Generic constructions of Banach spaces

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Main goal:

Study of generic constructions of directed limits of finite dimensional normed spaces (polyhedral spaces).
Equivalently, study of inverse limits of compact convex sets (simplexes).

Applications:

1. Lindenstrauss spaces (preduals of L_1)
2. Gurarij spaces
3. Distinction between different uncountable substructures of the generic Banach space (e.g. biorthogonal-like systems).

The construction uses the method of forcing, where conditions are a particular kind of finite dimensional normed spaces.

It turns out that by a completeness result of Magidor-Malitz all the generic constructions we obtain can also be done with the diamond principle.

Similar previous works:

1. Shelah's work (1985) on Banach spaces with few operators (done using diamond).
2. Bell-Ginsburg-Todorcevic (1982) construction of a generic non-metrizable compactum which is Hereditarily separable in all powers (done with the natural forcing for this with finite conditions).

Understanding the generic constructions

Notation

Given a set X and $0 \in Y \subseteq \mathbb{R}$, let $c_{00}(X, Y)$ be the set of all functions $f : X \to Y$ such that its *support* $\operatorname{supp} f = \{x \in X : f(x) \neq 0\}$ is finite.
Suppose that $0, 1 \in Y \subseteq \mathbb{R}$. Given $x \in X$, let $u_x \in c_{00}(X, Y)$ be the mapping with value 1 in x and 0 otherwise.
If Y is closed under $\vee$ and $\wedge$, then $c_{00}(X, Y)$ is a lattice.

Let $\mathbb{P}$ be a set of conditions $p = (D_p, \|\cdot\|_p)$, where

1. D_p is a finite subset of ω.
2. $\|\cdot\|_p$ is a norm on $c_{00}(D_p, \mathbb{R})$.

The extension $p \leq q$ is $D_q \subseteq D_p$ and

$$\|\cdot\|_p \restriction c_{00}(D_q) = \|\cdot\|_q.$$

Given a $\mathbb{G}$ be a generic filter of $\mathbb{P}$ we define

$$D_{\mathbb{G}} := \bigcup_{p \in \mathbb{G}} D_p,$$

and for each $x \in c_{00}(D_{\mathbb{G}})$ we define

$$\|x\|_{\mathbb{G}} := \|x\|_p$$

where $p \in \mathbb{G}$ is such that $x \in c_{00}(D_p)$.

Then $(c_{00}(D_{\mathbb{G}}), \|\cdot\|_{\mathbb{G}})$ is a normed space.
Let $X_{\mathbb{G}}$ be its completion.

Suppose that $\mathbb{P}_0$ consists of **all** finite dimensional spaces $(D_p, \|\cdot\|_p)$. Then $\mathbb{P}$ is not ccc:
For each real number $1 < r < \infty$ consider $p_r = (\{0,1\}, \|\cdot\|_r)$ where

$$\|(a,b)\|_r = \sqrt[r]{|a|^r + |b|^r}.$$

Since for $r \neq s$ one has that $\|\cdot\|_r \neq \|\cdot\|_s$, it follows that $\mathbb{P}_0$ is not ccc. Indeed it collapses ω_1.
The problem is that in $\mathbb{P}_0$ there are *too many* norms

Given $D \subseteq \mathbb{N}$ finite and $F \subseteq c_{00}(D, \mathbb{Q})$ also finite, we define for $x \in c_{00}(D)$,

$$\|x\|_F := \sup\{|\langle f, x\rangle| : f \in F\}.$$

This is always a seminorm.
If in addition the set F defines a norm, then we call $(c_{00}(D), F)$ a $\mathbb{Q}$*-finite dimensional space* ($\mathbb{Q}$-f.d.).
Observe that there are countably many of them, so, it is ccc.
A sufficient condition to ensure that $\|\cdot\|_F$ is a norm is for example that for every $n \in D$ there is some $f_n \in F$ such that $f_n(u_n) := \langle f_n, u_n\rangle = 1$ and $f_n(u_m) = 0$ for all $m < n$ in D.

Examples of $\mathbb{Q}$-f.d. spaces are

$$\ell_\infty^n := (c_{00}(n), \{u_i\}_{i<n})$$

for every $n \in \mathbb{N}$.

