Some questions on the models of MM

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PROBLEM

MM and PFA appears to produce models of set theory in which every "consistent" set of size $\aleph_1$ "exists".

How to formulate this in a suitable form?

For example in this way:

**Theorem 1 (Veličković)** Assume MM. Let $W$ be an inner model such that $\omega_2^W = \omega_2$. Then $P(\omega_1) \subseteq W$.

**Theorem 2 (Caicedo, Vel.)** Assume $W \subseteq V$ are models of BPFA such that $\omega_2^W = \omega_2$. Then $P(\omega_1) \subseteq W$. 
We would like to extend these results all over the cardinals:

**Conjecture 1 (Caicedo, Veličković)** Assume $W \subseteq V$ are models of MM with the same cardinals. Then $[\text{Ord}]^\leq\aleph_1 \subseteq W$.

This is almost best possible, since:

- There exist $W \subseteq V$ models of MM with the same cardinals such that $[\text{Ord}]^{\aleph_2} \not\subseteq W$.

- Using stationary tower forcing it is possible to produce two models of MM, $W \subseteq V$ such that $[\text{Ord}]^{\leq\aleph_1} \not\subseteq W$. However the two models have different cardinals.
FIRST PROBLEM TO MATCH: FIXING THE COFINALITIES.

This is solved by the following result which expands over works of Cummings, Schimmerling, Todorčević, Dzamonja, Shelah.

**Theorem 3 (V.)** Assume MM. Let $\kappa$ be singular (and strong limit). Let $W$ be an inner model such that $\kappa$ is regular in $W$ and $\kappa^+ = (\kappa^+)^W$. Then $\text{cf}(\kappa) > \omega_1$.

**Corollary 4** Let $V$ be a model of MM (such that every limit cardinal is strong limit). Let $W$ be an inner model with the same cardinals. If $\kappa$ is regular in $W$, $\text{cf}(\kappa) > \omega_1$. 
Back to the conjecture, the best result I have to time is the following:

**Corollary 5** Assume $W \subseteq V$ are models of ZFC with the same cardinals and:

- $V$ models MM,
- every limit cardinal is strong limit,
- $V$ is a set-forcing extension of $W$.

Then $[\text{Ord}]_{\leq \omega_1} \subseteq W$. 
Sketch of proof: if \( V = W[G] \) with \( G \) \( P \)-generic filter for some set \( P \in W \) of size \( \kappa \), new sets of ordinals appear already as subsets of \( \kappa \).

The assumptions entail that \( W \) and \( V \) have the same ordinals of cofinality at most \( \aleph_1 \).

Now the \( \kappa^+ \)-cc of \( P \) entails that \( W \)-stationary subsets of \( \kappa^+ \) remain stationary in \( V \). Fix in \( W \):

\[
\mathcal{E} = \{ S_\alpha : \alpha < \kappa^+ \} \in W
\]

partition of \( E^\omega_{\kappa^+} \) in \( W \)-stationary sets.
In $V$ this remains a partition in stationary sets of points of countable cofinality.

Now let $X \in [\kappa]^{\leq \aleph_1}$ in $V$,

Apply MM in $V$ to find an ordinal $\delta$ of cofinality $\aleph_1$ such that:

\[ S_\alpha \text{ reflects on } \delta \text{ iff } \alpha \in X. \]

Now $\delta$ has cofinality $\aleph_1$ also in $W$ and $P(\omega_1) \subseteq W$.

This is enough to get that the above property holds also in $W$. Thus $X \in W$. \qed
The first natural approach is to follow the same pattern of the proof of the previous theorem. In order to run the argument we need to find a way of generating indestructible partitions of stationary sets:

**Definition 6** Let $\lambda$ be a regular cardinal and $\Gamma$ a property.

$S$ is a $\Gamma$-indestructibly stationary subset of $\lambda$ if it remains stationary in any outer model where the property $\Gamma$ holds.

Let $S$ be a stationary subset of $\lambda$.

$\text{IP}(\Gamma, \kappa, S)$-holds if $S$ carries a partition in $\kappa$-many disjoint $\Gamma$-indestructibly stationary subsets.
We shall be interested in the following properties:

- $\Gamma = \text{Reg}(\lambda)$: $\lambda$ is a regular cardinal

- $\Gamma = \text{scale}(\mathcal{F}, \theta^+)$: for some increasing family $(\theta_i : i < \kappa)$ of regular cardinals, $\mathcal{F} = \{f_\alpha : \alpha < \theta^+\}$ is a scale on $\prod_{i<\kappa} \theta_i$ and $\theta = \sup_{i<\kappa} \theta_i$. 
Problem 1 Let $\kappa$ be an arbitrarily large cardinal.

Does $\text{IP}(\text{Reg}(\lambda), \kappa, E^\omega_\lambda)$ holds for some $\lambda \geq \kappa$?

Assume the answer is yes and let $V$ be a model of MM and $W$ be an inner model with the same cardinals.

We can use this property to show $[\text{Ord}]^{\leq \aleph_1} \subseteq W$ running the same proof sketched before.

We appeal to $\text{IP}(\text{Reg}(\lambda), \kappa, E^\omega_\lambda)$ to get a partition in $W$ of $E^\omega_\lambda$ into $\kappa$-many stationary subsets of $V$.

