Outline

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1 The virtues of first order categoricity in power

The virtues of first order categoricity in power

Detlefsen asked: **Detlefsen** 

Question A

Which view is the more plausible—that theories are the better the more nearly they are categorical, or that theories are the better the more they give rise to significant non-isomorphic interpretations?

Detlefsen also asked

Question B
Is there a single answer to the preceding question? Or is it rather the case that categoricity is a *virtue* in some theories but not in others? If so, how do we tell these apart, and how do we justify the claim that categoricity is or would be a virtue just in just former?

return

What is virtue?

What is virtue?
I take ‘ a virtuous property’ to be one which has significant mathematical consequences for a theory or its models.

Thus, a better property of theories has more mathematical consequences for the theory.

Is categoricity virtuous?

I will argue *categoricity* of a second order theory does not, by itself, shed any mathematical light on the categorical structure.

But *categoricity in uncountable power* for first order and infinitary logic yields significant structural information about models of theory.

This kind of structural analysis leads to a fruitful classification theory for complete first order theories. Indeed, fewer models usually indicates a better structure theory for models of the theory.

Choice of Logic matters

No *first order theory* is categorical.

There are important categorical second order *axiomatizations*.

Second Order Categoricity does not have structural consequences

Marek-Magidor/Ajtai (V=L) The second order theory of a countable structure is categorical.

H. Friedman (V=L) The second order theory of a Borel structure is categorical.

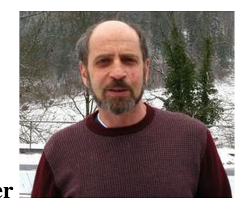
Solovay (V=L) A recursively axiomatizable complete 2nd order theory is categorical.

Solovay/Ajtai It is consistent with ZFC that there is a complete finitely axiomatizable second order theory that is not categorical.

Ali Enayat has nicely orchestrated this discussion on FOM and Mathoverflow. `http://mathoverflow.net/questions/72635/categoricity-in-second-order-logic`

Our Argument: First Order Logic

1. Categoricity in power implies strong structural properties of each categorical structure.
2. These structural properties can be generalized to all models of certain (syntactically described) complete first order theories.

$\aleph_1$-categorical theories **Morley**  **Lachlan**  **Zilber** 

Strongly minimal set are the building blocks of structures whose *first order* theories are categorical in uncountable power.

$\aleph_1$-categorical theories

Theorem (Morley/ Baldwin-Lachlan/Zilber) TFAE

1. T is categorical in one uncountable cardinal.
2. T is categorical in all uncountable cardinals.
3. T is ω-stable and has no two cardinal models.
4. Each model of T is prime over a strongly minimal set.
5. Each model of T can be decomposed by finite 'ladders'. Classical groups are first order definable in non-trivial categorical theories.

Item 3) implies categoricity in power is absolute.

Any theory satisfying these properties has either one or $\aleph_0$ models of cardinality $\aleph_0$.

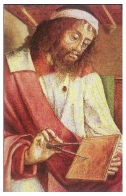
2 Axiomatization vrs Formalization

Section II Axiomatization vrs Formalization

Bourbaki on Axiomatization: **Dieudonne**  **Bourbaki**  **Cartan** 

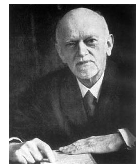
Bourbaki wrote:

> Many of the latter (mathematicians) have been unwilling for a long time to see in axiomatics anything other else than a *futile logical hairsplitting not capable of fructifying any theory whatever*.

Euclid-Hilbert formalization 1900: **Euclid**  **Hilbert** 

The Euclid-Hilbert (the Hilbert of the Grundlagen) framework has the notions of axioms, definitions, proofs and, with Hilbert, models.

But the arguments and statements take place in natural language.
For Euclid-Hilbert logic is a means of proof.