Indeed, the ℓ_∞^n's is a cofinal family on the class of $\mathbb{Q}$-f.d. spaces. Moreover the class of $\mathbb{Q}$-f.d. spaces is very rich:

For every finite dimensional normed space X and every $\varepsilon > 0$ there is a $\mathbb{Q}$-f.d. space Y and an $1+\varepsilon$-isomorphism onto $T : X \to Y$.

We consider $\mathbb{P}_1$ to be the class of all $p = (c_{00}(D_p), \{u_n\}_{n \in D_p})$. Let $\mathbb{G}$ be a generic filter for $\mathbb{P}_1$.

Question

What is the generic space $X_{\mathbb{G}}$?

Answer

$X_{\mathbb{G}}$ is $c_0(\mathbb{N})$, the space of converging sequences to zero $(a_n)_n$ with the suppremum norm $\|(a_n)_n\| = \sup_n |a_n|$.

We consider now the partial ordering $\mathbb{P}_2$ consisting on all the $\mathbb{Q}$-f.d. spaces. Let $\mathbb{G}$ be a generic filter for $\mathbb{P}_2$.

Question

What is the generic space $X_{\mathbb{G}}$?

Observe that the subfamily $\{\ell_\infty^n\}_n$ is cofinal in $\mathbb{P}_2$. However

Answer

$X_{\mathbb{G}}$ is *the separable Gurarij space* $\mathfrak{G}$.

This space is very rich:

For every $\varepsilon > 0$ every separable Banach space can be $1+\varepsilon$-isomorphically embedded into $\mathfrak{G}$.

The separable Gurarij space can be introduced as the closure of the limit of certain Fraisse class.

The basic forcing

Basic Forcing

Let $\mathbb{P}_{\mathrm{b}}$ be the set of $p = (D_p, F_p, \sigma_p, H_p)$, called *conditions*, with the following properties:

(C.0) $D_p \subseteq \omega_1$ and $\sigma_p \subseteq \mathbb{N} \times c_{00}(D_p, \mathbb{Q} \cap [-1, 1])$ are finite.

(C.1) $F_p \subseteq c_{00}(D_p, \mathbb{Q} \cap [-1, 1])$ is finite and it has in addition the following properties:

(C.2) $H_p = \{h_s^{(p)}\}_{s\in\sigma_p} \subseteq F_p$ is a subset of F_p indexed by σ_p such that
for every $s = (n, x) \in \sigma_p$ with $x \neq 0$ one has that

$$h_{(n,x)}^{(p)} \upharpoonright \min \operatorname{supp} x = 0.$$

(C.3) (Large density + separation) $\{0\} \times \{u_\gamma\}_{\gamma\in D_p} \subseteq \sigma_p$ and $(h_{(0,u_\gamma)}^{(p)})_\gamma = 1$ for every $\gamma \in D_p$.

Basic Forcing

The ordering $p \leq_{\mathrm{b}} q$ is defined by:

(O.0) $D_q \subseteq D_p$ and $\sigma_q \subseteq \sigma_p$.

(O.1) $F_q \subseteq F_p \restriction D_q = \{f \restriction D_q \,:\, f \in F_p\} \subseteq \mathrm{co}_{\mathbb{Q}}(\pm F_q)$.

(O.2) For every $s \in \sigma_q$ one has that

$$h_s^{(p)} \restriction D_q = h_s^{(q)}.$$

A *forcing notion* $\mathbb{P}$ is any subset of $\mathbb{P}_{\mathrm{b}}$ partially ordered by $\leq_{\mathrm{b}}$.

The norms

We define naturally the following seminorms on the f.d. vector space $c_{00}(D_p)$. Given $x \in c_{00}(D_p)$ let

$$\|x\|_p := \max\{|\langle f, x\rangle| \,:\, f \in F_p\} \tag{1}$$

The formula (1) defines clearly a seminorm.
Indeed, an easy inductive argument shows that (C.3) implies that $\|\cdot\|_p$ is a norm.
We denote this normed space by X_p.

The condition (C.3) also implies that that for every $\gamma < \eta$ in D_p one has that

$$\|u_\eta - u_\gamma\|_p \geq 1,$$

so $(u_\gamma)_{\gamma \in D_p}$ is a 1-separated sequence.
This will give as a consequence that the direct limit of these p's will be a non-separable space.

