CCC without random reals

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A topic and goal of this talk

Notation. *For a forcing notion $\mathbb{P}$, let $a(\mathbb{P})$ be the forcing notion consists of finite antichains in $\mathbb{P}$,*

$$\sigma \leq_{a(\mathbb{P})} \tau :\Longleftrightarrow \sigma \supseteq \tau.$$

Theorem (Zapletal). *Let T be an Aronszajn tree, N a countable elementary submodel N of $H(\theta)$ which has the set $\{T\}$, $\sigma \in a(T) \cap N$, and $f \in \omega^\omega$. If f is not captured by any slalom in N, then there exists $\tau \leq_{a(T)} \sigma$ which is $(N, a(T))$-generic such that*

$$\tau \Vdash_{a(T)} \text{“ } f \textit{ is not captured by any slalom in } N \text{ ”}.$$

That is, $a(T)$ keeps add$(\mathcal{N})$ *small (by the countable support iterations).*

We argue that $a(T)$ doesn't add random reals, so it keeps cov$(\mathcal{N})$ small.

Known examples of ccc forcing notions not adding random reals

There are many kinds of non-ccc forcing notions not adding random reals. But it seems that we don't know ccc forcing notions not adding random reals so much.

The following forcing notions are such examples.

- σ-centered forcing notions
- Suslin algebras (ccc complete Boolean algebras not adding new reals)
- ccc forcing notions with the Sacks property
- ? Talagrand's counterexample of the Control Measure Problem ?

Note that for a Suslin tree T, $a(T)$ doesn't have the property K, and we will see an example of the form $a(\mathbb{P})$ which doesn't add random reals and not ω^ω-bounding.

Properties of Aronszajn trees

Proposition. *For an ω_1-tree T, T is Aronszajn iff*

$$\forall I \in [T]^{\aleph_1}$$
$$\exists s_0, s_1 \in T \text{ such that } s_0 \perp_T s_1 \text{ and}$$
$$\text{both } \{u \in I; s_0 \leq_T u\} \text{ and } \{u \in I; s_1 \leq_T u\} \text{ are uncountable.}$$

Proof. If T is not Aronszajn, i.e. there exists an uncountable branch I through T, then for any s_0 and s_1 in T with $s_0 \perp_T s_1$, at least one of the sets $\{u \in I; s_0 \leq_T u\}$ and $\{u \in I; s_1 \leq_T u\}$ have to be countable.

If there exists an uncountable branch I through T such that for any s_0 and s_1 in T with $s_0 \perp_T s_1$, at least one of the sets $\{u \in I; s_0 \leq_T u\}$ and $\{u \in I; s_1 \leq_T u\}$ is countable, then the set

$$\{t \in T; \{u \in I; t \leq_T u\} \text{ is uncountable}\}$$

forms an uncountable branch thorugh T, so T is not Aronszajn. $\square$

Properties of Aronszajn trees

Proposition. *For an ω_1-tree T, T is Aronszajn iff*

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$$\exists\, s_0, s_1 \in T \text{ such that } s_0 \perp_T s_1 \text{ and}$$
$$\text{both } \{u \in I; s_0 \leq_T u\} \text{ and } \{u \in I; s_1 \leq_T u\} \text{ are uncountable.}$$

Corollary. *For an Aronszajn tree T,*

$$\forall I \in [T]^{\aleph_1}\ \forall J \in [T]^{\aleph_1}$$
$$\exists I' \in [I]^{\aleph_1}\ \exists J' \in [J]^{\aleph_1} \text{ such that } \forall p \in I', \forall q \in J', p \perp_T q.$$

Proof. For I and J in $[T]^{\aleph_1}$, there are s_0, s_1, t_0 and t_1 in T such that $s_0 \perp_T s_1$, $t_0 \perp_T t_1$ and for each $i \in \{0, 1\}$, both $\{u \in I; s_i \leq_T u\}$ and $\{u \in J; t_i \leq_T u\}$ are uncountable.

