Proper Translation

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Outline

Weak diamonds and Ostaszewski's club

- Weak Diamonds
- The club principle

Technique of proof

- Translating a forcing to a simpler one
- Computing generic conditions over guessed countable models in a coherent manner
- Playing with the variable argument of the Borel function giving a generic condition

Weak diamonds

Definition, Moore, Hrušák, Džamonja

Let $A, B \subseteq \mathbb{R}$ be Borel and let $E \subseteq A \times B$ be Borel in $\mathbb{R}^2$. $\diamondsuit(A, B, E)$ is the following principle:

$$(\forall \text{ Borel } F\colon 2^{<\omega_1} \to A)(\exists g_F\colon \omega_1 \to B)(\forall f\colon \omega_1 \to 2)$$
$$\{\alpha \in \omega_1 \,:\, F(f \restriction \alpha) E g_F(\alpha)\} \text{ is stationary.}$$

Begin proof
Guessing countable models
The end

Ostaszewski's guessing principle $\clubsuit$

Definition

$\clubsuit$ is the abbreviation of the following statement:

$$(\exists \langle A_\alpha : \alpha \in \omega_1, \lim(\alpha) \rangle) \quad (A_\alpha \text{ is cofinal in } \alpha \text{ and } \forall X \subseteq_{\mathrm{unc}} \omega_1 \{\alpha \in \omega_1 : A_\alpha \subseteq X\} \text{ is stationary}).$$

Theorem, Devlin

$\clubsuit + \mathrm{CH} \leftrightarrow \Diamond$.

Juhász' question

Question, Juhász

Does $\clubsuit$ imply the existence of a Souslin tree?

Stronger version of the question if heading for a negative answer

Is $\clubsuit$ together with "all Aronszajn trees are special" consistent relative to ZFC?

A version of Cichoń's diagramme

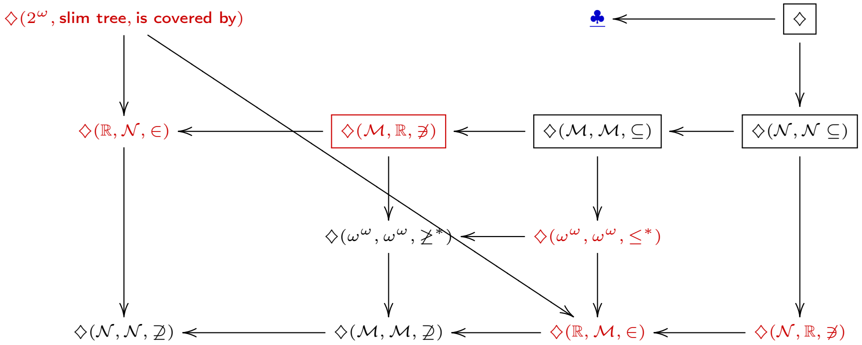

Figure: Just the framed weak diamonds imply the existence of a Souslin tree.

Large continuum and weak diamond and all Aronszajn trees special

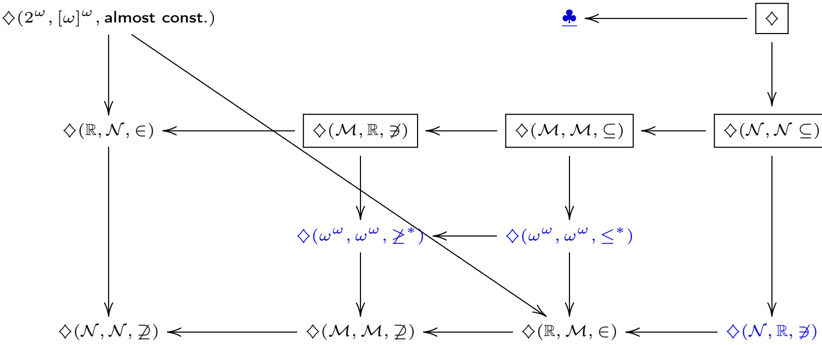

Figure: The blue weak diamonds allow $\mathfrak{c}$ large and all Aronszajn trees special.