Some remarks:

1. Each X_p is a $\mathbb{Q}$-f.d. space.
2. Observe that the condition (O.1) is equivalent to:

   (O.1') For every $f \in F_p$ one has that $f \upharpoonright D_q \in F_q$ and for every $x \in c_{00}(D_q)$ one has that $\|x\|_p = \|x\|_q$.
3. It follows from the Hahn-Banach theorem that

$$\mathrm{co}_{\mathbb{R}}(\pm F_p) = B_{(X_p)^*}, \tag{2}$$

   and so,

$$\mathrm{Ext}(B_{(X_p)^*}) \subseteq F_p. \tag{3}$$

The generic space

Given a generic filter $\mathbb{G}$ for a forcing notion $\mathbb{P}$, we define $X_{\mathbb{G}}$ as above: First we define

$$D_{\mathbb{G}} := \bigcup_{p \in \mathbb{G}} D_p$$

and then for $x \in c_{00}(D_{\mathbb{G}})$ we define

$$\|x\|_{\mathbb{G}} := \|x\|_p$$

where $p \in \mathbb{G}$ is such that $x \in c_{00}(D_p)$.

The generic space $X_{\mathbb{G}}$ is the completion of $(c_{00}(D_{\mathbb{G}}), \| \cdot \|_{\mathbb{G}})$.

For all the forcing notions $\mathbb{P}$ we consider one has that

For every $\gamma < \omega_1$ one has that

$$\mathcal{D}_\gamma := \{p \in \mathbb{P} \,:\, \gamma \in D_p\}$$

is dense.

It follows that $D_\mathbb{G} = \omega_1$ for every generic filter $\mathbb{G}$.
Another dense argument will give that most of the generic spaces $X_\mathbb{G}$ are *Gurarij*.

Property K

One also has that every forcing notion $\mathbb{P}$ considered has the property K: This follows from the fact that

Every uncountable sequence $(p_\alpha)_{\alpha<\omega_1}$ of conditions of $\mathbb{P}$ has an uncountable subsequence forming a Δ*-system of isomorphic conditions*.

Since the posets $\mathbb{P}$ we consider are always closed under a *minimal* amalgamation of finite such Δ-systems, it will follow that $\mathbb{P}$ has the property K.

To prove properties of the generic spaces we use two kind of arguments:

1. Dense arguments: This will give cofinal properties of the generic space, as for example the Gurarij property.
2. Study of uncountable Δ-system of isomorphic conditions. This will give a full understanding of *uncountable substructures* of the generic spaces.

Let $\mathbb{G}$ be a generic filter for a forcing notion $\mathbb{P}$.
We say that a configuration $\mathfrak{P}(\vec{v}_0, \dots, \vec{v}_k)$, $\vec{v}_i = (v_0^{(i)}, \dots, v_{n-1}^{(i)})$, is *unavoidable* for the sequences $(y_\alpha^{(0)})_{\alpha<\omega_1}, \dots, (y_\alpha^{(k)})_{\alpha<\omega_1}$ of points of $c_{00}(\omega_1, \mathbb{Q})$ if

for every uncountable $\Gamma \subseteq \omega_1$ there are $\xi_0 < \cdots < \xi_{n-1}$ in Γ such that $\mathfrak{P}((y_{\alpha_{\xi_i}}^{(0)})_{i<n}, \dots, (y_{\alpha_{\xi_i}}^{(k)})_{i<n})$ holds in $X_{\mathbb{G}}$., i.e. holds in $(\langle y_{\alpha_{\xi_i}}^{(j)} \rangle_{j\leq k, i<n}, \| \cdot \|_{\mathbb{G}})$.

A configuration $\mathfrak{P}(\vec{v}_0, \ldots, \vec{v}_k)$ is unavoidable in $X_{\mathbb{G}}$ if it is unavoidable for $k+1$ many uncountable sequences of points of $X_{\mathbb{G}}$ which are in $c_{00}(\omega_1, \mathbb{Q})$.
We say that the configuration $\mathfrak{P}(\vec{v}_0, \ldots, \vec{v}_k)$ is *unavoidable for* $\mathbb{P}$ if it is unavoidable in every generic space of $\mathbb{P}$.

Examples of configurations

Example 1

$$\|v_0 - \frac{1}{n}\sum_{i=1}^{n} v_i\| \le \frac{1}{n}.$$

Example 2

Given a $\mathbb{Q}$-f.d. space $X = (c_{00}(n), F)$ we consider the configuration

$$(v_i)_{i<n} \text{ is 1-equivalent to } (u_i)_{i<n} \text{ of } X.$$

The forcing Theorem.

