Universally measurable sets in generic extensions

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Let X be a topological space and let $\mathcal{B}$ be the collection of Borel subsets of X.

A *finite Borel measure* on X is a function

$$\mu \colon \mathcal{B} \to [0, \infty)$$

which is countably additive for pairwise disjoint families.

A set $A \subseteq X$ is *universally null* if A is contained in a Borel set of μ-measure 0, for each atomless finite Borel measure μ on X.

A is *universally measurable* if for each finite Borel measure μ on X there exist Borel sets B, N such that

$$\mu(N) = 0$$

and

$$A \triangle B \subset N.$$

A *Polish* space is a complete separable metric space.

Theorem 1. *If X and Y are Polish spaces and μ and ν are atomless Borel probability measures on X and Y respectively, then there is a Borel bijection $f\colon X \to Y$ such that*

$$\mu(I) = \nu(f[I])$$

for all Borel $I \subset X$.

Theorem 2. *If X is a Polish space and $A \subseteq X$, then A is universally measurable if and only if every continuous injective image of A in $\mathbb{R}$ is Lebesgue measurable.*

3 Fact. A universally null subset of a Polish space cannot contain a perfect set.

Theorem 4 (Lusin 1917)**.** *Every analytic set is universally measurable.*

Theorem 5 (Grzegorek, Ryll-Nardzewski 1979)**.** *There are analytic sets which don't have universally null symmetric difference with any Borel set.*

6 Question. Is there a nice pointclass Γ such that for every universally measurable set A there is a

$$B \in \Gamma$$

such that

$$A \triangle B$$

is universally null?

If proper class many Woodin cardinals exist, Γ has to contain the universally Baire sets.

Suppose that $\Gamma_0 \subset \Gamma_1$ are pointclasses which are closed under finite unions and complements, and suppose that

- Γ_0 is closed under countable changes,
- Γ_1 has the perfect set property,
- $A \in \Gamma_1 \setminus \Gamma_0$.

Then A does not have universally null symmetric difference with any member of Γ_0.

What if Γ is the sets with the Baire Property? Equivalent to asking if all universally measurable set can have the property of Baire.

7 Fact. (Larson-Neeman) Let $S \subset \mathbb{R}$ be a universally measurable set without the property of Baire. Then $S \times \mathbb{R}$ is universally measurable and does not have universally null symmetric difference with any set the property of Baire.

8 Definition. A function is *universally measurable* if all preimages of open sets are universally measurable.

9 Definition. A *medial limit* is a universally measurable function from $\mathcal{P}(\omega)$ to $[0, 1]$ which is finitely additive for disjoint sets, and maps singletons to 0 and ω to 1.

10 Question. Do medial limits necessarily exist?

Mokobodzki/Christensen : Yes if CH holds.

Fremlin: Yes if $cov(\mathcal{M}) = \mathfrak{c}$ holds.

11 Question. Is there necessarily a uniform universally measurable proper ideal on ω which contains all but countably many members of any almost disjoint family?

Not if every universally measurable ideal (equivalently, filter) has the property of Baire.

Yes if there exists a medial limit.

12 Definition. The *Filter Dichotomy* is the statement that for each nonmeager filter F on ω, there is a finite-to-1 $h\colon \omega \to \omega$ such that $\{h[x] \mid x \in F\}$ is an ultrafilter.

Shown consistent by Blass and Laflamme in 1989.

Theorem 13. *The Filter Dichotomy implies that universally null uniform filters on ω are meager.*

Proof: Let F be a nonmeager universally measurable uniform filter on ω, and let $h\colon \omega \to \omega$ be finite-to-1 such that $\{h[x] \mid x \in F\}$ is an ultrafilter. Let

$$S = \{ \bigcup_{n \in Z} h^{-1}[n] \mid Z \subset \omega \},$$

and let $G\colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ be defined by $G(x) = h[x]$.

- S is a perfect subset of $\mathcal{P}(\omega)$.

- $F \cap S$ is a universally measurable subset of S.

- $G \restriction S\colon S \to \mathcal{P}(\omega)$ is a Borel isomorphishm.

- $G[F \cap S] = G[F]$ is not Lebesgue measurable.

14 Question. (Mauldin 1978) Do there exist more than continuum many universally measurable sets?

Larson-Shelah: consistently, no.

Theorem 15. *(Hausdorff 1908) There exists a universally null subset of* $\mathbb{R}$ *of cardinality* $\aleph_1$*.*

Theorem 16. *(Recław) If* R *is a Borel binary relation on a Polish space and* X *is well-ordered by* R*, then* X *is universally null.*

Theorem 17. *(Laver 1970's) Consistently there exist just continuum many universally null subsets of* $\mathbb{R}$*.*

For any nonempty set X, let μ_X denote the standard product measure on ${}^X2$, in which, for each finite $Y \subseteq X$ and each function $a\colon Y \to 2$,

$$\mu_X(\{f \in {}^X2 \mid f \restriction Y = a\}) = 2^{-|Y|}.$$

The *Baire* sets are those in the σ-algebra generated by the sets

$$\{f \in {}^X2 \mid f(x) = i\},$$

for $x \in X$ and $i \in 2$.