We then argue by induction on $\kappa$, that for no $\kappa$ new elements of $[\kappa]^{\leq \aleph_1}$ are added.
This leads us to partition relations:

**Definition 7** Let $\mathcal{F}$ be a filter on $\kappa$

$$
\lambda \rightarrow_{\mathcal{F}} [\kappa]_\lambda^2
$$

holds if for every $f : [\lambda]^2 \rightarrow \kappa$, there is $H \subseteq \lambda$ of size $\lambda$ such that $f[[H]^2] \not\in \mathcal{F}$.

We are interested in the failure of this partition relation for the filter of cobounded subsets of $\kappa$. 
Definition 8

\[ \lambda \not\rightarrow_{\mathcal{F}}^{\Gamma} [\kappa]^2_{\lambda} \]

*If there is* \( f : [\lambda]^2 \to \kappa \) *which witness the failure of the partition relation in every outer model in which* \( \Gamma \) *holds.*

To avoid too many subscripts we shall not mention \( \mathcal{F} \) when \( \mathcal{F} \) is the filter of cobounded subsets of \( \kappa \).

This is a slight abuse of notation...
We are interested in this partition relation mainly for this observation:

**Lemma 9** *Larson? Assume* \[ \lambda \notightarrow_{[\kappa]^2_\lambda}. \]

*Then IP(\Gamma, \kappa, S) holds for any \Gamma\text{-indestructibly stationary subset } S \text{ of } \lambda.*
Moreover our approach is not without hope since:

**Theorem 10** Todorčević

$$\omega_1 \not\leadsto^{\text{Reg} (\omega_1)} [\omega]_{\omega_1}^2.$$  

Basic observations coming from pcf-theory give also:

**Fact 1** *If $\theta$ is singular, then:*

$$\theta^+ \not\leadsto^{\text{scale} (\mathcal{F}, \theta^+)} [\text{cf} (\theta)]_{\theta^+}^2.$$  

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As a corollary of the fact we get....

**Corollary 11** Assume $W \subseteq V$ are models of ZFC with the same cardinals and:

- $V$ models MM,
- every limit cardinal is strong limit,
- There are arbitrarily large cardinals $\kappa$ such that for some increasing sequence $(\theta_i : i < \kappa) \in W$ of regular cardinals larger than $\kappa$, there is $F \in W$ scale on $\prod_{i<\kappa} \theta_i$ in $V$.

Then $[\text{Ord}]^{\leq \omega_1} \subseteq W$. 
Proof: For arbitrarily large $\kappa$ we get that $scale(\mathcal{F}, \lambda)$ holds in $V$ for some $\mathcal{F} \in W$ and for some $\lambda$ successor of a singular cardinal of cofinality $\kappa$. This is enough to run the usual arguments. □

**Corollary 12** Assume $V$ models MM and $W$ is an inner model with the same cardinals such that $[\kappa]^{\aleph_1} \not\subseteq W$. Then any $\mathcal{F} \in W$ scale in $W$ on $\prod_{i<\kappa} \theta_i \in W$ increasing sequence of regular cardinals has a new exact upper bound in $V$. 

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On the other hand: for club many singular \( \theta \) of cofinality at most \( \aleph_1 \) there are scales \( F \in W \) of type \( \theta^+ \) which remain scales in \( V \).

**Fact 2 (Silver?, Shelah?)** Assume \( \kappa \) is regular and \( \{ \theta_i : i < \kappa \} \) is a club of singular cardinals larger than \( \kappa \). Let

\[
F = \{ f_\alpha : \alpha < \theta^+ \} \subseteq \prod_{i<\kappa} \theta_i^+
\]

be a family of functions increasing modulo bounded.

Then there is \( D \) club subset of \( \kappa \) such that

\[
F \upharpoonright D = \{ f_\alpha \upharpoonright D : \alpha < \theta^+ \} \subseteq \prod_{i<\kappa} \theta_i^+
\]

has exact upper bound \( \prod_{i \in D} \theta_i^+ \).
Thus:

*If* \( \kappa \geq \aleph_1 \) *is regular and* \((\theta_i : i < \kappa) \in W\) is a sequence of regular cardinals larger than \( \kappa \), *there is* \( F \in W \) *and* \( D \in V \) *club subset of* \( \kappa \) *such that* \( F \upharpoonright D \) *is a scale in* \( V \) *on* \( \prod_{i \in D} \theta_i \).

So if \( V \) and \( W \) witness the failure of the conjecture, on one hand the pcf-structure of \( W \) and \( V \) diverge completely, while on the other hand the two pcf-structures must still be very close to each other.
Other approaches to solve the conjecture
Fact 3 Assume the conjecture fails for $W \subseteq V$ and $\kappa$ is the least such that $[\kappa]^{\aleph_1} \not\subseteq W$. Then for any finite set $\{\lambda_i : i < n\}$ of regular cardinals larger than $\kappa$ there is:

$$j : N \to H(\lambda_{n-1})^W$$

elementary and such that:

- $[N]^{\aleph_1} \subseteq N$,

- $\omega_2 < \text{crit}(j) < \kappa$,

- $j(\kappa) = \kappa$ and $j(\lambda_i) = \lambda_i$ for all $i < n$,

- for each $i < n$ the set of $\delta < \lambda_i$ such that $j(\delta) = \delta$ is closed under all sequences of length at most $\aleph_1$. 
One may try to argue that if $W$ is a "nice" inner model, then it is the case that $N = H(\lambda_{n-1})^W$.

Ideas coming from inner model theory may then lead to a contradiction.