Theorem

Let $\mathbb{P}$ be an arbitrary forcing notion, and let $\mathfrak{P}((v_j^{(0)})_{j<n},\dots,(v_j^{(k)})_{j<n})$ be a configuration. TFAE

(a) $\mathfrak{P}((v_j^{(0)})_{j<n},\dots,(v_j^{(k)})_{j<n})$ *is unavoidable for* $\mathbb{P}$.

(b) *Whenever* $(p_j,(v_j^{(i)})_{i\le k})_{j<n}$ *is a* Δ*-sequence of extended isomorphic conditions in* $\mathbb{P}$ *there is a condition* $p\in\mathbb{P}$ *such that*

  (b.0) $p\le p_j$ *for every* $j<n$ *and*

  (b.1) $\mathfrak{P}((v_j^{(0)})_{j<n},\dots,(v_j^{(k)})_{j<n})$ *holds in* X_p.

No uncountable biorthogonal

Let X be a Banach space, and κ be a cardinal. A sequence $(x_\alpha, f_\alpha)_{\alpha<\kappa} \in (X \times X^*)^\kappa$ is called a *biorthogonal system* if

$$f_\alpha(x_\beta) = \delta_{\alpha,\beta}.$$

Let $X_{\mathbb{G}}$ be a generic space for the basic forcing $\mathbb{P}_{\mathrm{b}}$. Then $X_{\mathbb{G}}$ does not have uncountable biorthogonal systems:

We consider the configuration $\mathfrak{P}_n(v_0,\dots,v_n)$:

$$\|v_0 - \frac{1}{n}\sum_{i=1}^{n} v_i\| \leq \frac{1}{n}.$$

It is clear that if $X_{\mathbb{G}}$ has an uncountable biorthogonal sequence then there is an uncountable sequence $(x_\alpha)_\alpha$ of points in $c_{00}(\omega_1)$ and $m \in \mathbb{N}$ such that

$$\|x_{\alpha_0} - \frac{1}{n}\sum_{i=1}^{n} x_{\alpha_i}\| > \frac{1}{m} \text{ for every } \alpha_0 < \cdots < \alpha_n.$$

This means that $\mathfrak{P}_m$ is **not** unavoidable.
By using the Forcing Theorem one proves that indeed each $\mathfrak{P}_n$ is unavoidable:
We fix n, and a Δ-system of isomorphic conditions $(p_i, v_i)_{i\le n}$ with root R. Let $t = (N, F, \sigma, H, v)$ be its *type*, and $\theta_i : N \to D_{p_i}$ be the corresponding isomorphisms.
Let $p = (D_p, F_p, \sigma_p, H_p)$ be the following condition:

1. $D_p = \bigcup_{i\le n} D_{p_i}$, $\sigma_p = \bigcup_{i\le n} \sigma_{p_i}$.
2. F_p is formed by:
   - $\bigvee_{i\le n} \theta_i(f)$ for every $f \in F$.
   - $h^{(p_i)}_{(m,x)}$ for every $(m, x) \in \sigma_{p_i}$ with $i \ge 1$ and $R < x$.

3. For every $1 \leq i \leq n$ and $s = (m, x) \in \sigma_i$ with $R < x$ one has that $h_s^{(p)} = h_s^{(p_i)}$.
4. For every $0 \leq i \leq n$ and $s = (m, x) \in \sigma_i$ with $\min \operatorname{supp} x \in R$ one has that $h_s^{(p)} = \bigvee_{i \leq n} \theta_i(h)$.
5. For every $s = (m, x) \in \sigma_0$ with $R < x$ one has that $h_s^{(p)} = \bigvee_{i \leq n} \theta_i(h)$, where h is the type of $h_{(m,x)}^{(p_0)}$.

Then p is a basic condition, $p \leq p_i$ for every $i \leq n$ and $\mathfrak{P}_n$ holds in X_p.

No supported sets

Recall that a sequence $(x_\alpha, f_\alpha)_{\alpha<\kappa} \in (X \times X^*)^\kappa$ is called semi-biorthogonal system if $f_\alpha(x_\alpha) = 1$, $f_\alpha(x_\beta) = 0$ and $f_\beta(x_\alpha) \geq 0$ for $\alpha < \beta$.
The generic space $X_{\mathbb{G}}$ for the basic forcing does not have supported sets, i.e. no uncountable semibiorthogonal sequences.
This is a consequence of the fact that for every n the configuration $\mathfrak{P}_n(v_0, \ldots, v_{3n})$

$$\| - \sum_{i=0}^{2n-1} v_i + n \cdot v_{2n} + \sum_{i=2n+1}^{3n} v_i \| \leq 2$$

is unavoidable.