Large continuum and weak diamond and all Aronszajn trees special II

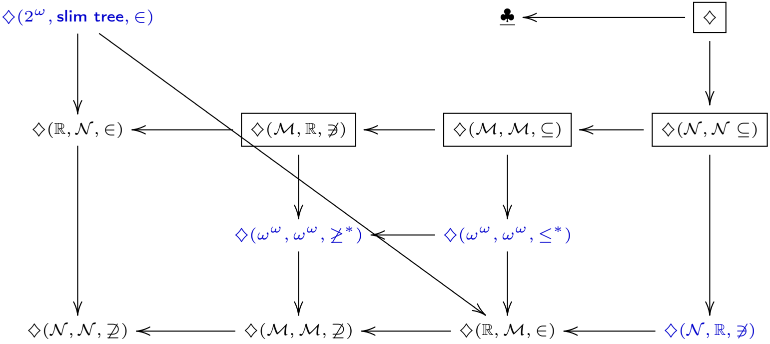

Figure: The blue weak diamonds allow $\mathfrak{c}$ large and all Aronszajn trees special.

A new theorem

Theorem

Let $r\colon \omega \to \omega$ *such that* $\lim \frac{r(n)}{2^n} = 0$. *Then the conjunction of the following weak diamonds together with* $2^\omega = \aleph_2$ *and with "all Aronszajn trees are special" is consistent relative to ZFC:*

- $\diamondsuit(2^\omega, \{\lim(T) : T \subseteq 2^\omega \textit{ perfect } \wedge (\forall n)|\{\eta \restriction n : \eta \in \lim(T)\}| \leq r(n)\}, \in)$,
- $\diamondsuit(\mathbb{R}, F_\sigma \textit{ null sets}, \in)$,
- $\diamondsuit(\mathbb{R}, G_\delta \textit{ meagre sets}, \in)$.

The forcing

Assume that the ground model fulfils $2^{\omega_1} = \omega_2$ and $\diamondsuit$.

We take a countable support iteration

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$$

of the following proper iterands:

$\mathbb{Q}_{2\alpha}$ specialises an Aronszajn tree without adding reals (a forcing of size $2^{\aleph_1}$ with uncountable conditions)

$\mathbb{Q}_{2\alpha+1}$ is just the Sacks forcing (for the weak diamond) or any ω^ω-bounding $< \omega_1$-proper forcing $\subseteq \omega^\omega$ such that being a condition and $\leq$ are Σ^1_1 (if we want only proper translation).

Towards the weak diamond in the extension

Since the evenly indexed iterands do not add reals and since the oddly indexed iterands are Σ_1-definable subsets of the reals, we could have that

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \,:\, \alpha \leq \omega_2, \beta < \omega_2 \rangle$$

is equivalent to a forcing in which $\mathbb{Q}_{2\alpha+1}$ has a

$$\mathbb{P}_{*,2\alpha+1} = \langle \underset{\sim}{\mathbb{Q}}_{2\beta+1} \,:\, \beta < \alpha \rangle\text{-name.}$$

Handling the large NNR iterands

We show that below $(M, \mathbb{P})$-generic conditions have names in the simpler iteration

$$\mathbb{P}_* = \langle \mathbb{P}_{*,2\alpha+1}, \underset{\sim}{\mathbb{Q}}_{2\beta+1} : \alpha \leq \omega_2, \beta < \omega_2 \rangle$$

and that the $(M, \mathbb{P})$-generic conditions force that conditions in $M \cap \mathbb{P}$ can be translated to $M \cap \mathbb{P}_*$.

