mu)$ on $(X, \mu)$ has a spatial model. Thus under $\text{CH}$ every near-action has a spatial model.
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Thus under CH every near-action has a spatial model.
The problem is at least superficially similar to the classical lifting problem for the measure algebra:

\[ h : \text{MALG}(X, \mu) \rightarrow B(X) \] is a "lifting" if

1. \( h \) is a Boolean algebra homomorphism into the \( \sigma \)-algebra of Borel sets on \( X \);
2. \( h(A) \in A \) for all \( A \in \text{MALG}(X, \mu) \).

i.e, \( h \) splits the identity \( \text{Id} : \text{MALG}(X, \mu) \rightarrow \text{MALG}(X, \mu) \):

\[ \text{Id} = \kappa \circ h, \] where \( \kappa : B(X) \rightarrow \text{MALG}(X, \mu) \) is the canonical homomorphism with \( \ker(\kappa) = \text{Im}z = \text{the ideal of measure zero sets} \).
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**Theorem** (Shelah, circa 1980) *There is a model of ZFC in which* $\text{Id} : \text{MALG}(X, \mu) \to \text{MALG}(X, \mu)$ *does not split.*
Let $G(X, \mu)$ denote the group of measure preserving Borel bijections, and $I(X, \mu)$ the (normal) subgroup of those $T \in G(X, \mu)$ such that $T(x) = x$ a.e.
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1. $h(T) \in T$ for all $T$

2. $h$ splits the identity $\text{Id} : \text{Aut}(X, \mu) \rightarrow \text{Aut}(X, \mu)$ as follows:

   $$\text{Id} = \kappa \circ h,$$

where $\kappa : G(X, \mu) \rightarrow \text{Aut}(X, \mu)$ is the canonical homomorphism with $\ker(\kappa) = I(X, \mu)$. 
The lifting problem for Aut($X, \mu$), II

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$$H_\alpha = \langle T_\beta : \beta < \alpha \rangle,$$

(the group generated by the $T_\beta$, $\beta < \alpha < \omega_1$; for convenience, $H_0 = \{\text{Id}_X\}$.)
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W.m.a. \( \beta < \alpha \implies H_\beta \not\subseteq H_\alpha \), after possibly thinning out the sequence \( (T_\alpha : \alpha < \omega_1) \).
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**Theorem 1’.** *Assuming CH, the identity homomorphism $Id : Aut(X, \mu) \to Aut(X, \mu)$ splits.*

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(the group generated by the $T_\beta$, $\beta < \alpha < \omega_1$; for convenience, $H_0 = \{Id_X\}$.)

W.m.a. $\beta < \alpha \implies H_\beta \subsetneq H_\alpha$, after possibly thinning out the sequence $(T_\alpha : \alpha < \omega_1)$. Also, we assume that $|H_1| = \aleph_0$. 
We will define $h_\alpha : H_\alpha \to G(X, \mu)$ that is a lifting on $H_\alpha$, for $\alpha < \omega_1$. 
Proof of Theorem 1

We will define \( h_\alpha : H_\alpha \rightarrow G(X, \mu) \) that is a lifting on \( H_\alpha \), for \( \alpha < \omega_1 \). We will make sure that \( \beta < \alpha \implies h_\alpha \upharpoonright H_\beta = h_\beta \).
Proof of Theorem 1

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At first one might try to arbitrarily choose some \( g_0 \in T_0 \), and let \( h_1(T_0) = g_0 \). But if we then choose \( g_1 \in T_1 \) arbitrarily and let \( h_2(T_1) = g_1 \), then \( h_2 \) will most likely only induce an action of \( H_2 \) almost everywhere, but fail to induce a \( H_2 \) action everywhere.
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The idea is to make sure that we have chosen the $g_\beta$, $\beta < \alpha$, in such a way that for a given choice of $g \in T_\alpha$, there is some reasonably easy way to adjust $g$ on a null-set so that it becomes fully compatible with $h_\alpha : H_\alpha \to G(X, \mu)$, thus allowing the induction to proceed.
Master actions, I

