Rado's Conjecture, Saturation of NS_{ω_1} and Diamonds

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Outline

Saturation of NS_{ω_1}

Definition (Saturation of NS_{ω_1})

Let W be a collection of stationary sets in ω_1 such that for every S and T in W, $S \cap T$ is nonstationary. Then $|W| \leq \omega_1$.

Rado's Conjecture (RC)

Definition (Rado's Conjecture)

A family of intervals of a linearly ordered set is the union of countably many disjoint subfamilies (σ-disjoint) if and only if every subfamily of size $\aleph_1$ is σ-disjoint.

Definition (Rado's Conjecture in Todorčević's equivalent version, 1983)

A tree T of height ω_1 is the union of countably many antichains (special) if and only if every subtree of T of size $\aleph_1$ is special.

Some applications of RC

Theorem (Todorčević, 1993)

Rado's Conjecture implies (some examples):

1. $\theta^{\aleph_0} = \theta$ *for all regular* $\theta \geq \aleph_2$,
2. *the Singular Cardinal Hypothesis,*
3. $2^{\aleph_0} \leq \omega_2$,
4. $\square_\kappa$ *fails for every uncountable cardinal* κ.

Theorem (Feng, 1999)

Rado's Conjecture implies the presaturation of the nonstationary ideal on ω_1.

Stationary sets in two cardinals version

Definition

We say that a set $S \subseteq [\lambda]^{\mu}$ is stationary if for every function $f : \lambda^{<\omega} \to \lambda$, there is $X \in S$ such that $f[X^{<\omega}] \subseteq X$.

Diamond in two cardinals version

Definition

Let $\langle \mathcal{G}_Z \rangle_{Z \in [\lambda]^\mu}$ be a sequence such that $\mathcal{G}_Z \subseteq P(Z)$ and

$$|\mathcal{G}_Z| \leq \mu$$

for all $Z \in [\lambda]^\mu$. Then $\langle \mathcal{G}_Z \rangle_{Z \in [\lambda]^\mu}$ is a $\diamondsuit_{[\lambda]^\mu}$-sequence if for all $W \subseteq \lambda$, the set

$$\{Z \in [\lambda]^\mu : W \cap Z \in \mathcal{G}_Z\}$$

is stationary. The principle $\diamondsuit_{[\lambda]^\mu}$ states that there is a $\diamondsuit_{[\lambda]^\mu}$-sequence.

Main Theorem

Theorem

Rado's Conjecture together with the saturation of NS_{ω_1} *imply* $\diamondsuit_{\omega_2}\{\delta < \omega_2 : \mathrm{cof}(\delta) = \omega_1\}$.

Some tools used in the proof...

Notation

For an ordinal α, the set

$$\mathrm{lev}_\alpha(T) = \{t \in T : \mathrm{ht}_T(t) = \alpha\}$$

is the α-th level of T.
The height of the tree T, $\mathrm{ht}(T)$ is the minimal ordinal α such that $\mathrm{lev}_\alpha(T) = \emptyset$.
For $E \subseteq \mathrm{Ord}$ and a tree T, let

$$T\restriction_E = \bigcup_{\alpha \in E} \mathrm{lev}_\alpha(T).$$

Some tools used in the proof...

Definition

Let T be a tree and $S \subseteq T$. A function $f : S \to T$ is *regressive* if for all $t \in S$ such that $\mathrm{ht}(t) > 0$, we have $f(t) <_T t$.

Definition

For a tree T of height $\leq \omega_1$ and a set $E \subseteq \omega_1$ we say that E is *T-nonstationary* if there is a regressive mapping $f : T{\restriction}_E \to T$ such that $f^{-1}(t)$ is a special subtree of T for all $t \in T$.

Let $\mathrm{NS}_T = \{E \subseteq \omega_1 : E \text{ is } T\text{-nonstationary}\}$.

Some tools used in the proof...

The follwoing two results are from Todorčević (1981):

Theorem

For every T of height ω_1, NS_T is a normal ideal on ω_1.

Theorem (Pressing Down Lemma for Trees)

For every nonspecial tree T and for any regressive function $f : T \to T$, there is a nonspecial subtree U of T such that $f\restriction_U$ is constant.