For each Baire set $E \subseteq {}^X 2$, let $[E]_{\mu_X}$ be the set of Baire sets $F \subseteq {}^X 2$ such that

$$\mu_X(E \triangle F) = 0.$$

The *random algebra* $\mathbb{B}(X)$ consists of all sets of the form $[E]_{\mu_X}$, with E a non-μ_X-null Baire subsets of ${}^X2$, with the order of mod-μ_X-null containment.

$\mathbb{B}(X)$ is c.c.c., and forcing with $\mathbb{B}(X)$ adds a generic function $F\colon X \to 2$.

If κ is a cardinal such that $\kappa^{\aleph_1} = \kappa$, then

$$2^{\aleph_0} = 2^{\aleph_1} = \kappa$$

in the $\mathbb{B}(\kappa)$-extension.

Theorem 18. *(Larson-Shelah) Every universally measurable subset of $\mathbb{R}$ in the $\mathbb{B}(X)$-extension is the union of $\mathfrak{c}^V$ many Borel sets.*

Baire sets and Borel sets have codes which induce reinterpretations in generic extensions.

A function $F\colon X \to 2$ is V-generic for $\mathbb{B}(X)$ if and only if F is in (the reinterpretation of) every measure 1 Baire set in V.

If $Y \subseteq X$ then $\mathbb{B}(X) \sim \mathbb{B}(Y) * \mathbb{B}(X \setminus Y)$.

For every $\mathbb{B}(X)$-name η for an element of $\mathbb{R}$, if Y is a subset of X and

$$H \subseteq \mathbb{B}(Y)$$

is a V-generic filter, then there is in $V[H]$ a finite Borel measure

$$\nu(H, \eta)(I) = \mu_{X \setminus Y}([\![(\eta/\check{H}) \in \check{I}]\!]).$$

Every name for a countable set of ordinals has countable support, i.e., depends only on $F \restriction Y$, for some countable $Y \subseteq X$.

Every $\mathbb{B}(X)$-name for a finite Borel measure is induced by a $\mathbb{B}(Y)$-name, for some countable $Y \subseteq X$ (i.e., the name has support Y).

For each countable $Y \subseteq X$, there is a set of $\mathfrak{c}^V$ many $\mathbb{B}(X)$-names which represent all finite Borel measures on $\mathbb{R}$ in the $\mathbb{B}(Y)$ extension.

Letting $\dot{A}$ be a $\mathbb{B}(X)$-name for a universally measurable set, for each $\mathbb{B}(X)$-name $\dot{m}$ for a finite Borel measure, there are names $\dot{B}$, $\dot{N}$ for Borel sets for which every condition forces that

$$\dot{A} \triangle \dot{B}$$

will be contained in the $\dot{m}$-null set $\dot{N}$.

Then there is a $Y \subseteq X$ of cardinality $\mathfrak{c}^V$ such that for every $\mathbb{B}(X)$-name for a finite Borel measure with support contained in Y, there exist such names $\dot{B}$, $\dot{N}$ with support contained in Y.

Say that such a Y is $\dot{A}$-*closed*.

Lemma 19. *If η is a $\mathbb{B}(X)$-name for an element of $\mathbb{R}$,*

- $Y \subseteq X$*,*

- $\dot{B}$ *and* $\dot{N}$ *are* $\mathbb{B}(X)$*-names with support* Y *for Borel sets,*

- *every condition forces that* $\dot{N}$ *is* $\nu(G \restriction Y, \eta)$*-null and* $\dot{A} \triangle \dot{B} \subseteq \dot{N}$,

then every condition forces that $\eta \notin \dot{N}$ *and thus that* $\eta \in \dot{A} \leftrightarrow \eta \in \dot{B}$*.*

It follows that if Y is $\dot{A}$-closed, then every element of $\dot{A}$ in the $\mathbb{B}(X)$-extension will be an element of a Borel set contained in $\dot{A}$ (of the form $\dot{B} \setminus \dot{N}$) in the $\mathbb{B}(Y)$-extension.

If $|Y| = \mathfrak{c}^V$, then there are $\mathfrak{c}^V$ many such Borel sets.

Under MA_λ every union of λ many Borel sets is universally measurable.

20 Question. Can the universally measurable sets be the unions of $\aleph_1$-many Borel sets?

No:

Theorem 21 (Grzegorek)**.** *If κ is the smallest cardinality of a nonmeasurable set of reals, then there is a universally null set of cardinality κ.*

Theorem 22. *(Larson-Neeman-Shelah) If* $|X| > \mathfrak{c}^V$ *then in the* $\mathbb{B}(X)$ *extension* A *is universally measurable if and only if* A *and its complement are unions of* $\mathfrak{c}^V$ *many Borel sets.*

23 Definition. The *Borel reinterpretation* of a set A in a generic extension is the union of all reinterpreted ground model Borel sets contained in A.

A is universally measurable if and only if the Borel reinterpretations of A and its complement in any random algebra extension are complements.