The role of H_p

Both $C(K)$ for K being the Kunen space and the Shelah space S (which is our $X_{\mathbb{G}}$ for the basic forcing) are hereditarily Lindelöf in all powers, when considered with their weak topology.
By declaring the appropriate additional structure in H_p, we have examples of generic spaces $X_{\mathbb{G}}$ such that

1. $(X_{\mathbb{G}}, w)$ is hereditarily Lindelöf but $(X_{\mathbb{G}}, w)^2$ is not. This is the configuration $\mathfrak{P}(v_0, \dots, v_n)$:

For every $f_0, \dots, f_{m-1} \in B_{X^*}$ there is $1 \leq i \leq n$ with $\max_{j<m} |f_j(v_0) - f_j(v_i)| \leq \varepsilon \max_{k \leq n} \|v_k\|$.

2. $(X_{\mathbb{G}}, w)$ is not HL, i.e. it has an uncountable right separated family, but it has no uncountable convexly separated family, i.e. a sequence $(x_\alpha)_\alpha$ such that $x_\alpha \notin \overline{\mathrm{co}(x_\beta)_{\beta \neq \alpha}}$.

Other examples

1. A generic space with a K-basis but no uncountable K'-basic sequence for $K' < K$.
2. A generic space with uncountable 1^+-basic sequences but no uncountable monotone basic sequence.

More properties

All the generic spaces $X_{\mathbb{G}}$ we consider have the following properties:

1. Every operator from a subspace $X \subseteq X_{\mathbb{G}}$ into $X_{\mathbb{G}}$ is a multiple of the identity plus a separable range operator.
2. Every basic sequence can be 1^+-finitely block represented in any uncountable separated sequence of $X_{\mathbb{G}}$

Generic Compacta

The dual balls of our generic spaces are also interesting compacta:

1. $C(K)$ without supported sets and such that $(C(K), w)^n$ is Hereditarily Lindelöf.
2. Poulsen and Bauer simplexes.
3. ...

Question 1:

Is it possible to have these generic constructions with density $> \omega_1$ and not having uncountable biorthogonal systems?

Question 2:

Is it possible to have these generic constructions being *Asplund*?
In general, explore forcing notions with a restricted family of norms, e.g. asymptotic ℓ_1, with non-trivial co-type,...

Definition

Let X be a Banach space, and κ be a cardinal. A sequence $(x_\alpha, f_\alpha)_{\alpha<\kappa} \in (X \times X^*)^\kappa$ is called a *biorthogonal system* if $f_\alpha(x_\beta) = \delta_{\alpha,\beta}$.
The sequence $(x_\alpha, f_\alpha)_\alpha$ is called *fundamental* if $\overline{\langle x_\alpha \rangle_{\alpha<\kappa}} = X$.
The sequence $(x_\alpha, f_\alpha)_\alpha$ is called *total* if $\overline{\langle f_\alpha \rangle_\alpha}^{w^*} = X^*$.
A fundamental and total biorthogonal system is called a *Markushevich* basis of X (or simply M-basis).

Theorem (Markushevich)

Every separable Banach space has a M-basis.

Every Schauder basis is a Markushevich basis.

Theorem (Enflo)

It is not true that every separable Banach space has a Schauder basis.

Question 3

Is it true that if X has an uncountable Markushevich basis then X has an uncountable basic sequence?

Definition

A sequence $(x_\alpha)_{\alpha<\kappa}$ in X is called ω-independent if for every sequence increasing $(\alpha_n)_n$ and every sequence $(a_n)_n$ of scalars if $(\sum_{n=1}^m a_n x_n)_n$ converges in norm to 0, then $a_n = 0$ for every n.

Remark

If $(x_\alpha, f_\alpha)_\alpha$ is a biorthogonal sequence then $(x_\alpha)_\alpha$ is an ω-independent family.

Remark

A separable Banach space does not have uncountable independent families (Fremlin and Sersouri).

Question 4:

Is it true that if a Banach space have an uncountable ω-independent family, then it has an uncountable biorthogonal system?

Remark

If X has an uncountable ω-sequence then it has uncountable ε-biorthogonal sequences for every $\varepsilon > 0$.