We recall: Weak diamonds

Definition, Moore, Hrušák, Džamonja

Let $A, B \subseteq \mathbb{R}$ be Borel and let $E \subseteq A \times B$ be Borel in $\mathbb{R}^2$.
$\diamondsuit(A, B, E)$ is the following principle:

$$(\forall \text{ Borel } F\colon 2^{<\omega_1} \to A)(\exists g_F\colon \omega_1 \to B)(\forall f\colon \omega_1 \to 2)$$
$$\{\alpha \in \omega_1 \,:\, F(f \restriction \alpha) E g_F(\alpha)\} \text{ is stationary.}$$

Which branches of T have continuation on the level $\mu = M \cap \omega_1$?

We let η stand for functions from ω to ω.

We assume that every level of the Aronszajn tree is identified with ω. For $y \in T_\mu$ we set $h_{y,\bar{\beta}}(n)$ be the $x \in T_{\beta_n}$ such that $x <_T y$.

Computing bounded generic filters by Borel functions

Lemma

There is a Borel function $\mathbf{B}_1 \colon \omega^\omega \times \mathcal{P}(\omega) \to \mathcal{P}(\omega)$*, such that if* $p \in Q_{\mathbf{T}} \cap M$*,* $\mu = \mathrm{otp}(M \cap \omega_1) = \sup\langle \beta_n \,:\, n < \omega\rangle$*,* $\beta_{n+1} > \beta_n$*, and* $c\colon \omega \to M$ *is a bijection with* $c(0) = Q_{\mathbf{T}}$*,* $c(1) = p$*,* $c(2n+2) = \beta_n$*, and*

$$U = U(M, Q_{\mathbf{T}}, p) = \{2e(n_1, n_2) \,:\, c(n_1) \in c(n_2)\} \cup \{2e(n_1, n_2) + 1 \,:\, c(n_1) <^*_\chi c(n_2)\}$$

is a description of the isomorphism type then and if

$$(\forall y \in T_\mu)(h_{y,\bar\beta} \leq^* \eta),$$

Computing bounded generic filters II

Continuation of the Lemma

then for

$$G = \{c(n) : n \in \mathbf{B}_1(\eta, U)\}$$

the following holds: G is $(M, Q_{\mathbf{T}})$-generic and $p \in G$ and there is an upper bound r of G.

Version of the previous lemma for iterated forcing

Theorem

Let $\mathbb{P}_{\omega_2} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ *be a countable support iteration of iterands of the form* $Q_{\mathbf{T}}$*. Let* χ *be sufficiently large. There is a sequence of Borel functions* $\langle \mathbf{B}_\alpha : \alpha < \omega_1 \rangle$ *such that* $\mathbf{B}_\alpha \colon (\omega^\omega)^\alpha \times \mathcal{P}(\omega) \to \mathcal{P}(\omega)$*, such that the following conditions hold*

(a) $\mathbb{P} \in M$*,*

(b) $p \in \mathbb{P} \cap M$*,*

(c) $\alpha = \mathrm{otp}(M_0 \cap \omega_2)$*,*

(d) Let $\bar{\beta}$ *be cofinal in* $M \cap \omega_1$*. Let* $c \colon \omega \to M$ *be a bijection with* $c(0) = \mathbb{P}$*,* $c(1) = p$*,* $c(2n+2) = \beta_n$*, and set*

Continuation

$$U = U(M, \mathbb{P}, p) = \{2e(n_1, n_2) : c(n_1) \in c(n_2)\} \\ \cup \{2e(n_1, n_2) + 1 : c(n_1) <^*_\chi c(n_2)\}.$$

Then in the following games $\Game_{(\bar{M}, \mathbb{P}, p)}$ the generic player has a winning strategy σ, which depends only on the isomorphism type of $(\bar{M}, \in, <^*_\chi, \mathbb{P}, p, \bar{\beta})$:

Continuation

(α) a play lasts α moves,

(β) in the ε-th move the generic player chooses some real ν_ε and the antigeneric player chooses some $\eta_\varepsilon \in \omega^\omega$, such that $\eta_\varepsilon \geq^* \nu_\varepsilon$,

(γ) in the end the generic player wins iff the following is true:

$$G_\alpha = \{c(n) : n \in \mathbf{B}_\alpha(\langle \eta_\varepsilon : \varepsilon < \alpha \rangle, U)\}$$

is an $(M, \mathbb{P})$-generic filter and

$p \in G_\alpha$ and

there is a $\mathbb{P}_*$-name for G_α.

