Indestructible Strong Compactness but not Supercompactness

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I will speak about joint work with Joel Hamkins and Grigor Sargsyan. Throughout, we assume some familiarity with the large cardinal notions of measurable, strong, strongly compact, and supercompact cardinal, along with related forcing techniques.

We begin with a brief discussion of forcing indestructibility for supercompact cardinals. This was first done by Richard Laver, who proved the following theorem.

Theorem 1 *(***Laver***, Israel J. Math. 1978)*

Let $V \models$ *"ZFC +* κ *is supercompact". There is then a partial ordering* $\mathbb{P} \in V$, $|\mathbb{P}| = \kappa$ *such that* $V^{\mathbb{P}} \models$ *"*κ *is supercompact". Further, if* $\mathbb{Q} \in V^{\mathbb{P}}$ *is* κ*-directed closed, then* $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ *"*κ *is supercompact".*

Note that a partial ordering $\mathbb{Q}$ is *κ-directed closed* iff every directed set of conditions of size less than κ has a common extension. A supercompact cardinal such as the above κ in $V^{\mathbb{P}}$ is called *Laver indestructible* or simply *indestructible*. The terminology comes from the fact that κ's supercompactness is preserved whenever any κ-directed closed forcing is done. Laver's forcing easily iterates, and it is possible to create a universe in which each supercompact cardinal is Laver indestructible.

Laver indestructibility is one of the most powerful tools used in large cardinals and forcing. Its first application was given by Magidor, who used it to construct a model in which, for every $n \in \omega$, $2^{\aleph_n} = \aleph_{n+1}$, yet $2^{\aleph_\omega} = \aleph_{\omega+2}$.

Notice that Laver's result says nothing about whether it is possible to force a supercompact cardinal to have its strong compactness, yet not its supercompactness, indestructible under

κ-directed closed forcing. In fact, this can be done, as witnessed by the following theorem.

Theorem 2 *(***AA, Hamkins, Sargsyan***)*

Let $V \models$ "ZFC + κ is supercompact". There is then a partial ordering $\mathbb{P} \in V$, $|\mathbb{P}| = \kappa$ such that $V^{\mathbb{P}} \models$ "κ is both supercompact and the least strongly compact cardinal". For any $\mathbb{Q} \in V^{\mathbb{P}}$ which is κ-directed closed, $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is strongly compact". Further, there is $\mathbb{R} \in V^{\mathbb{P}}$ which is κ-directed closed and nontrivial such that $V^{\mathbb{P} * \dot{\mathbb{R}}} \models$ "κ is not supercompact". Moreover, for this $\mathbb{R}$, $V^{\mathbb{P} * \dot{\mathbb{R}}} \models$ "κ has trivial Mitchell rank".

The rest of the lecture will be devoted to a discussion of Theorem 2 and some further possibilities. We begin with the following definition:

Suppose $\mathfrak{A}$ is a collection of partial orderings. Then the *lottery sum of* $\mathfrak{A}$ is the partial ordering $\oplus\mathfrak{A} = \{\langle \mathbb{P}, p\rangle : \mathbb{P} \in \mathfrak{A}$ and $p \in \mathbb{P}\} \cup \{0\}$, ordered with 0 weaker than everything and $\langle \mathbb{P}, p\rangle \leq \langle \mathbb{P}', p'\rangle$ iff $\mathbb{P} = \mathbb{P}'$ and $p \leq p'$.

The terminology (although not the definition) of the lottery sum of a collection of partial orderings is due to Hamkins. Intuitively, if G is V-generic over $\oplus\mathfrak{A}$, then G first selects an element of $\mathfrak{A}$ (or, as Hamkins puts it, "holds a lottery among the posets in $\mathfrak{A}$") and then forces with it.

We will also need the concept of a *Gitik iteration of Prikry-like forcings*. Intuitively, this is an iteration with Easton supports that, for our purposes, intermixes directed closed forcing with Prikry forcing. (These iterations can also include strategically closed forcing, Magidor and Radin forcing, etc., but that won't be needed here.) Roughly speaking, q is stronger than p iff q extends p as in a reverse Easton iteration, except that only finitely many stems of Prikry conditions in p can be extended non-trivially.