Hilbert-Gödel-Tarski-Vaught formalization 1918-1956: **Hilbert**  **Gödel**  **Tarski**

 **Vaught** 

In the Hilbert (the founder of proof theory)-Gödel-Tarski framework, logic is a mathematical subject.

There are explicit rules for defining a formal language and proof.
Semantics is defined set-theoretically.

First order logic is complete. The theory of the real numbers is complete and easily axiomatized. The first order Peano axioms are not complete.

Formalism freeness

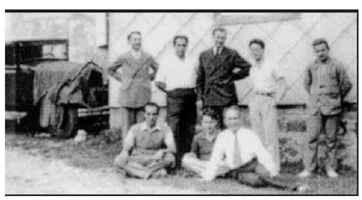
This paper is a counterpoint to discussions of trends away from fully formalized theories in model theory.

Baldwin,J.T., *Formalization, Primitive Concepts, and Purity*, 2013, Review of Symbolic Logic.

Juliette Kennedy, *On formalism freeness*, 2013 , to appear Bulletin of Symbolic Logic

The current paper is on-line:
`http://homepages.math.uic.edu/~jbaldwin/pub/catcomsub.pdf` longer version of slides

`http://homepages.math.uic.edu/~jbaldwin/catcompucsli.pdf`

Bourbaki Again **Bourbaki** 

Bourbaki distinguishes between 'logical formalism' and the 'axiomatic method'.

'We emphasize that it (logical formalism) is but one aspect of this (the axiomatic) method, indeed the least interesting one'.

We reverse this aphorism:
The axiomatic method is but one aspect of logical formalism.

And the foundational aspect of the axiomatic method is the least important for mathematical practice.

Two roles of formalization

1. Building a piece or all of mathematics on a firm ground specifying the underlying assumptions

2. When mathematics is organized by studying first order (complete) theories, syntactic properties of the theory induce profound similarities in the structures of models. These are tools for mathematical investigation.

3 First Order Logic/Formalization as a Mathematical Tool

Section III First Order Logic/Formalization as a Mathematical Tool
Mathematical Applications of Completeness

The significance of theories
The breakthroughs of model theory as a tool for organizing mathematics come in several steps.

1. The significance of (complete) first order theories

From a mathematician: **Kazhdan** 

> On the other hand, the Model theory is concentrated on gap between an abstract definition and a concrete construction. Let T be a complete theory. On the first glance one should not distinguish between different models of T, since all the results which are true in one model of T are true in any other model. One of main observations of the Model theory says that our decision to ignore the existence of differences between models is too hasty.

Kazhdan continued

> Different models of complete theories are of different flavors and support different intuitions. So an attack on a problem often starts [with] a choice of an appropriate model. Such an approach lead to many non-trivial techniques for constructions of models which all are based on the compactness theorem which is almost the same as the fundamental existence theorem.
>
> On the other hand the novelty creates difficulties for an outsider who is trying to reformulate the concepts in familiar terms and to ignore the differences between models.

3.1 Classes of Theories

Classes of Theories

3.2 The stability hierarchy as a mathematical tool

The stability hierarchy as a mathematical tool

The significance of *classes* of Theories

The breakthroughs of model theory as a tool for organizing mathematics come in several steps.

1. The significance of (complete) first order theories.
2. The significance of classes of (complete) first order theories: *Quantifier reduction*
3. The significance of classes of (complete) first order theories: *syntactic dividing lines*

Shelah on Dividing Lines: Shelah 

> I am grateful for this great honour. While it is great to find full understanding of that for which we have considerable knowledge, I have been attracted to trying to find some order in the darkness, more specifically, finding meaningful dividing lines among general families of structures. This means that there are meaningful things to be said on both sides of the divide: characteristically, understanding the tame ones and giving evidence of being complicated for the chaotic ones.

Shelah on Dividing Lines

> It is expected that this will eventually help in understanding even specific classes and even specific structures. Some others see this as the aim of model theory, not so for me. Still I expect and welcome such applications and interactions. It is a happy day for me that this line of thought has received such honourable recognition. Thank you

on receiving the Steele prize for seminal contributions.