Version of the previous theorem for the iteration adding reals

Theorem

Let $\mathbb{P}_{\omega_2} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \,:\, \alpha \leq \omega_2, \beta < \omega_2 \rangle$ *be a countable support iteration of iterands of the form* $Q_{\mathbf{T}}$ *and of* ω^ω*-bounding* $< \omega_1$*-proper iterands that are* Σ^1_1*-subsets of* ω^ω*. Let* χ *is sufficiently large and regular.*

There is a coherent sequence $\langle \mathbf{B}_\alpha \,:\, \alpha < \omega_1 \rangle$ *of Borel functions* $\mathbf{B}_\alpha \colon (\omega^\omega)^\alpha \times \mathcal{P}(\omega) \to \mathcal{P}(\omega)$*, such that the following conditions hold:*

(a) $\bar{M} \prec (H(\chi), \in, <^*_\chi)$ *is a tower of countable elementary submodels,*

(b) $\mathbb{P} \in M_0$*,* $\gamma \leq \omega_2$*,*

(c) $p \in \mathbb{P} \cap M_0$*,*

Continuation

(d) $\alpha = \mathrm{otp}(M_0 \cap \omega_2)$,

(e) $\bar{\beta}$ is cofinal in $M_0 \cap \omega_1$. Let $c\colon \omega \to M_\alpha$ be a bijection with $c(0) = \mathbb{P}$, $c(1) = p$, $c(3n+2) = \beta_n$, $c(3n+1) = M\alpha_n$, $\alpha = \{\alpha_n : n \in \omega\}$ and set

$$U = U(M, P_\gamma, p) = \{2e(n_1, n_2) : c(n_1) \in c(n_2)\} \\ \cup \{2e(n_1, n_2) + 1 : c(n_1) <^*_\chi c(n_2)\}.$$

Then in the following game $\Game_{(\bar{M},\mathbb{P},p)}$ the generic player has a winning strategy σ, which depends only on the isomorphism type of $(\bar{M}, \in, <^*_\chi, P_\gamma, p, \bar{\beta})$:

Continuation

(α) a play lasts α moves,

(β) in the ε-th move the generic player chooses some real ν_ε and the antigeneric player chooses some $\eta_\varepsilon \in \omega^\omega$, such that $\eta_\varepsilon \geq^* \nu_\varepsilon$,

(γ) in the end the generic player wins iff the following is true: $p \leq q_\alpha = \{c(n) : n \in \mathbf{B}_\alpha(\langle \eta_\varepsilon : \varepsilon < \alpha \rangle, U)\}$ is a α-Sacks name for a $(M_0, \mathbb{P})$-generic condition.

Choosing a suitable argument $\bar{\eta}$

A lemma from the ancient paradise.

Lemma

Suppose that

(α) $\gamma < \omega_1$, and

(β) $\mathbf{B}'$ is a Borel function from $(\omega^\omega)^\gamma$ to 2^ω.

Then *we can find some $S = S_{\mathbf{B}'}$ such that*

(a) S is a small slalom,

(b) in the following game $\partial_{(\gamma,\mathbf{B}')}$ between two players, IN and OUT, the player IN has a winning strategy, the play lasts γ moves and in the ε-th move OUT chooses $\nu_\varepsilon \in \omega^\omega$ and then IN chooses $\eta_\varepsilon \geq^* \nu_\varepsilon$. In the end IN wins iff $\mathbf{B}'(\langle \eta_\varepsilon : \varepsilon < \gamma \rangle)$ is covered by S.

