The lifting problem for Aut(X, μ)

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Measure preserving transformations

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Warning: We usually write T for $[T]$ if there is no danger of confusion.

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$\mathrm{Aut}(X, \mu)$ is a Polish group in the topology induced by the sub-neighbourhood basis

$$N(T_0, \varepsilon, A) = \{T \in \mathrm{Aut}(X, \mu) : \mu(T(A) \triangle T_0(A)) < \varepsilon\}$$

where $A \subseteq X$ is Borel, $\varepsilon > 0$ and $T_0 \in \mathrm{Aut}(X, \mu)$.

Near actions vs. spatial actions, I

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(2) A *spatial model* for a near action h is an (actual, pointwise) action $\sigma : G \times X \to X$ such that for each $g \in G$

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Very recently Kwiatkowska and Solecki (2009) have generalized this to a new and much larger class of groups.

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Theorem 1. (T., 2009) *If CH holds, then the near-action of* $\mathrm{Aut}(X,\mu)$ *on* (X,μ) *has a spatial model.*

Thus under CH **every** *near-action has a spatial model.*

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i.e, h splits the identity $\mathrm{Id} : \mathrm{MALG}(X, \mu) \to \mathrm{MALG}(X, \mu)$:

$$\mathrm{Id} = \kappa \circ h,$$

where $\kappa : \mathcal{B}(X) \to \mathrm{MALG}(X, \mu)$ is the canonical homomorphism with $\ker(\kappa) = \mathcal{I}_{mz}$ =the ideal of measure zero sets.

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Theorem (Shelah, circa 1980) *There is a model of ZFC in which* $\mathrm{Id} : \mathrm{MALG}(X, \mu) \to \mathrm{MALG}(X, \mu)$ *does not split.*

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(2) h splits the identity $\mathrm{Id} : \mathrm{Aut}(X, \mu) \to \mathrm{Aut}(X, \mu)$ as follows:

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(the group generated by the T_β, $\beta < \alpha < \omega_1$; for convenience, $H_0 = \{\mathrm{Id}_X\}$.)

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The idea is to make sure that we have chosen the g_β, $\beta < \alpha$, in such a way that for a given choice of $g \in T_\alpha$, there is some reasonably easy way to adjust g on a null-set so that it becomes fully compatible with $h_\alpha : H_\alpha \to G(X,\mu)$, thus allowing the induction to proceed.

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(*i*) For some $n \in \mathbb{N}$, the initial segment $\{m : m <^* n\}$ is isomorphic to α;

(*ii*) There is a monomorphism $\varphi : H_\alpha \to G$ such that $\mathrm{rank}_{<^*}(\varphi(T_\beta)) = \beta$.

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Thus for $(<^*, G) \in \mathcal{S}_\alpha$, we may identify H_α with a subgroup of G in a canonical way.

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$$\sigma_\alpha(T_\beta)(<_0^*, G_0, x) = (<_1^*, G_1, y) \iff$$
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That is: For every $(<^*, G) \in \mathcal{S}_\alpha$, we identify H_α with a subgroup of G, and let H_α act on $(2^{\mathbb{N}})^{\mathbb{N}}$ by a left-shift (where we think of $(2^{\mathbb{N}})^{\mathbb{N}}$ as $(2^{\mathbb{N}})^G$.)

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Lemma

If $\alpha < \omega_1$ is a limit ordinal, it holds for the master action σ_α that

$$\sigma_\alpha = \bigcup_{\beta<\alpha} \sigma_\beta \restriction \mathcal{M}_\alpha$$

Universality property

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Universality Property. (Folklore.) *If* $\Lambda \leq \Gamma$ *are countable groups and* $\tau : \Lambda \curvearrowright X$ *is a Borel action of* Λ *on a standard Borel space, then there is a shift invariant Borel set* $B \subseteq (2^{\mathbb{N}})^{\Gamma}$ *and a Borel bijection* $\psi : X \to B$ *such that*

$$(\forall g \in \Lambda)\psi(\sigma(g)(x)) = \beta(g)(\psi(x)),$$

where β *denotes the shift-action* $\beta : \Gamma \curvearrowright (2^{\mathbb{N}})^{\Gamma}$*.*

One step extensions, I

Given a pair of countable groups $\Lambda \leq \Gamma$, and a Borel action $\sigma : \Lambda \curvearrowright X$ on a standard Borel space X, the Universality Property gives us a way of extending the action: Embed X into $(2^{\mathbb{N}})^{\Gamma}$ using ψ. Unfortunately, $\psi(X) \subsetneq (2^{\mathbb{N}})^{\Gamma}$ may happen.