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**Definition.** For $\alpha < \omega_1$ we define $S_\alpha \subseteq \textbf{LO} \times \textbf{GP}$ to consists of all $(<^*, G) \in \textbf{LO} \times \textbf{GP}$ such that

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(i) For some $n \in \mathbb{N}$, the initial segment $\{m : m <^* n\}$ is isomorphic to $\alpha$;

(ii) There is a monomorphism $\varphi : H_\alpha \rightarrow G$ such that $\text{rank}_{<^*}(\varphi(T_\beta)) = \beta$.
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For $(<^*, G) \in S_\alpha$, the unique monomorphism $\varphi : H_\alpha \to G$ satisfying (ii) in the definition of $S_\alpha$ will be called the \textit{canonical} monomorphism $H_\alpha \to G$. 
Lemma. The set $S_\alpha$ is Borel for all $\alpha < \omega_1$.

For $(\prec^*, G) \in S_\alpha$, the unique monomorphism $\varphi : H_\alpha \to G$ satisfying (ii) in the definition of $S_\alpha$ will be called the canonical monomorphism $H_\alpha \to G$.

Thus for $(\prec^*, G) \in S_\alpha$, we may identify $H_\alpha$ with a subgroup of $G$ in a canonical way.
Definition. For $\alpha < \omega_1$, the $\alpha$'th master action $\sigma_\alpha : H_\alpha \curvearrowright M_\alpha$ is defined by

\begin{align*}
M_\alpha &= S_\alpha \times \left(2^N\right)^N; \\
\text{For } \beta < \omega_1, \quad \sigma_\alpha(T_\beta)(<^*0, G_0, x) = (<^*1, G_1, y) \iff <^*0 = <^*1 \land G_0 = G_1 \land (\forall m) \text{ rank } <^*0(m) = \beta = \Rightarrow (\forall n) y(n) = x(m - 1 \cdot G_0 n).}
\end{align*}
**Definition.** For $\alpha < \omega_1$, the $\alpha$’th master action $\sigma_\alpha : H_\alpha \curvearrowleft M_\alpha$ is defined by

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\sigma_\alpha(T_\beta)(<^*, G_0, x) &= (<^*, G_1, y) \iff \\
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1. $M_\alpha = S_\alpha \times (2^\mathbb{N})^\mathbb{N}$;
2. For $\beta < \omega_1$,

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\sigma_\alpha(T_\beta)(<^*, G_0, x) = (<_1^*, G_1, y) \iff
<_0^* = <_1^* \wedge G_0 = G_1 \wedge
(\forall m) \quad \text{rank}_{<^*}(m) = \beta \implies (\forall n) y(n) = x(m^{-1} \cdot G_0 n)).
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That is: For every $(<_^*, G) \in S_\alpha$, we identify $H_\alpha$ with a subgroup of $G$, and let $H_\alpha$ act on $(2^\mathbb{N})^\mathbb{N}$ by a left-shift (where we think of $(2^\mathbb{N})^\mathbb{N}$ as $(2^\mathbb{N})^G$.)
We can think of the master action $\sigma_\alpha$ as a Borel action of $H_\alpha$ that contains a copy of all shift-actions of $H_\alpha$ on $(2^\mathbb{N})^G$, for any countable group $G$ containing $H_\alpha$. 

Lemma

If $\alpha < \omega_1$ is a limit ordinal, it holds for the master action $\sigma_\alpha$ that $\sigma_\alpha = \bigcup_{\beta < \alpha} \sigma_\beta \upharpoonright M_\alpha$. 

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**Lemma**

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The next idea for the proof of Theorem 1 is now to use the following universality property of shift actions:

Universality Property. (Folklore.)

