Some questions on the models of MM

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PROBLEM

MM and PFA appears to produce models of set theory in which every *"consistent"* set of size $\aleph_1$ *"exists"*.

How to formulate this in a suitable form?

For example in this way:

Theorem 1 (Veličković) *Assume* MM. *Let* W *be an inner model such that* $\omega_2^W = \omega_2$. *Then* $P(\omega_1) \subseteq W$.

Theorem 2 (Caicedo, Vel.) *Assume* $W \subseteq V$ *are models of* BPFA *such that* $\omega_2^W = \omega_2$. *Then* $P(\omega_1) \subseteq W$.

We would like to extend these results all over the cardinals:

Conjecture 1 (Caicedo, Veličković) *Assume $W \subseteq V$ are models of* MM *with the same cardinals. Then $[Ord]^{\leq\aleph_1} \subseteq W$.*

This is almost best possible, since:

- There exist $W \subseteq V$ models of MM with the same cardinals such that $[Ord]^{\aleph_2} \not\subseteq W$.

- Using stationary tower forcing it is possible to produce two models of MM, $W \subseteq V$ such that $[Ord]^{\leq\aleph_1} \not\subseteq W$. However the two models have different cardinals.

FIRST PROBLEM TO MATCH: FIXING THE COFINALITIES.

This is solved by the following result which expands over works of Cummings, Schimmerling, Todorčević, Dzamonja, Shelah.

Theorem 3 (V.) *Assume* MM. *Let* κ *be singular (and strong limit). Let* W *be an inner model such that* κ *is regular in* W *and* $\kappa^+ = (\kappa^+)^W$. *Then* $\mathrm{cf}(\kappa) > \omega_1$.

Corollary 4 *Let* V *be a model of* MM *(such that every limit cardinal is strong limit). Let* W *be an inner model with the same cardinals.*
If κ *is regular in* W, $\mathrm{cf}(\kappa) > \omega_1$.

Back to the conjecture, the best result I have to time is the following:

Corollary 5 *Assume* $W \subseteq V$ *are models of* ZFC *with the same cardinals and:*

- V *models* MM,
- *every limit cardinal is strong limit,*
- V *is a set-forcing extension of* W.

Then $[Ord]^{\leq\omega_1} \subseteq W$.

Sketch of proof: if $V = W[G]$ with G $\mathbb{P}$-generic filter for some set $\mathbb{P} \in W$ of size κ, new sets of ordinals appear already as subsets of κ.

The assumptions entail that W and V have the same ordinals of cofinality at most $\aleph_1$.

Now the κ^+-cc of P entails that W-stationary subsets of κ^+ remain stationary in V. Fix in W:

$$\mathcal{E} = \{S_\alpha : \alpha < \kappa^+\} \in W$$

partition of $E^\omega_{\kappa^+}$ in W-stationary sets.

In V this remains a partition in stationary sets of points of countable cofinality.

Now let $X \in [\kappa]^{\leq \aleph_1}$ in V,

Apply MM in V to find an ordinal δ of cofinality $\aleph_1$ such that:

S_α reflects on δ iff $\alpha \in X$.

Now δ has cofinality $\aleph_1$ also in W and $P(\omega_1) \subseteq W$.

This is enough to get that the above property holds also in W. Thus $X \in W$. $\square$

The first natural approach is to follow the same pattern of the proof of the previous theorem. In order to run the argument we need to find a way of generating indestructible partitions of stationary sets:

Definition 6 *Let λ be a regular cardinal and* Γ *a property.*

S is a Γ*-indestructibly stationary subset of λ if it remains stationary in any outer model where the property* Γ *holds.*

Let S be a stationary subset of λ.

IP(Γ, κ, S)*-holds if S carries a partition in κ-many disjoint* Γ*-indestuctibly stationary subsets.*

We shall be interested in the following properties:

- $\Gamma = Reg(\lambda)$: *λ is a regular cardinal*

- $\Gamma = scale(\mathcal{F}, \theta^+)$: for some increasing family $(\theta_i : i < \kappa)$ of regular cardinals,

  $$F = \{f_\alpha : \alpha < \theta^+\}$$

  is a scale on $\prod_{i<\kappa} \theta_i$ and $\theta = \sup_{i<\kappa} \theta_i$.

Problem 1 *Let κ be an arbitrarily large cardinal.*

Does $\mathsf{IP}(Reg(\lambda), \kappa, E^{\omega}_{\lambda})$ *holds for some* $\lambda \geq \kappa$*?*

Assume the answer is yes and let V be a model of MM and W be an inner model with the same cardinals.

We can use this property to show $[Ord]^{\leq \aleph_1} \subseteq W$ running the same proof sketched before.

We appeal to $\mathsf{IP}(Reg(\lambda), \kappa, E^{\omega}_{\lambda})$ to get a partition in W of E^{ω}_{λ} into κ-many stationary subsets of V.

We then argue by induction on κ, that for no κ new elements of $[\kappa]^{\leq \aleph_1}$ are added.