Turning to the proof of Theorem 2, let $V \models$ "ZFC + κ is supercompact". Without loss of generality, we assume that $V \models$ GCH as well. The partial ordering $\mathbb{P}$ to be used in the proof of Theorem 2 is now defined as follows. For any ordinal δ, let δ' be the least V-strong cardinal above δ. $\mathbb{P} = \langle \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle : \alpha < \kappa \rangle$ is the Gitik iteration of length κ which (possibly) does nontrivial forcing only at those ordinals δ which are, in V, Mahlo limits of strong cardinals. At such a stage δ, $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathbb{L}}_\delta * \dot{\mathbb{R}}_\delta$, where $\dot{\mathbb{L}}_\delta$ is a term for the lottery sum of all δ-directed closed partial orderings having rank below δ'. If either $V^{\mathbb{P}_\delta * \dot{\mathbb{L}}_\delta} = V^{\mathbb{P}_\delta}$, i.e., the lottery selects trivial forcing at stage δ, or $V^{\mathbb{P}_\delta * \dot{\mathbb{L}}_\delta} \models$ "δ is not measurable", then $\dot{\mathbb{R}}_\delta$ is a term for trivial forcing. If $V^{\mathbb{P}_\delta * \dot{\mathbb{L}}_\delta} \models$ "δ is measurable" and $V^{\mathbb{P}_\delta * \dot{\mathbb{L}}_\delta} \neq V^{\mathbb{P}_\delta}$, i.e., the lottery selects nontrivial forcing at stage δ, then $\dot{\mathbb{R}}_\delta$ is a term for Prikry forcing defined with respect to some normal measure over δ.

The intuition behind the above definition of $\mathbb{P}$ is as follows. The fact that nothing is done at stage δ when the lottery selects trivial forcing, i.e., that no Prikry sequence is added, ensures that $V^{\mathbb{P}} \models$ "κ is supercompact". Since a Prikry sequence is added when a nontrivial forcing at stage δ preserves the measurability of δ, there will be a partial ordering $\mathbb{R} \in V^{\mathbb{P}}$ such that $V^{\mathbb{P} * \dot{\mathbb{R}}} \models$ "κ is not supercompact". The lottery sum at stage δ, in conjunction with the Prikry forcing, will allow us to show that in $V^{\mathbb{P}}$, κ's strong compactness is preserved by nontrivial forcing. Because unboundedly many in κ Prikry sequences will have been added by

$\mathbb{P}$, $V^{\mathbb{P}} \models$ "No cardinal below κ is strongly compact", i.e., $V^{\mathbb{P}} \models$ "κ is the least strongly compact cardinal".

The following lemmas show that $\mathbb{P}$ is as desired.

Lemma 1: $V^{\mathbb{P}} \models$ "κ is supercompact".

Proof Sketch: Let $\lambda > \kappa$ be any cardinal. Let $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of κ such that $M \models$ "κ is not λ supercompact". In M, κ is a Mahlo limit of strong cardinals, meaning by the definition of $\mathbb{P}$ that it is possible to opt for trivial forcing in the stage κ lottery held in M in the definition of $j(\mathbb{P})$. Further, since $M \models$ "No cardinal $\delta \in (\kappa, \lambda]$ is strong" (otherwise, κ is in M supercompact up to a strong cardinal and hence fully supercompact), the next nontrivial forcing in the definition of $j(\mathbb{P})$ takes place well above λ. An argument of Gitik now shows that $V^{\mathbb{P}} \models$ "κ is λ supercompact". Since λ was arbitrary, $V^{\mathbb{P}} \models$ "κ is supercompact". $\square$

Thus, the key idea in the proof of Lemma 1 is to choose a sufficiently large λ and associated supercompactness embedding $j : V \to M$ such that at stage κ in the definition of the forcing in M, we are able to opt for trivial forcing. Since no additional nontrivial forcing takes place in M until well after λ, we may then run an argument of Gitik for the preservation of λ supercompactness.