If $\Lambda \leq \Gamma$ are countable groups and $\tau : \Lambda \curvearrowright X$ is a Borel action of $\Lambda$ on a standard Borel space, then there is a shift invariant Borel set $B \subseteq (2^\mathbb{N})^\Gamma$ and a Borel bijection $\psi : X \to B$ such that $(\forall g \in \Lambda) \psi(\sigma(g)(x)) = \beta(g)(\psi(x))$, where $\beta$ denotes the shift-action $\beta : \Gamma \curvearrowright (2^\mathbb{N})^\Gamma$. 

Asger Törnquist (Vienna)
The next idea for the proof of Theorem 1 is now to use the following universality property of shift actions:

**Universality Property.** (Folklore.) If $\Lambda \leq \Gamma$ are countable groups and $\tau : \Lambda \curvearrowright X$ is a Borel action of $\Lambda$ on a standard Borel space, then there is a shift invariant Borel set $B \subseteq (2^\mathbb{N})^\Gamma$ and a Borel bijection $\psi : X \rightarrow B$ such that

$$\left( \forall g \in \Lambda \right) \psi(\sigma(g)(x)) = \beta(g)(\psi(x)),$$

where $\beta$ denotes the shift-action $\beta : \Gamma \curvearrowright (2^\mathbb{N})^\Gamma$. 
Given a pair of countable groups $\Lambda \leq \Gamma$, and a Borel action $\sigma : \Lambda \curvearrowright X$ on a standard Borel space $X$, the Universality Property gives us a way of extending the action: Embed $X$ into $(2^\mathbb{N})\Gamma$ using $\psi$. Unfortunately, $\psi(X) \varsubsetneq (2^\mathbb{N})\Gamma$ may happen.
Given a pair of countable groups \( \Lambda \leq \Gamma \), and a Borel action \( \sigma : \Lambda \curvearrowright X \) on a standard Borel space \( X \), the Universality Property gives us a way of extending the action: Embed \( X \) into \( (2^\mathbb{N})^\Gamma \) using \( \psi \). Unfortunately, \( \psi(X) \not\subseteq (2^\mathbb{N})^\Gamma \) may happen.

If, however, \( X = X_0 \sqcup X_1 \sqcup X_2 \sqcup \cdots \) and \( \sigma|_{X_i} \simeq \beta : \Lambda \curvearrowright (2^\mathbb{N})^\Gamma \) for all \( i > 0 \), then we may:

(i) map \( X_0 \) into \( X_i \) using some \( \psi_i \) that conjugates the action

(ii) extend the \( \Lambda \)-action to \( \Gamma \) on \( X_0 \sqcup X_1 \setminus \psi_1(X_0) \)

(iii) extend the \( \Lambda \)-action to \( \Gamma \) on \( \psi_1(X_0) \sqcup X_2 \setminus \psi_2(\psi_1(X_0)) \)

(iv) etc...

More precisely:
Given a pair of countable groups $\Lambda \leq \Gamma$, and a Borel action $\sigma : \Lambda \curvearrowright X$ on a standard Borel space $X$, the Universality Property gives us a way of extending the action: Embed $X$ into $(2^\mathbb{N})^\Gamma$ using $\psi$. Unfortunately, $\psi(X) \subsetneq (2^\mathbb{N})^\Gamma$ may happen.

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One step extensions, I

Given a pair of countable groups $\Lambda \leq \Gamma$, and a Borel action $\sigma : \Lambda \curvearrowright X$ on a standard Borel space $X$, the Universality Property gives us a way of extending the action: Embed $X$ into $({2^\mathbb{N}})^\Gamma$ using $\psi$. Unfortunately, $\psi(X) \subsetneq ({2^\mathbb{N}})^\Gamma$ may happen.