This leads us to partition relations:

Definition 7 *Let $\mathcal{F}$ be a filter on κ*

$$\lambda \rightarrow_{\mathcal{F}} [\kappa]^2_\lambda$$

holds if for every $f : [\lambda]^2 \rightarrow \kappa$, there is $H \subseteq \lambda$ of size λ such that $f[[H]^2] \notin \mathcal{F}$.

We are interested in the failure of this partition relation for the filter of cobounded subsets of κ.

Definition 8

$$\lambda \not\to^{\Gamma}_{\mathcal{F}} [\kappa]^2_\lambda$$

If there is $f : [\lambda]^2 \to \kappa$ which witness the failure of the partition relation in every outer model in which Γ holds.

To avoid too many subscripts we shall not mention $\mathcal{F}$ when $\mathcal{F}$ is the filter of cobounded subsets of κ.

This is a slight abuse of notation...

We are interested in this partition relation mainly for this observation:

Lemma 9 *Larson? Assume*

$$\lambda \not\to^{\Gamma} [\kappa]^2_{\lambda}.$$

Then $\mathsf{IP}(\Gamma, \kappa, S)$ *holds for any* Γ*-indestructibly stationary subset* S *of* λ*.*

Moreover our approach is not without hope since:

Theorem 10 *Todorčević*

$$\omega_1 \not\to^{Reg(\omega_1)} [\omega]^2_{\omega_1}.$$

Basic observations coming from pcf-theory give also:

Fact 1 *If θ is singular, then:*

$$\theta^+ \not\to^{scale(\mathcal{F},\theta^+)} [\mathrm{cf}(\theta)]^2_{\theta^+}.$$

As a corollary of the fact we get....

Corollary 11 *Assume $W \subseteq V$ are models of* ZFC *with the same cardinals and:*

- *V models* MM,
- *every limit cardinal is strong limit,*
- *There are arbitrarily large cardinals κ such that for some increasing sequence $(\theta_i : i < \kappa) \in W$ of regular cardinals larger than κ, there is $\mathcal{F} \in W$ scale on $\prod_{i<\kappa} \theta_i$ in V.*

Then $[Ord]^{\leq\omega_1} \subseteq W$.

Proof: For arbitrarily large κ we get that $scale(\mathcal{F}, \lambda)$ holds in V for some $\mathcal{F} \in W$ and for some λ successor of a singular cardinal of cofinality κ. This is enough to run the usual arguments. $\square$

Corollary 12 *Assume V models* MM *and W is an inner model with the same cardinals such that $[\kappa]^{\leq\aleph_1} \not\subseteq W$. Then any $\mathcal{F} \in W$ scale in W on $\prod_{i<\kappa} \theta_i \in W$ increasing sequence of regular cardinals has a new exact upper bound in V.*

On the other hand: for club many singular θ of cofinality at most $\aleph_1$ there are scales $\mathcal{F} \in W$ of type θ^+ which remain scales in V.

Fact 2 *(Silver?, Shelah?) Assume κ is regular and $\{\theta_i : i < \kappa\}$ is a club of singular cardinals larger than κ. Let*

$$\mathcal{F} = \{f_\alpha : \alpha < \theta^+\} \subseteq \prod_{i<\kappa} \theta_i^+$$

be a family of functions increasing modulo bounded.

Then there is D club subset of κ such that

$$\mathcal{F} \restriction D = \{f_\alpha \restriction D : \alpha < \theta^+\} \subseteq \prod_{i<\kappa} \theta_i^+$$

has exact upper bound $\prod_{i \in D} \theta_i^+$.

Thus:

If $\kappa \geq \aleph_1$ is regular and $(\theta_i : i < \kappa) \in W$ is a sequence of regular cardinals larger than κ, there is $\mathcal{F} \in W$ and $D \in V$ club subset of κ such that $\mathcal{F} \restriction D$ is a scale in V on $\prod_{i \in D} \theta_i$.

So if V and W witness the failure of the conjecture, on one hand the pcf-structure of W and V diverge completely, while on the other hand the two pcf-structures must still be very close to each other.

Other approaches to solve the conjecture

Fact 3 *Assume the conjecture fails for* $W \subseteq V$ *and* κ *is the least such that* $[\kappa]^{\leq \aleph_1} \not\subseteq W$*. Then for any finite set* $\{\lambda_i : i < n\}$ *of regular cardinals larger than* κ *there is:*

$$j : N \to H(\lambda_{n-1})^W$$

elementary and such that:

- $[N]^{\leq \aleph_1} \subseteq N$,
- $\omega_2 < crit(j) < \kappa$,
- $j(\kappa) = \kappa$ *and* $j(\lambda_i) = \lambda_i$ *for all* $i < n$,
- *for each* $i < n$ *the set of* $\delta < \lambda_i$ *such that* $j(\delta) = \delta$ *is closed under all sequences of length at most* $\aleph_1$.

One may try to argue that if W is a "nice" inner model, then it is the case that $N = H(\lambda_{n-1})^W$.

Ideas coming from inner model theory may then lead to a contradiction.