Different ways to construct non-special ω_2-Aronszajntrees

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If $\mathbb{P}_\kappa$ is the Mitchell collapse of a large cardinal κ to ω_2, then

- $V^{\mathbb{P}_\kappa} \models$ "there are no ω_2-Aronszajntrees", if κ is weakly compact
- $V^{\mathbb{P}_\kappa} \models$ "there are no special ω_2-Aronszajntrees", if κ is Mahlo

This was first proved by Mitchell.

Let $\square(\omega_2)$ be the statement that there is a sequence $(C_\alpha : \alpha < \omega_2)$ such that

- $C_\alpha \subseteq \alpha$ is club
- if α is a limit point of C_β then $C_\beta \cap \alpha = C_\alpha$
- there is no "trivializing" club $C \subseteq \omega_2$ such that $C \cap \alpha = C_\alpha$ for all limit points α of C

We know that the failure of $\square(\omega_2)$ has consistency strength exactly a weakly compact (one direction given by Mitchell's result above). Then we have the following theorem of Todorcevic:

Theorem

Assume $\square(\omega_2)$. Then there is a non-special ω_2-Aronszajntree.

This with the above mentioned independence result shows that the statement

- "all ω_2-Aronszajntrees are special"
  (or "there are no non-special ω_2-Aronszajntrees")

has consistency strength exactly a weakly compact. The consistency strength of

- "all ω_2-Aronszajntrees have an unbounded antichain"
  (or "there are no ω_2-Suslintrees")

is still open.
[Attention: the consistency strength of "there are no *special* ω_2-Aronszajntrees" is just a Mahlo.]

We want to prove some new equiconsistency results, so we look at a certain type of ω_2-Aronszajntrees with some helpful properties.

Definition

Say that an ω_2-Aronszajntree is *fin-coherent* if it is generated by a sequence of functions

$$(f_\alpha : \alpha < \omega_2)$$

in the following sense

- $f_\alpha : \alpha \longrightarrow 2$ for all $\alpha < \omega_2$
- if $\alpha < \beta$ then f_α and $f_\beta \restriction \alpha$ differ in only finitely many values
- the fin-coherent tree T is the set of all functions $f : \alpha \longrightarrow 2$ that differ from f_α in only finitely many values (for some $\alpha < \omega_2$).

Fin-coherent trees are *strongly homogeneous* in the sense that's there is a family $\{h_{t_0,t_1} : t_0, t_1 \in T_\alpha, \alpha < \kappa\}$ of automorphisms with the following properties:

- h_{t_0,t_1} moves T^{t_0} to T^{t_1} and vice versa, so t_0 is mapped to t_1. h_{t_0,t_1} is the identity in all other parts of the tree. $h_{t,t}$ is the identity on T.
- (commutativity) $h_{s_0,s_2}(t_0) = h_{s_1,s_2}(h_{s_0,s_1}(t_0))$ holds for all $s_0, s_1, s_2 \in T_\alpha$ with $s_0 \leq t_0$.
- (uniformity) If $s_0, s_1 \in T_\alpha$ with $s_0 \leq t_0$ and $s_1 \leq h_{s_0,s_1}(t_0) = t_1$ then $h_{t_0,t_1} \upharpoonright T^{t_0} = h_{s_0,s_1} \upharpoonright T^{t_0}$.
- (transitivity) If α is a limit ordinal and $t_0, t_1 \in T_\alpha$, then there exist $s_0, s_1 \in T_{<\alpha}$ such that $h_{s_0,s_1}(t_0) = t_1$.

We actually get an equivalence result:

Theorem (K.)

Strongly homogeneous trees are fin-coherent and vice versa.

It was shown that there can be non-special fin-coherent ω_2-Aronszajntrees, even stronger:

Theorem (Velickovic)

Assume a strong combinatorial principle true in the constructible universe (called "square with built-in-diamond"). Then there is a fin-coherent ω_2-Suslintree.

What about fin-coherent ω_2-Aronszajntrees?

Theorem (K.)

Assume $\square(\omega_2)$. *Then there is a fin-coherent* ω_2*-Aronszajntree* T. *Even stronger, we can construct a* T *generated by the sequence* $(f_\alpha : \alpha < \omega_2)$ *such that*

(+) $f_\alpha = f_\beta \restriction \alpha$ *iff* α *is a limit point of* C_β

The property (+) *then implies straightforwardly that*

- *The tree* T *of the statement above is special iff the* $\square(\omega_2)$*-sequence used in its construction is special.*

So we get:

Corollary

Assume $\square(\omega_2)$. *Then there is a non-special strongly homogeneous* ω_2*-Aronszajntree.*

It turns out that fin-coherent trees have some additional helpful properties.

Lemma

Assume T is strongly homogeneous and it has a club antichain. Then T is special.

So we get:

Corollary

Assume $\square(\omega_2)$*. Then there is a strongly homogeneous* ω_2*-Aronszajntree without any club antichains.*

Proof.

We know we can get a non-special one by the previous results. So it cannot have a club antichain by the last lemma. □

This gives the following equiconsistency result:

Corollary

The following is equiconsistent with a weakly compact:

- *"every ω_2-Aronszajntree has a club antichain"*

We end with a "coffeehouse" result: another non-standard construction of a non-special ω_2-Aronszajntree.

Theorem

Assume CH *and* NS_{ω_1} *is saturated. Then there is an* ω_2*-Suslintree.*

PS: Assaf Rinot notified me that that this theorem has already been pointed out by Shelah in the 1980s.

Proof.

By classical results of Solovay/Ketonen we have $2^{\aleph_1} = \aleph_2$. So we can deduce $\diamondsuit_{\omega_2}(\mathrm{cof}\ \omega)$. Now split into two cases:

1. either we have a non-reflecting stationary subset of $\omega_2 \cap \mathrm{cof}\ \omega$. Then we're done by classical results of Gregory.
2. if we have no such non-reflecting set, then we can do a Gregory type argument to step up $\diamondsuit_{\omega_2}(\mathrm{cof}\ \omega)$ to $\diamondsuit_{\omega_2}(\mathrm{cof}\ \omega_1)$ (this uses saturation). But then CH and $\diamondsuit_{\omega_2}(\mathrm{cof}\ \omega_1)$ easily construct an ω_2-Suslintree (seal off antichains at uncountable cofinalities).

In either case, we have an ω_2-Suslintree. □