Key definition

For every $Z \in [\lambda]^{\omega_1}$, we define the sequence:

$$\mathscr{G}_Z = \{Y \subseteq Z : \text{ there is a guessing subtree } T \subseteq T_{[Z]^{\omega}} \text{ that guesses Y}\}$$

Guessing trees

We will use the following theorem of Todorčević:

Theorem

$\diamondsuit_{[\lambda]^\omega}$ *holds for every cardinal* $\lambda \geq \omega_2$.

Guessing trees

Let $\langle S_N : N \in [\lambda]^\omega \rangle$ be a $\diamondsuit_{[\lambda]^\omega}$-sequence.
Then for every set $W \subseteq \lambda$, the set

$$\mathscr{S}_W = \{N \in [\lambda]^\omega : W \cap N = S_N\}$$

is a stationary (actually, it is a projective stationary) subset of $[\lambda]^\omega$.

Guessing trees

For a subset $\mathscr{S} \subseteq [\lambda]^\omega$, let $T_{\mathscr{S}}$ denote the tree built with countable continuously strictly increasing chains $t = \langle N^t_\xi : \xi \leq \alpha(t) \rangle$ of elements of $\mathscr{S}$, and such that for every $\xi < \eta$,

$$N_\xi \cap \omega_1 < N_\eta \cap \omega_1$$

and

$$\sup(N_\xi \cap \lambda) < \sup(N_\eta \cap \lambda).$$

For $t = \langle N^t_\xi : \xi \leq \alpha(t) \rangle$, in order to have simpler notation, we let

$$N_t = N^t_{\alpha(t)}.$$

Guessing trees

For every $Z \in [\lambda]^{\omega_1}$, we fix

$$Z = \bigcup_{\gamma \in \omega_1} Z_\gamma,$$

a continuous increasing decomposition of Z into countable sets Z_γ.

Guessing trees

Definition

For $Z \in [\lambda]^{\omega_1}$, we call a subtree $T \subseteq T_{[Z]^\omega}$ a *guessing subtree* if

1. T is a nonspecial tree but for every $\alpha \in Z$

$$\{t \in T : \alpha \notin N_t\}$$

   is special.
2. $\forall u, t \in T$ and $\forall \gamma \in \omega_1$, if $N_u \supseteq Z_\gamma$, $N_t \supseteq Z_\gamma$, then

$$S_{N_u} \cap Z_\gamma = S_{N_t} \cap Z_\gamma.$$

Guessing trees

Definition

Suppose T is a guessing subtree of $T_{[Z]^\omega}$ and $Y \subseteq Z$. We say that T *guesses* Y iff

$$(\forall \gamma < \omega_1)(\forall t \in T) N_t \supseteq Z_\gamma \rightarrow Y \cap Z_\gamma = S_{N_t} \cap Z_\gamma.$$

Saturation of NS implies $|\mathscr{G}_Z| \leq \aleph_1$ for every $Z \in [\lambda]^\mu$

Proposition

Assume that NS_{ω_1} is a saturated ideal. Then $|\mathscr{G}_Z| \leq \aleph_1$ for every $Z \in [\lambda]^{\omega_1}$.

RC implies $\langle \mathscr{G}_Z \rangle_{Z \in [\omega_2]^{\omega_1}}$ is actually a $\diamondsuit_{\omega_2}\{\delta < \omega_2 : \operatorname{cof}(\delta) = \omega_1\}$

Proposition

For every $W \subseteq \omega_2$ and every $h : \omega_2^{<\omega} \to \omega_2$, there is $\delta \in \omega_2$ such that

1. $W \cap \delta \in \mathscr{G}_\delta$,
2. $h[\delta^{<\omega}] \subseteq \delta$ *and*
3. $\operatorname{cof}(\delta) = \omega_1$.

We got actually a more general result:

Theorem

Rado's Conjecture together with the saturation of NS_{ω_1} imply $\diamondsuit_{[\lambda]^{\omega_1}}$ for $\omega_2 \leq \lambda < \omega_\omega$.