Then there are $i \in \{0, 1\}$ and $j \in \{0, 1\}$ such that $s_i \perp_T t_j$, and then let

$$I' := \{u \in I; s_i \leq_T u\} \text{ and } J' := \{u \in J; t_i \leq_T u\}.$$

□

Properties of Aronszajn trees

Definition (Y.). *A forcing notion $\mathbb{P}$ has the anti-rectangle refining property (arec) if $\mathbb{P}$ is uncountable and*

$$\forall I \in [\mathbb{P}]^{\aleph_1}\ \forall J \in [\mathbb{P}]^{\aleph_1}$$
$$\exists I' \in [I]^{\aleph_1}\ \exists J' \in [J]^{\aleph_1} \ \text{ such that } \forall p \in I',\ \forall q \in J',\ p \perp_{\mathbb{P}} q.$$

Proposition. *If $\mathbb{P}$ has the arec, then*

$$\forall I \in [a(\mathbb{P})]^{\aleph_1}\ \forall J \in [a(\mathbb{P})]^{\aleph_1}, \text{ if } I \cup J \text{ forms a } \Delta\text{-system},$$
$$\text{then } \exists I' \in [I]^{\aleph_1}\ \exists J' \in [J]^{\aleph_1} \ \text{ such that } \forall \sigma \in I'\ \forall \tau \in J',\ \sigma \not\perp_{a(\mathbb{P})} \tau\ .$$

Proof. Let I and J in $[a(\mathbb{P})]^{\aleph_1}$ be such that $I \cup J$ forms a Δ-system with root ν. By shrinking I and J if necessary, we may assume that there are $m, n \in \omega$ such that for every $\sigma \in I$ and $\tau \in J$, $|\sigma \setminus \nu| = m$ and $|\tau \setminus \nu| = n$.

Using the arec $m \cdot n$ many times, we can find $I' \in [I]^{\aleph_1}$ and $J' \in [J]^{\aleph_1}$ such that for every $\sigma \in I'$, $\tau \in J'$, $i \in m$ and $j \in n$,

$$(i\text{-th member of } \sigma \setminus \nu) \perp_{\mathbb{P}} (j\text{-th member of } \tau \setminus \nu).$$

$\square$

Properties of Aronszajn trees

Proposition. *For an ω_1-tree T, T is Aronszajn iff*

$$\forall I \in [T]^{\aleph_1}$$
$$\exists s_0, s_1 \in T \text{ such that } s_0 \perp_T s_1 \text{ and}$$
$$\text{both } \{u \in I; s_0 \leq_T u\} \text{ and } \{u \in I; s_1 \leq_T u\} \text{ are uncountable.}$$

Corollary. *For an Aronszajn tree T,*

$$\forall \text{ countable } N \prec H(\aleph_2) \text{ with } T \in N \quad \forall I \in [T]^{\aleph_1} \cap N \quad \forall p \in T \setminus N$$
$$\exists I' \in [I]^{\aleph_1} \cap N \text{ such that } \forall q \in I',\ p \perp_T q.$$

Definition (Y.). *A forcing notion $\mathbb{P}$ has the anti-$\mathrm{R}_{1,\aleph_1}$ (the anti-R) if $\mathbb{P}$ is uncountable and*

$$\forall \text{ countable } N \prec H(\aleph_2) \text{ with } \mathbb{P} \in N \quad \forall p \in \mathbb{P} \setminus N \quad \forall I \in [\mathbb{P}]^{\aleph_1} \cap N$$
$$\exists I' \in [I]^{\aleph_1} \cap N \text{ such that } \forall q \in I',\ p \perp_{\mathbb{P}} q.$$

Proposition. *If $\mathbb{P}$ has the anti-R, then*

$$\forall \text{ countable } N \prec H(\aleph_2) \text{ with } \mathbb{P} \in N\ \forall \sigma \in a(\mathbb{P}) \setminus N$$
$$\forall I \in [a(\mathbb{P})]^{\aleph_1} \cap N \text{ which forms a } \Delta\text{-system with root } \sigma \cap N$$
$$\exists I' \in [I]^{\aleph_1} \cap N \text{ such that } \forall \tau \in I', \sigma \not\perp_{a(\mathbb{P})} \tau.$$