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If, however, $X = X_0 \sqcup X_1 \sqcup X_2 \sqcup \cdots$ and $\sigma | X_i \simeq \beta : \Lambda \curvearrowright (2^{\mathbb{N}})^{\Gamma}$ for all $i > 0$, then we may:

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If, however, $X = X_0 \sqcup X_1 \sqcup X_2 \sqcup \cdots$ and $\sigma | X_i \simeq \beta : \Lambda \curvearrowright (2^{\mathbb{N}})^{\Gamma}$ for all $i > 0$, then we may:

(i) map X_0 into X_i using some ψ_i that conjugates the action

(ii) extend the Λ-action to Γ on $X_0 \sqcup X_1 \setminus \psi_1(X_0)$

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More precisely:

One step extensions, II

One step extension Lemma. *Let* $\Lambda < \Gamma$ *be countable groups such that there is an element* $\gamma \in \Gamma \setminus \Lambda$ *such that* $\Gamma = \langle \Lambda \cup \{\gamma_0\} \rangle$, *and suppose there are countable groups* Γ_i, $i \in \mathbb{N}$ *such that* $\Gamma \leq \Gamma_i$ *for all* i.

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$$X = X_0 \sqcup \bigsqcup_{i \in \mathbb{N}} (2^{\mathbb{N}})^{\Gamma_i},$$

that is, X *is the disjoint union of* X_0 *and* $(2^{\mathbb{N}})^{\Gamma_i}, (i \in \mathbb{N})$, X_0 *is Borel, and* $(2^{\mathbb{N}})^{\Gamma_i}$ *carries its usual Borel structure for all* $i \in \mathbb{N}$.

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$$\rho \upharpoonright \Lambda \times (2^{\mathbb{N}})^{\Gamma_i}$$

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is the shift action. Then there is a Borel action $\hat{\rho} : \Gamma \curvearrowright X$ *such that* $\hat{\rho} \restriction \Lambda \times X = \rho$*.*

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If this can be done then we get a lifting $h : \mathrm{Aut}(X, \mu) \to G(X, \mu)$ by letting

$$h = \bigcup_{\alpha < \omega_1} h_\alpha.$$

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If α is a limit ordinal then $h_\alpha = \bigcup_{\beta<\alpha} h_\beta$ may easily be seen to work, using the fact that

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In this case, we let

$$Y_\alpha = \bigcap_{\beta<\alpha} Y_\beta$$

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First find some $Z \subseteq X_0$ of full measure and $\theta \in T_\beta$ such that Z is $h_\beta(H_\beta) \cup \{\theta\}$-invariant, and $h_\beta \restriction Z, \theta \restriction Z$ implements an action of H_α.

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Pick a countable sequence $(y_i \in Y_\beta : i \in \mathbb{N})$ of distinct elements in Y_β. Also pick a sequence $(<_i^*, G_i) \in \mathcal{S}_\alpha, i \in \mathbb{N}$, distinct.

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Then on W the action of induced by h_β has the form required in the one step extension Lemma to be extended to a H_α-action. Finally, we let h_α act like the master-action along each section on $Y_\alpha \times \mathcal{M}_\alpha$. □

A burning question, I

Recall that Shelah showed that in the case of the measure algebra that there is a model of ZFC in which there is *no* lifting $h : \mathrm{MALG}(X, \mu) \to \mathcal{B}(X)$.

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Glasner-Tsirelson-Weiss' result shows that in the case of $\mathrm{Aut}(X, \mu)$, there is no *uniformly Borel* lifting. So it is natural to ask:

Question 1. *Is there a model of ZFC in which there is* **no** *lifting of $h : \mathrm{Aut}(X, \mu) \to G(X, \mu)$?*

A burning question, II

One can go a step further. Glasner, Tsirelson and Weiss showed that a so-called *Lévy groups* cannot act pointwise in a (uniformly) Borel way and induce a non-trivial measure preserving action.

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Question 2. *Is it consistent with ZFC that* **no** *Lévy group admits a non-trivial spatial measure preserving action (by Borel automorphisms, non-uniformly)?*

Another question

While Question 1 is undoubtedly the most important, it should be noted that the construction of the lifting in the proof of Theorem 1 gives us Borel automorphisms of arbitrarily high rank in the Borel hierarchy.

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In any case, what can be proved from CH?

Thank you!