Choosing a suitable argument $\bar{\eta}$ when there are new reals

Lemma

Suppose that

(α) $\gamma < \omega_1$*, and*

(β) $\mathbf{B}'$ *is a Borel function from* $(\omega^\omega)^\gamma$ *to* γ*-Sacks names for elements of* 2^ω*.*

Then *we can find some* $S = S_{\mathbf{B}'}$ *such that*

(a) S *is a small slalom,*

(b) in the following game $\Game_{(\gamma,\mathbf{B}')}$ *between two players, IN and OUT, the player IN has a winning strategy, the play lasts* γ *moves and in the* ε*-th move OUT chooses* $\nu_\varepsilon \in \omega^\omega$ *and then IN chooses* $\eta_\varepsilon \geq^* \nu_\varepsilon$*. In the end IN wins iff* γ*-Sacks forcing forces that* $\mathbf{B}'(\langle \eta_\varepsilon : \varepsilon < \gamma \rangle)$ *is covered by* S*.*

Guessing the countable situation and choosing a suitable $\bar{\eta}$

Let G be P_{ω_2}-generic over $\mathbf{V}$. We use the $\diamondsuit_S$-sequence $\langle A_\delta : \delta \in S \rangle$ in the following manner:

We recall again: Weak diamonds

Definition, Moore, Hrušák, Džamonja

Let $A, B \subseteq \mathbb{R}$ be Borel and let $E \subseteq A \times B$ be Borel in $\mathbb{R}^2$.
$\diamondsuit(A, B, E)$ is the following principle:

$$(\forall \text{ Borel } F\colon 2^{<\omega_1} \to A)(\exists g_F\colon \omega_1 \to B)(\forall f\colon \omega_1 \to 2)$$
$$\{\alpha \in \omega_1 \,:\, F(f \restriction \alpha) E g_F(\alpha)\} \text{ is stationary.}$$

Guessing the countable situation and choosing a suitable $\bar{\eta}$

We have $\langle (N^\delta, \bar{\beta}^\delta, \underset{\sim}{f}^\delta, \underset{\sim}{F}^\delta, \underset{\sim}{C}^\delta, P^\delta_{\omega_2}, p^\delta, <^\delta) : \delta \in S \rangle$ such that

Countable components

(a) $\bar{N}^\delta$ is a transitive collapse of a tower $\bar{M} \prec H(\chi, \in, <^*_\chi)$, $<^\delta$ is a well-ordering of $\bigcup \bar{N}^\delta$, U^δ codes the isomorphism type of $(\bar{N}^\delta, P^\delta_{\omega_2}, p^\delta, \bar{\beta}^\delta)$.

(b) $N^\delta_0 \models P^\delta_{\omega_2} = \langle P^\delta_\alpha, \underset{\sim}{Q}^\delta_\beta : \alpha \leq \omega_2^{N^\delta}, \beta < \omega_2^{N^\delta} \rangle$ is our chosen forcing iteration,

(c) $N^\delta_0 \models (p^\delta \in P^\delta_{\omega_2}$, $\underset{\sim}{f}^\delta$ is a $P^\delta_{\omega_2}$-name of a member of ${}^{\omega_1}2$ $\underset{\sim}{F}^\delta : 2^{<\omega_1} \to 2^\omega)$.

Continuation of the list

(d) If $p \in P_{\omega_2}$,

$$p \Vdash_{P_{\omega_2}} \underset{\sim}{f} \in 2^{\omega_1} \wedge \underset{\sim}{F} \colon 2^{<\omega_1} \to 2^\omega \text{ is Borel}, \underset{\sim}{C} \subseteq \omega_1 \text{ is club},$$

and p, P_{ω_2}, $\underset{\sim}{F}$, $\underset{\sim}{f}$, $\underset{\sim}{C} \in H(\chi)$, then