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More precisely:
One step extension Lemma. Let $\Lambda < \Gamma$ be countable groups such that there is an element $\gamma \in \Gamma \setminus \Lambda$ such that $\Gamma = \langle \Lambda \cup \{\gamma_0\} \rangle$, and suppose there are countable groups $\Gamma_i, i \in \mathbb{N}$ such that $\Gamma \leq \Gamma_i$ for all $i$. 
One step extension Lemma. Let $\Lambda < \Gamma$ be countable groups such that there is an element $\gamma \in \Gamma \setminus \Lambda$ such that $\Gamma = \langle \Lambda \cup \{\gamma_0\} \rangle$, and suppose there are countable groups $\Gamma_i$, $i \in \mathbb{N}$ such that $\Gamma \leq \Gamma_i$ for all $i$. Let $X$ be a standard Borel space which is partitioned into Borel pieces,

$$X = X_0 \sqcup \bigsqcup_{i \in \mathbb{N}} (2^\mathbb{N})^{\Gamma_i},$$

that is, $X$ is the disjoint union of $X_0$ and $(2^\mathbb{N})^{\Gamma_i}$, $(i \in \mathbb{N})$, $X_0$ is Borel, and $(2^\mathbb{N})^{\Gamma_i}$ carries its usual Borel structure for all $i \in \mathbb{N}$. 
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$$\rho \upharpoonright \Lambda \times (2^\mathbb{N})^{\Gamma_i}$$

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is the shift action. Then there is a Borel action $\hat{\rho} : \Gamma \curvearrowright X$ such that $\hat{\rho} \upharpoonright \Lambda \times X = \rho$. 
Finishing Theorem 1.

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Let $X = X_0 \sqcup 2^\mathbb{N} \times \mathcal{M}_0$ (disjoint union) and let $\mu$ be a measure such that $\mu(X_0) = 1$. We construct by induction on $\alpha < \omega_1$ homomorphisms $h_\alpha : H_\alpha \to G(X, \mu)$ and uncountable Borel sets $Y_\alpha \subseteq 2^\mathbb{N}$ such that

1. $h_0(I) = \text{Id}$, $Y_0 = 2^\mathbb{N}$;
2. $h_\alpha : H_\alpha \to G(X, \mu)$ is a homomorphism such that $h_\alpha(T) \in T$ for all $T \in H_\alpha$;
3. If $\beta < \alpha$ then $Y_\beta \supseteq Y_\alpha$ and $Y_\beta \setminus Y_\alpha$ is countable;
4. For $(y, x) \in Y_\alpha \times M_\alpha$ we have $h_\alpha(T)(y, x) = (y, \sigma_\alpha(T)(x))$ for all $T \in H_\alpha$;
5. If $\beta < \alpha$ then $h_\beta = h_\alpha|_{H_\beta}$.

If this can be done then we get a lifting $h : \text{Aut}(X, \mu) \to G(X, \mu)$ by letting $h = \bigcup_{\alpha < \omega_1} h_\alpha$. 

Asger Törnquist (Vienna)
The lifting problem for $\text{Aut}(X, \mu)$
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We can now put the pieces together. Let $X = X_0 \sqcup 2^N \times M_0$ (disjoint union) and let $\mu$ be a measure such that $\mu(X_0) = 1$. We construct by induction on $\alpha < \omega_1$ homomorphisms $h_\alpha : H_\alpha \to G(X, \mu)$ and uncountable Borel sets $Y_\alpha \subseteq 2^N$ such that

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If this can be done then we get a lifting $h : \text{Aut}(X, \mu) \to G(X, \mu)$ by letting

$$h = \bigcup_{\alpha < \omega_1} h_\alpha.$$
If $\alpha$ is a limit ordinal then $h_\alpha = \bigcup_{\beta < \alpha} h_\beta$ may easily be seen to work, using the fact that

$$\sigma_\alpha = \bigcup_{\beta < \alpha} \sigma_\beta \upharpoonright M_\alpha.$$
If $\alpha$ is a limit ordinal then $h_\alpha = \bigcup_{\beta<\alpha} h_\beta$ may easily be seen to work, using the fact that

$$\sigma_\alpha = \bigcup_{\beta<\alpha} \sigma_\beta \restriction M_\alpha.$$ 

In this case, we let

$$Y_\alpha = \bigcap_{\beta<\alpha} Y_\beta$$
Finishing Theorem 1...