If $A \subset \mathbb{R}$ is universally measurable, then for any Borel $f\colon {}^{\omega}2 \to \mathbb{R}$ there are Borel B, N such that $f^{-1}[N]$ is μ_ω-null and $A \triangle B \subset N$.

If ν is an atomless finite Borel measure on $\mathbb{R}$, let

$$f\colon {}^{\omega}2 \to \mathbb{R}$$

be a Borel isomorphism mapping μ_ω to ν. Then if $[B]_{\mu_\omega}$ is the Boolean value that $f(G)$ is in a Borel subset of A in the ground model, then $f[B]$ is a Borel set whose symmetric difference with A is ν-null.

So: if $\mathcal{A}$ and $\mathcal{B}$ are collection of Borel sets whose unions are complements, and $\bigcup \mathcal{A}$ is not universally measurable, then there exist a Borel function $f\colon {}^{\omega}2 \to \mathbb{R}$ and a Borel set $E \subseteq {}^{\omega}2$ such that for no Borel $E' \subset E$ is $f[E']$ contained in a Borel subset of either $\bigcup \mathcal{A}$ or $\bigcup \mathcal{B}$.

By genericity, there is a countable $Y \subset X$ such that $\mathcal{A}, \mathcal{B}, f$ are in $V[G \restriction X \setminus Y]$ and $G \restriction Y$ is "in" E. But $f(G \restriction Y)$ is in some member of $\mathcal{A} \cup \mathcal{B}$.

The Borel reinterpretation of a universally measurable set in a random algebra extension is universally measurable.

Given a Borel function $x \mapsto \rho_x$ (for $x \in {}^{\omega}2$) representing a Borel measure, consider the measure

$$\nu(E) = \int \rho_x(E)\, d(\mu_\omega).$$

24 Definition (Ciesielski-Pawlikowski). A *cube* is a continuous injection from $\Pi_{n\in\omega}C_n$ to X, where X is a Polish space and each C_n is a perfect subset of ${}^{\omega}2$.

Let $Perf(X)$ denote the set of perfect subsets of X.

25 Definition. $\mathcal{E} \subseteq Perf(X)$ is $\mathcal{F}_{cube}$-*dense* if for each cube f there is a cube g such that $g \subseteq f$ and $range(g) \in \mathcal{E}$.

26 Definition. The axiom $\mathrm{CPA}_{cube}(X)$ says that for every $\mathcal{F}_{cube}$-dense $\mathcal{E} \subseteq Perf(X)$ there is a $\mathcal{E}_0 \subseteq \mathcal{E}$ such that $|\mathcal{E}_0| \leq \aleph_1$ and $|X \setminus \bigcup \mathcal{E}_0| \leq \aleph_1$.

Theorem 27. *If X is a Polish space and CPA$_{cube}(X)$ holds, then every universally measurable set A is the union of at most $\aleph_1$ many sets, each of which is either a perfect set or a singleton.*

Proof. Let $\mathcal{E}$ be the collection of perfect subsets of X which are either contained in or disjoint from A. Let $f\colon \Pi_{n\in\omega}{}^{\omega}2 \to X$ be a continuous injection, and let μ be the Borel measure on X defined by letting $\mu(I)$ be the measure of $f^{-1}[I]$. Then there exist Borel subsets B, N of X such that $A \triangle B \subset N$ and $\mu(N) = 0$. Then one of $B \setminus N$ and $(X \setminus B) \setminus N$ has positive μ-measure. $\square$

Lemma 28. *If D is a Borel subset of $\Pi_{n\in\omega}{}^{\omega}2$ and D has positive measure in the usual product measure, then D contains a set of the form $\Pi_{n\in\omega}C_n$, where each $C_n \in Perf(X)$.*

Under $\mathrm{CPA}^{game}_{cube}$, the perfect sets can be taken to be disjoint.

(Debs) An ideal I on $\mathbb{R}$ is *polar* if for some (Borel) set Σ of Borel measures on $\mathbb{R}$, I is the set of Borel sets which are null for all members of Σ.

29 Question. Can the arguments given above for the random algebra be carried out under the countable support iteration of any (iterable, proper) polar ideal?

30 Definition. A *probability transition kernel* from a Polish space X to a Polish space Y a function that associates to each $x \in X$ a Borel measure ρ_x on Y, in such a way that

$$x \mapsto \rho_x(B)$$

is a Borel function, for each Borel $B \subseteq Y$.

31 Definition. Probability measures p, q are *orthogonal* if there is a set which has p-measure 1 and q-measure 0.

32 Definition. A probability transition kernel is *orthogonality preserving* if

$$\int \rho_x(E)\, dp$$

and

$$\int \rho_x(E)\, dq$$

are orthogonal measures on Y, whenever p and q are orthogonal measures on X.

33 Question (Mauldin, Preiss, Wiezsacker). If $x \mapsto \rho_x$ is orthogonality preserving, is there a universally measurable $f\colon Y \to X$ such that

$$\rho_x(f^{-1}\{x\}) = 1$$

for all $x \in X$?

- Replace "universally measurable" with Borel, then no, though the converse holds.

- Yes if there is a medial limit.

- No if all universally measurable sets have property of Baire.