$$\begin{aligned} S(p, \underset{\sim}{F}, \underset{\sim}{f}) := \{\delta \in S \ :& \text{there is a tower } \bar{M} \prec (H(\chi), \in, <^*_\chi) \\ & \text{such that } \underset{\sim}{f}, \underset{\sim}{F}, \underset{\sim}{C}, P_{\omega_2}, p \in M \text{ and} \\ & \text{there is an isomorphism } h^\delta \text{ from } \bar{N}^\delta \text{ onto } \bar{M} \\ & \text{mapping } P^\delta_{\omega_2} \text{ to } P_{\omega_2}, \underset{\sim}{f}^\delta \text{ to } \underset{\sim}{f}, \\ & \underset{\sim}{F}^\delta \text{ to } \underset{\sim}{F}, \underset{\sim}{C}^\delta \text{ to } \underset{\sim}{C}, p^\delta \text{ to } p, <^\delta \text{ to } <^*_\chi \restriction M_\delta\} \end{aligned}$$

is a stationary subset of ω_1.

Continuation of the list II

(e) Choose $\langle \mathbf{B}_{\gamma(\delta)} : \delta \in S \rangle$ such that $\gamma(\delta) = \mathrm{otp}(N_0^\delta \cap \omega_2)$ and

$$\mathbf{B}_{\gamma(\delta)} \colon (\omega^\omega)^{\gamma(\delta)} \times \mathcal{P}(\omega) \to$$
$$\gamma(\delta)\text{-Sacks names for } (N^\delta, \mathbb{P})\text{-generic conditions}$$

with $U^\delta = U(\bar{N}^\delta, P^\delta_{\omega_2}, p^\delta, \bar{\beta}^\delta)$.

Computing over guessed countable models

We assume that $N_0^\delta \cap \omega_1 = \delta$. Since this holds on a club set of $\delta \in \omega_1$, this is no restriction.

Now assume the $p \in G$ and $\underset{\sim}{F}$, $\underset{\sim}{f}$, $\underset{\sim}{C}$ are as in (d).
We define a function $\mathbf{B}'_{\delta,U_\delta}$ with domain $(\omega^\omega)^{\gamma(\delta)}$.

$$\mathbf{B}'_{\delta,U^\delta}(\langle \eta_\varepsilon : \varepsilon < \gamma(\delta)\rangle) = \begin{cases} \underset{\sim}{F}^\delta(\underset{\sim}{f}^\delta \restriction \delta)[\mathbf{B}_{\gamma(\delta)}(\langle \eta_\varepsilon : \varepsilon < \gamma(\delta)\rangle, U^\delta)], \\ \text{if the argument } \bar{\eta} \text{ is sufficiently large;} \\ \langle 0, 0, \dots, \rangle \in 2^\omega, \\ \text{otherwise.} \end{cases}$$

Applying the second game to the value of the Borel function at the guessed argument

$$\mathbf{B}'_{\delta,U_\delta}(\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle) \in S_{\mathbf{B}'_{\delta,U^\delta}}. \tag{3.1}$$

Note that $S_{\mathbf{B}'_{\delta,U^\delta}}$ does not depend on $\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$. So (3.1) also holds for $\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ that are the answers of player IN in the game $\Game_{(\gamma(\delta),\mathbf{B}'_{\delta,U^\delta})}$ to any winning sequence $\langle \nu^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ given by the generic player in the first game that is so fast growing ν^δ_ε that $\mathbf{B}_{\delta,U_\delta}(\langle \nu^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle)$ computes a Sacks name for a generic filter over M_0.

The next level in the tree above $N^\delta_\alpha \cap \omega_1$

This is important, since the isomorphism h^δ does not preserve the knowledge (that is which branches are continued and what are the values of the promises in these continuations) about the level $\omega_1 \cap M^\delta_{\gamma(\delta)}$ for the Aronszajn trees in $P \cap M^\delta_{\gamma(\delta)}$.

The diamond function giving a small slalom

We set

$$S_{\mathbf{B}'_{\delta,U^\delta}} =: g(\delta).$$

The order of the quantifiers in the weak diamonds

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$$(\forall \text{ Borel } F\colon 2^{<\omega_1} \to A)(\exists g_F\colon \omega_1 \to B)(\forall f\colon \omega_1 \to 2)$$
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