So assume that $\alpha = \beta + 1$. 
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First find some $Z \subseteq X_0$ of full measure and $\theta \in T_\beta$ such that $Z$ is $h_\beta(H_\beta) \cup \{\theta\}$-invariant, and $h_\beta \upharpoonright Z, \theta \upharpoonright Z$ implements an action of $H_\alpha$. 
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$$W = X \setminus (Z \cup Y_\alpha \times M_\alpha).$$
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Then on $W$ the action of induced by $h_\beta$ has the form required in the one step extension Lemma to be extended to a $H_\alpha$-action.
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Finally, we let $h_\alpha$ act like the master-action along each section on $Y_\alpha \times M_\alpha$. 

\[\square\]
A burning question, I

Recall that Shelah showed that in the case of the measure algebra that there is a model of ZFC in which there is no lifting $h : \text{MALG}(X, \mu) \to \mathcal{B}(X)$.

Glasner-Tsirelson-Weiss' result shows that in the case of $\text{Aut}(X, \mu)$, there is no uniformly Borel lifting. So it is natural to ask:

Question 1. Is there a model of ZFC in which there is no lifting $h : \text{Aut}(X, \mu) \to \mathcal{G}(X, \mu)$?
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**Question 1.** *Is there a model of ZFC in which there is no lifting of \( h : \text{Aut}(X, \mu) \to G(X, \mu) \)?*
One can go a step further. Glasner, Tsirelson and Weiss showed that a so-called Lévy groups cannot act pointwise in a (uniformly) Borel way and induce a non-trivial measure preserving action.
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Some examples of Levy groups are: \text{Aut}(X, \mu), \text{Inn}(E_0), \mathcal{U}(\ell_2(\mathbb{N})), L_0([0, 1], \mathbb{T}).
One can go a step further. Glasner, Tsirelson and Weiss showed that a so-called Lévy groups cannot act pointwise in a (uniformly) Borel way and induce a non-trivial measure preserving action.

Some examples of Levy groups are: Aut($X, \mu$), Inn($E_0$), $\mathcal{U}(\ell_2(\mathbb{N}))$, $L_0([0, 1], \mathbb{T})$.

**Question 2.** Is it consistent with ZFC that no Lévy group admits a non-trivial spatial measure preserving action (by Borel automorphisms, non-uniformly)?
Another question

While Question 1 is undoubtedly the most important, it should be noted that the construction of the lifting in the proof of Theorem 1 gives us Borel automorphisms of arbitrarily high rank in the Borel hierarchy.
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Another question

While Question 1 is undoubtedly the most important, it should be noted that the construction of the lifting in the proof of Theorem 1 gives us Borel automorphisms of arbitrarily high rank in the Borel hierarchy.

This prompts the question:

**Question 3.** Is it consistent with ZFC that there is some fixed $\gamma < \omega_1$ and a lifting $h : \text{Aut}(X, \mu) \to G(X, \mu)$ such that $h(T) \in \Pi^0_\gamma$ for all $T \in \text{Aut}(X, \mu)$? Or must any lifting have unbounded range in the Borel hierarchy?
While Question 1 is undoubtedly the most important, it should be noted that the construction of the lifting in the proof of Theorem 1 gives us Borel automorphisms of arbitrarily high rank in the Borel hierarchy.

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**Question 3.** *Is it consistent with ZFC that there is some fixed $\gamma < \omega_1$ and a lifting $h : \text{Aut}(X, \mu) \to G(X, \mu)$ such that $h(T) \in \Pi^0_\gamma$ for all $T \in \text{Aut}(X, \mu)$? Or must any lifting have unbounded range in the Borel hierarchy?*

*In any case, what can be proved from CH?*
Thank you!