Lemma 2: Suppose $\mathbb{Q} \in V^{\mathbb{P}}$ is a partial ordering which is κ-directed closed. Then $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is strongly compact".

Proof Sketch: Suppose $\mathbb{Q} \in V^{\mathbb{P}}$ is κ-directed closed. Let $\sigma > \lambda > \max(|\mathrm{TC}(\dot{\mathbb{Q}})|, \kappa)$ be sufficiently large regular cardinals, and let $j : V \to$

M be an elementary embedding witnessing the σ supercompactness of κ such that $M \models$ "κ is not σ supercompact". By the choice of σ, it is possible to opt for $\mathbb{Q}$ in the stage κ lottery held in M in the definition of $j(\mathbb{P})$. Further, as in Lemma 1, since $M \models$ "No cardinal $\delta \in (\kappa, \sigma]$ is strong", the next nontrivial forcing in the definition of $j(\mathbb{P})$ takes place well above σ. Thus, above the appropriate condition, $j(\mathbb{P} * \dot{\mathbb{Q}})$ is forcing equivalent in M to $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$. This means that it is now possible to use an argument of Gitik to show that $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is λ strongly compact". Since λ was arbitrary, $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is strongly compact". $\square$

Thus, the key idea in the proof of Lemma 2 is to choose sufficiently large λ and σ and associated σ supercompactness embedding $j : V \to M$ such that at stage κ in the definition of the forcing in M, we are able to opt for $\mathbb{Q}$ as our partial ordering. Prikry forcing may or may not occur at stage κ in M in the definition of $j(\mathbb{P})$, but this is irrelevant. Since no additional non-trivial forcing in M takes place until well after σ, we may therefore run an argument due to Gitik for the preservation of λ strong compactness.

Lemma 3: $V^{\mathbb{P}} \models$ "No cardinal $\delta < \kappa$ is strongly compact".

Proof: Let $\lambda \geq 2^\kappa$ be any cardinal, and let $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of κ. Suppose $\mathbb{Q} \in V^{\mathbb{P}}$ is Add$(\kappa, 1)$, i.e., the partial ordering for adding one Cohen subset of κ. By Lemma 2, $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is measurable" (since $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is strongly compact"). Because λ has been chosen large enough, it therefore follows that $M^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is measurable". Consequently, by reflection, for unboundedly many

$\delta < \kappa$, $V^{\mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta} \models$ "δ is measurable". By the definition of $\mathbb{P}$, a Prikry sequence is now added to δ. Hence, $V^{\mathbb{P}} \models$ "Unboundedly many $\delta < \kappa$ contain Prikry sequences". By a theorem of Cummings, Foreman, and Magidor, $V^{\mathbb{P}} \models$ "Unboundedly many $\delta < \kappa$ (i.e., the successors of those cardinals having Prikry sequences) contain non-reflecting stationary sets of ordinals of cofinality ω". By a theorem of Solovay, it thus follows that $V^{\mathbb{P}} \models$ "No cardinal $\delta < \kappa$ is strongly compact". This completes the proof of Lemma 3. $\square$

Thus, the key idea in the proof of Lemma 3 is to show that after forcing with $\mathbb{P}$, there are unboundedly many $\delta < \kappa$ which contain Prikry sequences. This allows us to use a theorem of Solovay to infer that in $V^{\mathbb{P}}$, no cardinal $\delta < \kappa$ is strongly compact.

Lemma 4: For $\mathbb{R} = \text{Add}(\kappa, 1)$, $V^{\mathbb{P} * \dot{\mathbb{R}}} \models$ "κ is not supercompact". In fact, in $V^{\mathbb{P} * \dot{\mathbb{R}}}$, κ has trivial

Mitchell rank, i.e., there is no normal measure μ over κ in $V^{\mathbb{P}*\dot{\mathbb{R}}}$ such that for $j : V^{\mathbb{P}*\dot{\mathbb{R}}} \to M^{j(\mathbb{P}*\dot{\mathbb{R}})}$ the elementary embedding generated by the ultrapower via μ, $M^{j(\mathbb{P}*\dot{\mathbb{R}})} \models$ "κ is measurable".