Motivations of two properties

Definition (Larson–Todorčević). *A partition* $K_0 \cup K_1$ *on* $[\omega_1]^2$ *has the rectangle refining property if*

$$\begin{aligned}&\forall I \in [\omega_1]^{\aleph_1}\ \forall J \in [\omega_1]^{\aleph_1}\\ &\exists I' \in [I]^{\aleph_1}\ \exists J' \in [J]^{\aleph_1}\ \text{such that}\ \forall \alpha \in I'\ \forall \beta \in J'\ \text{if}\ \alpha < \beta,\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{then}\ \{\alpha, \beta\} \in K_0.\end{aligned}$$

Theorem (Y.). *TFAE:*

- *Every partition* $K_0 \cup K_1$ *on* $[\omega_1]^2$ *with the rectangle refining property has an uncountable* K_0*-homogeneous subset of* ω_1*.*
- *For every forcing notion* $\mathbb{P}$ *with the arec,* $a(\mathbb{P})$ *has the property K.*

This partially answers a question of Todorčević's fragments of $\mathrm{MA}_{\aleph_1}$:
If every ccc partition on $[\omega_1]^2$ has an uncountable homogeneous sets, then every ccc forcing notion has the property K?

Motivations of two properties

Theorem (Y.). *It is consistent that there exists a non-special Aronszajn tree and for every $\mathbb{P}$ with the anti-*R*, $a(\mathbb{P})$ has precaliber $\aleph_1$, i.e.*

$$\forall I \in [a(\mathbb{P})]^{\aleph_1}$$
$\exists\, I' \in [I]^{\aleph_1}$ *with the finite compatibility property, i.e.*

any finite subsets of I' has a common extension.

This partially answers a question of Todorčević's fragments of $\mathrm{MA}_{\aleph_1}$:

$$\begin{tikzcd}
\mathrm{MA}_{\aleph_1} \arrow[r] & \cdots \arrow[r] & \mathcal{K}_{n+1} \arrow[r] & \mathcal{K}_n \arrow[r] \arrow[lll, bend right, "?"'] & \cdots \arrow[r] & \mathcal{K}_2 \arrow[lllll, bend right, "?"']
\end{tikzcd}$$

where a forcing notion $\mathbb{Q}$ has the property K_n if

$$\forall\, I \in [\mathbb{Q}]^{\aleph_1}$$
$\exists I' \in [I]^{\aleph_1}$ n-linked i.e.

any subset of I' of size n has a common extension in $\mathbb{Q}$,

and $\mathcal{K}_n$ says that every ccc forcing notion has the property K_n.

Motivations of two properties

Theorem (Y.). *It is consistent that there exists a non-special Aronszajn tree and for every* $\mathbb{P}$ *with the anti-*R, $a(\mathbb{P})$ *has precaliber* $\aleph_1$*, i.e.*

$$\forall I \in [a(\mathbb{P})]^{\aleph_1}$$
$$\exists\, I' \in [I]^{\aleph_1} \quad \textit{with the finite compatibility property, i.e.}$$
$$\textit{any finite subsets of } I' \textit{ has a common extension.}$$

Theorem (Todorčević–Veličković). $\mathrm{MA}_{\aleph_1}$ *is equivalent to the statement that every ccc forcing notion has precaliber* $\aleph_1$*.*

Examples: (ω_1, ω_1)-gaps

Definition. *An (ω_1, ω_1)-pregap is a sequence $\langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$ of infinite sets of natural numbers such that*

- $\forall \alpha < \beta$, $a_\alpha \subseteq^* a_\beta$ *and* $b_\alpha \subseteq^* b_\beta$, *and both* $a_\alpha \cap b_\beta$ *and* $a_\beta \cap b_\alpha$ *are finite,*
- *for every* $\alpha \in \omega_1$, $a_\alpha \cap b_\alpha = \emptyset$,
- *it is closed under finite modifications, that is,*