Properties of classes of theories

The Stability Hierarchy

Every complete first order theory falls into one of the following 4 classes.

1. ω-stable
2. superstable but not ω-stable
3. stable but not superstable
4. unstable

Stability is Syntactic

Definition
T is stable if no formula has the order property in any model of T.

ϕ is unstable in T just if for every n the sentence $\exists x_1, \ldots x_n \exists y_1, \ldots y_n \bigwedge_{i<j} \phi(x_i, y_i) \wedge \bigwedge_{j \geq i} \neg\phi(x_i, y_i)$ is in T.

This formula changes from theory to theory.

1. dense linear order: $x < y$;
2. real closed field: $(\exists z)(x + z^2 = y)$,
3. $(\mathbb{Z}, +, 0, \times)$:$(\exists z_1, z_2, z_3, z_4)(x + (z_1^2 + z_2^2 + z_3^2 + z_4^2) = y)$.
4. infinite boolean algebras: $x \neq y$ & $(x \wedge y) = x$.

Model Theoretic Consequences: Main Gap
Shelah proved:

Main Gap
For every first order theory T, either

1. Every model of T is decomposed into a tree of countable models with uniform bound on the depth of the tree, or
2. The theory T has the maximal number of models in all uncountable cardinalities.

Sample Mathematical Applications

1. Differential Algebra
2. Arithmetic algebraic geometry
3. Real algebraic geometry and o-minimality
4. Motivic integration Asymptotic classes and finite groups
5. Groups of finite Morley rank

Mordell Lang for function fields
Ingredients to Hrushovski's proof of Mordell-Lang for function fields include:

1. Shelah's notions of orthogonality and p-regularity
2. such notions from geometric stability theory as one-based
3. stability of separably closed fields
4. tools of arithmetic algebraic geometry.

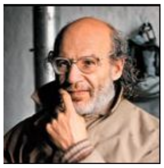
Summation: Hrushovski 

Hrushovski ICM talk 1998

> Instead of defining the abstract context for the [stability] theory, I will present a number of its results in a number of special and hopefully more familiar, guises: compact complex manifolds, ordinary differential equations, difference equations, highly homogeneous finite structures. Each of these has features of its own and the transcription of results is not routine; they are nonetheless *readily recognizable as instances of a single theory*.

4 Back to Foundations

Back to Foundations

Set theory and mathematics Grothendieck 

Grothendieck introduced the notion of a universe explicitly having cardinality of an inaccessible. And the major results of cohomology are written under the hypotheses of universes.

McLarty

> Large-structure tools like toposes and derived categories in cohomology never go far from arithmetic in practice, yet existing foundations for them are stronger than ZFC.

Mclarty's program McLarty 

Mclarty's program is to develop cohomology theory in weak set theories.

Zermelo set theory with choice (ZC) is ZFC without foundation or replacement but with the separation axiom scheme.

MC (Maclane Set theory) is ZC with bounded separation.

Sample Theorem In MC, every sheaf of modules M on any small site embeds in an injective, and thus has injective resolutions of any given finite length.

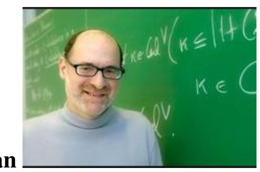
Is replacement necessary? B. Kim 

More Precisely

Theorem 1 (Kim)**.** *For a simple first order theory non-forking is equivalent to non-dividing.*

The known proof relies on Morley's use of Erdos-Rado.

Question 2. *Is Kim's theorem provable in ZC?*

Inner Model Hypothesis Sy Friedman 

Inner Model Hypothesis
If a statement ϕ without parameters holds in an inner universe of outer universe of V then it holds in an inner universe of V

Some consequences

1. Singular cardinals hypothesis

2. There are no inaccessible cardinals.