Proof: Let $G * H$ be V-generic over $\mathbb{P} * \dot{\mathbb{R}}$. If $V[G * H] \models$ "κ does not have trivial Mitchell rank", then let $j : V[G * H] \to M[j(G * H)]$ be an elementary embedding generated by a normal measure over κ in $V[G * H]$ such that $M[j(G * H)] \models$ "κ is measurable". Let $I = j(G*H)$. Because $j \restriction \kappa = \mathrm{id}$, we may infer that $(V_\kappa)^V = (V_\kappa)^M$, and hence that $j(\mathbb{P}) \restriction \kappa = \mathbb{P}_\kappa = \mathbb{P}$ and $I_\kappa = G$. We may further infer that $M \models$ "κ is a Mahlo limit of strong cardinals", since V and M must have the same strong cardinals below κ, and forcing can't create a new Mahlo cardinal. Also, as $V[G * H] \models$ "$M[I]^\kappa \subseteq M[I]$", $H \in M[I]$. It cannot be the case that $H \in M[G_\delta]$ for any $\delta < \kappa$, since H codes the generic added at stage δ for unboundedly many $\delta < \kappa$. We know in addition that in M, $\Vdash_{\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa}$ "The forcing beyond stage κ adds no new subsets of 2^κ" and κ is a stage at which nontrivial forcing in $j(\mathbb{P})$ can take place. Consequently, H must be added by the stage κ forcing done in $M[G] = M[I_\kappa]$, i.e., the stage κ lottery held in $M[I_\kappa]$ must opt for some nontrivial forcing. It also follows that $M[I_{\kappa+1}] \vDash$ "κ is measurable". By the definition of $\mathbb{P}$ and $j(\mathbb{P})$, we must then have that $M[I_{\kappa+1}] \vDash$ "κ contains a Prikry sequence and hence has cofinality ω". This contradiction completes the proof of Lemma 4. $\square$

Thus, the key idea in the proof of Lemma 4 is that if κ has nontrivial Mitchell rank after forcing with $\mathbb{P} * \dot{\mathbb{R}}$, then the Cohen generic H for $\mathbb{R}$ must be added by nontrivial forcing at stage κ in $M^{\mathbb{P}} = M^{\mathbb{P}_\kappa}$. κ is then both measurable and singular in $M^{\mathbb{P}_{\kappa+1}}$, a contradiction.

Theorem 2 now follows from Lemmas 1-4. By Lemma 1, κ is supercompact in $V^{\mathbb{P}}$, and by Lemma 2, in $V^{\mathbb{P}}$, κ's strong compactness is indestructible under κ-directed closed forcing. By Lemma 3, $V^{\mathbb{P}} \vDash$ "κ is the least strongly compact cardinal". By Lemma 4, there is a nontrivial κ-directed closed forcing $\mathbb{R} \in V^{\mathbb{P}}$ such that $V^{\mathbb{P} * \dot{\mathbb{R}}} \vDash$ "κ has trivial Mitchell rank". This completes the proof sketch of Theorem 2. $\square$

We close by listing a few questions for further exploration related to Theorem 2. In particular:

1. Is it possible to get a model witnessing the conclusions of Theorem 2 in which κ is not the least strongly compact cardinal? Since Prikry forcing above a strongly compact cardinal destroys strong compactness, this would require a different sort of iteration from the one used in the proof of Theorem 2. A weak version of

Theorem 2 seems to be possible, using techniques due to Sargsyan.

2. Is it possible to prove an analogue for Theorem 2 in a model containing more than one supercompact cardinal? A weak version of Theorem 2 for two supercompact cardinals seems possible, by first forcing as in the proof of Theorem 2 just discussed, and then using a variant of this forcing due to Sargsyan which replaces Prikry sequences with non-reflecting stationary sets of ordinals of high enough cofinality. At this point, it is unclear how far this idea can be extended. This contrasts sharply with Laver's forcing, which easily iterates.