  $\forall \alpha \in \omega_1 \ \forall \langle c, d \rangle$, *if* $c \setminus n = a_\alpha \setminus n$ *and* $d \setminus n = b_\alpha \setminus n$ *for some* $n \in \omega$,
  then $\exists \beta$ *such that* $\langle c, d \rangle = \left\langle a_\beta, b_\beta \right\rangle$,

and an (ω_1, ω_1)-pregap is called a gap if there are no $c \subseteq \omega$ such that

$$\forall \alpha \in \omega_1, a_\alpha \subseteq^* c \text{ and } b_\alpha \cap c \text{ finite.}$$

Definition. *For an (ω_1, ω_1)-pregap $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$, the forcing notion $\mathcal{S}(\mathcal{S}, \mathcal{B}) := \left(\omega_1, \leq_{\mathcal{S}(\mathcal{A},\mathcal{B})}\right)$ is defined such that*

$$\alpha \leq_{\mathcal{S}(\mathcal{A},\mathcal{B})} \beta :\iff a_\beta \subseteq a_\alpha \text{ and } b_\beta \subseteq b_\alpha.$$

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$$\alpha \leq_{\mathcal{S}(\mathcal{A},\mathcal{B})} \beta :\Longleftrightarrow a_\beta \subseteq a_\alpha \text{ and } b_\beta \subseteq b_\alpha.$$

Proposition (Y.). *For an (ω_1, ω_1)-pregap $(\mathcal{A}, \mathcal{B})$, $(\mathcal{A}, \mathcal{B})$ is a gap iff $\mathcal{S}(\mathcal{A}, \mathcal{B})$ has the arec iff $\mathcal{S}(\mathcal{A}, \mathcal{B})$ has the anti-*R.

We note that $a(\mathcal{S}(\mathcal{A}, \mathcal{B}))$ is a forcing notion adds an uncountable subset I of ω_1 such that for every α and β in I with $\alpha \neq \beta$,

$$\left(a_\alpha \cap b_\beta\right) \cup \left(a_\beta \cap b_\alpha\right) \neq \emptyset,$$

i.e. $a(\mathcal{S}(\mathcal{A}, \mathcal{B}))$ forces $(\mathcal{A}, \mathcal{B})$ to be indestructible.

Example: Unbounded families

Theorem (Todorčević). *For an $<^*$-increasing sequence $F = \langle f_\alpha ; \alpha \in \omega_1 \rangle$ of members of $\omega^{\uparrow\omega}$, if F is unbounded, then the following partition $K_0 \cup K_1$ on $[\omega_1]^2$ is ccc*

$$\{\alpha, \beta\} \in K_0 :\iff \alpha < \beta \text{ and } \exists n \in \omega \text{ such that } f_\alpha(n) > f_\beta(n)$$

Therefore $\mathcal{K}_2$ (for partitions) implies $\mathfrak{b} > \aleph_1$.

Theorem (Y.). *For an $<^*$-increasing sequence $F = \langle f_\alpha ; \alpha \in \omega_1 \rangle$ of members of $\omega^{\uparrow\omega}$, define the forcing notion (ordered by superset)*

$$\mathbb{P}(F) := \Big\{ \sigma \in [\omega_1]^{<\aleph_0} ; \forall\, \alpha \in \sigma\ \forall\, n \in \omega$$
$$\max \big\{ f_\xi(n) ; \xi \in \sigma \cap \alpha \big\} < f_\alpha(n) \text{ or } f_\alpha(n) \in \big\{ f_\xi(n) ; \xi \in \sigma \cap \alpha \big\} \Big\}.$$

*Then F is unbounded, then $\mathbb{P}(F)$ has the arec and the anti-*R *and ccc.*
Therefore, e.g. $\mathcal{K}_2$(rec) *implies $\mathfrak{b} > \aleph_1$.*

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*Then F is unbounded, then $\mathbb{P}(F)$ has the arec and the anti-*R *and ccc. Therefore, e.g. $\mathcal{K}_2$*(rec) *implies $\mathfrak{b} > \aleph_1$.*

Question. *What is any other example of forcing notions with the arec or the anti-*R*? And are the arec and the anti-*R *different?*

Theorems

Theorem (Y.). *Let $\mathbb{P}$ be a forcing notion with the arec or the anti-*R*, N a countable elementary submodel N of $H(\theta)$ which has the set $\{\mathbb{P}\}$, $\sigma \in a(\mathbb{P}) \cap N$, and $f \in \omega^\omega$.*
If f is not captured by any slalom in N, then there exists $\tau \leq_{a(\mathbb{P})} \sigma$ which is $(N, a(\mathbb{P}))$-generic such that

$$\tau \Vdash_{a(\mathbb{P})} \text{“ } f \text{ is not captured by any slalom in } N \text{ ”.}$$

That is, $a(\mathbb{P})$ keeps $\mathrm{add}(\mathcal{N})$ *small (by the countable support iterations).*

Theorem (Y.). *Let $\mathbb{P}$ be a forcing notion with the arec or the anti-*R*.*
Then $a(\mathbb{P})$ doesn't add random reals.

A proof that $a(\mathbb{P})$ adds no random reals

Let $\mathbb{P}$ be a forcing notion $\langle \omega_1, \leq_{\mathbb{P}} \rangle$ with the arec or the anti-R, $\dot{r}$ be an $a(\mathbb{P})$-name for a real in 2^ω, and $\sigma \in a(\mathbb{P})$.

Let N be a countable elementary submodel of $H(\aleph_2)$ with $\{\mathbb{P}, \dot{r}, \sigma, \omega_1\} \in N$, and $\langle U_n ; n \in \omega \rangle$ a sequence of open subsets of 2^ω such that for each $n \in \omega$, the Lebesgue measure of U_n is less than 2^{-n} and

$$2^\omega \cap N \subseteq \bigcap_{n \in \omega} \bigcup_{m \geq n} U_m.$$

We show that

$$\sigma \not\Vdash_{a(\mathbb{P})} \text{“ } \dot{r} \notin \bigcap_{n \in \omega} \bigcup_{m \geq n} U_m \text{ ”}.$$

A proof that $a(\mathbb{P})$ adds no random reals

$\mathbb{P} = \langle \omega_1, \leq_{\mathbb{P}} \rangle$, $\dot{r}:$ $a(\mathbb{P})$-name, $\sigma \in a(\mathbb{P})$, $\{\mathbb{P}, \dot{r}, \sigma, \omega_1\} \in N$, $2^\omega \cap N \subseteq \bigcap_{n\in\omega} \bigcup_{m\geq n} U_m$.

A proof that $a(\mathbb{P})$ adds no random reals

$\mathbb{P} = \langle \omega_1, \leq_{\mathbb{P}} \rangle$, $\dot{r}$: $a(\mathbb{P})$-name, $\sigma \in a(\mathbb{P})$, $\{\mathbb{P}, \dot{r}, \sigma, \omega_1\} \in N$, $2^\omega \cap N \subseteq \bigcap_{n\in\omega} \bigcup_{m\geq n} U_m$.

Suppose that

$$\sigma \Vdash_{a(\mathbb{P})} \text{“ } \dot{r} \notin \bigcap_{n\in\omega} \bigcup_{m\geq n} U_m \text{ ”},$$

and take $\tau \leq_{a(\mathbb{P})} \sigma$ and $n \in \omega$ such that

$$\tau \Vdash_{a(\mathbb{P})} \text{“ } \forall m \geq n \, (\dot{r} \notin U_m) \text{ ”}.$$

Then, $n \in N$ and $\tau \cap N \in N$, and maybe $\tau \notin N$. So by strengthning τ if necessary, we may assume that $\tau \notin N$.

Let for each $k \in N$,

$$S_k := \Big\{ s \in 2^k ; \exists \alpha \in \omega_1 \text{ such that } \forall \mu \in a(\mathbb{P}) \text{ with } \mu \supseteq \tau \cap N, \\ \text{if } \mu \Vdash_{a(\mathbb{P})} \text{“ } \dot{r} \restriction k \neq s \text{ ”, then } \min(\mu \setminus (\tau \cap N)) \leq \alpha \Big\}.$$

Note that $\langle S_k ; k \in \omega \rangle \in N$.

A proof that $a(\mathbb{P})$ adds no random reals

$\mathbb{P} = \langle \omega_1, \leq_{\mathbb{P}} \rangle$, $\dot{r}$: $a(\mathbb{P})$-name, $\sigma \in a(\mathbb{P})$, $\{\mathbb{P}, \dot{r}, \sigma, \omega_1\} \in N$, $2^\omega \cap N \subseteq \bigcap_{n\in\omega} \bigcup_{m \geq n} U_m$.
$\tau \leq_{a(\mathbb{P})} \sigma$, $\tau \notin N$, $\tau \Vdash_{a(\mathbb{P})}$ " $\forall m \geq n \, (\dot{r} \notin U_m)$ ".

$$S_k := \Big\{ s \in 2^k ; \exists \alpha \in \omega_1 \text{ such that } \forall \mu \in a(\mathbb{P}) \text{ with } \mu \supseteq \tau \cap N, \\ \text{if } \mu \Vdash_{a(\mathbb{P})} \text{" } \dot{r} \restriction k \neq s \text{ ", then } \min(\mu \setminus (\tau \cap N)) \leq \alpha \Big\}.$$

A proof that $a(\mathbb{P})$ adds no random reals

$\mathbb{P} = \langle \omega_1, \leq_{\mathbb{P}} \rangle$, $\dot{r}$: $a(\mathbb{P})$-name, $\sigma \in a(\mathbb{P})$, $\{\mathbb{P}, \dot{r}, \sigma, \omega_1\} \in N$, $2^\omega \cap N \subseteq \bigcap_{n\in\omega} \bigcup_{m\geq n} U_m$.
$\tau \leq_{a(\mathbb{P})} \sigma$, $\tau \notin N$, $\tau \Vdash_{a(\mathbb{P})}$ " $\forall m \geq n\, (\dot{r} \notin U_m)$ ".
$S_k := \Big\{ s \in 2^k ; \exists \alpha \in \omega_1$ such that $\forall \mu \in a(\mathbb{P})$ with $\mu \supseteq \tau \cap N$,
if $\mu \Vdash_{a(\mathbb{P})}$ " $\dot{r}{\restriction}k \neq s$ ", then $\min(\mu \setminus (\tau \cap N)) \leq \alpha \Big\}$.

Claim. *For every $k \in \omega$, S_k is not empty.*

Proof of Claim. If S_k is empty, i.e.

$\forall s \in 2^k\, \forall \alpha \exists \mu \in a(\mathbb{P}) \Big(\mu \supseteq \tau \cap N \;\&\; \mu \Vdash_{a(\mathbb{P})}$ " $\dot{r}{\restriction}k \neq s$ " $\&\; \min(\mu \setminus (\tau \cap N)) > \alpha \Big)$,

then construct uncountable subsets $\left\langle I_s ; s \in 2^k \right\rangle$ of $a(\mathbb{P})$ in N such that

- the set $\bigcup_{s\in 2^k} I_s$ forms a Δ-system with root $\tau \cap N$, and
- for any $s \in 2^k$ and $\mu \in I_s$, $\mu \Vdash_{a(\mathbb{P})}$ " $\dot{r}{\restriction}k \neq s$ ".

By the property of $\mathbb{P}$, we can find $\left\langle \mu_s ; s \in 2^k \right\rangle \in \prod_{s\in 2^k} I_s$ such that $\bigcup_{s\in 2^k} \mu_s \in a(\mathbb{P})$. And then

$$\bigcup_{s\in 2^k} \mu_s \Vdash_{a(\mathbb{P})} \text{“ } \dot{r}{\restriction}k \notin 2^k \text{ ”},$$

which is a contradiction. $\dashv$

Remember that

Proposition. *If* $\mathbb{P}$ *has the arec, then*

$\forall I \in [a(\mathbb{P})]^{\aleph_1}\ \forall J \in [a(\mathbb{P})]^{\aleph_1}$, *if* $I \cup J$ *forms a* Δ*-system,*
then $\exists I' \in [I]^{\aleph_1}\ \exists J' \in [J]^{\aleph_1}$ *such that* $\forall \sigma \in I'\ \forall \tau \in J'$, $\sigma \not\perp_{a(\mathbb{P})} \tau$.

Proposition. *If* $\mathbb{P}$ *has the anti-*R*, then*

$\forall$ *countable* $N \prec H(\aleph_2)$ *with* $\mathbb{P} \in N\ \forall \sigma \in a(\mathbb{P}) \setminus N$
$\forall I \in [a(\mathbb{P})]^{\aleph_1} \cap N$ *which forms a* Δ*-system with root* $\sigma \cap N$
$\exists I' \in [I]^{\aleph_1} \cap N$ *such that* $\forall \tau \in I', \sigma \not\perp_{a(\mathbb{P})} \tau$.

A proof that $a(\mathbb{P})$ adds no random reals

$\mathbb{P} = \langle \omega_1, \leq_{\mathbb{P}} \rangle$, $\dot{r}$: $a(\mathbb{P})$-name, $\sigma \in a(\mathbb{P})$, $\{\mathbb{P}, \dot{r}, \sigma, \omega_1\} \in N$, $2^\omega \cap N \subseteq \bigcap_{n\in\omega} \bigcup_{m\geq n} U_m$.
$\tau \leq_{a(\mathbb{P})} \sigma$, $\tau \notin N$, $\tau \Vdash_{a(\mathbb{P})}$ " $\forall m \geq n \, (\dot{r} \notin U_m)$ ".
$S_k := \Big\{ s \in 2^k ; \exists \alpha \in \omega_1$ such that $\forall \mu \in a(\mathbb{P})$ with $\mu \supseteq \tau \cap N$,
if $\mu \Vdash_{a(\mathbb{P})}$ " $\dot{r} \restriction k \neq s$ ", then $\min(\mu \setminus (\tau \cap N)) \leq \alpha \Big\}$.

So $\bigcup_{k\in\omega} S_k$ forms an infinite subtree of 2^ω in N.

Take $u \in 2^\omega \cap N$ such that for every $k \in \omega$, $u \restriction k \in S_k$, and
let $m \geq n$ and $k \geq m$ such that $[u \restriction k] := \{v \in 2^\omega ; u \restriction k \subseteq v\} \subseteq U_m$.

Then there exists $\alpha \in \omega_1 \cap N$ such that for every $\mu \in a(\mathbb{P})$ with $\mu \supseteq \tau \cap N$,
if $\mu \Vdash_{a(\mathbb{P})}$ " $\dot{r} \restriction k \neq u \restriction k$ ", then $\min(\mu \setminus (\tau \cap N)) \leq \alpha$.

Since $\min(\tau \setminus (\tau \cap N)) \geq \omega_1 \cap N > \alpha$, $\tau \nVdash_{a(\mathbb{P})}$ " $\dot{r} \restriction k \neq u \restriction k$ ".
Thus there is $\nu \leq_{a(\mathbb{P})} \tau$ such that $\nu \Vdash_{a(\mathbb{P})}$ " $\dot{r} \restriction k = u \restriction k$ ".
Then since $\nu \Vdash_{a(\mathbb{P})}$ " $[\dot{r} \restriction k] = [u \restriction k] \subseteq U_m$ ", it follows that $\nu \Vdash_{a(\mathbb{P})}$ " $\dot{r} \in U_m$ ",
which is a contradiction. $\square$