Parity Games and Resolution

Arnold Beckmann

Department of Computer Science
College of Science
Swansea University, Wales, UK

SDF-60, 10 July 2013

(Joint work with Pavel Pudlák and Neil Thapen)

Overview

Parity Games

Parity Games

Infinite two-player games played on finite directed leafless graphs.

Deciding winner in a parity game is significant

- in verification (ptime-equivalent to model checking problem for modal μ-calculus)
- in automata theory (ptime-equivalent to emptiness problem for alternating tree automata)
- from complexity-theoretic point of view (in NP $\cap$ coNP, not known to be in P)

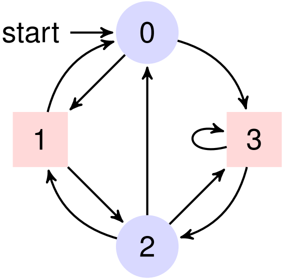
Any parity game can be transformed (in linear time) into equivalent simple graph game.

Simple Graph Games

Played on a directed graph with vertices

$$V = V_A \cup V_B = \{0, 1, \ldots, n-1\}$$

owned by player A or B, with at least one outgoing edge for each vertex.



$$V_A = \{0, 2\}, \quad V_B = \{1, 3\}$$

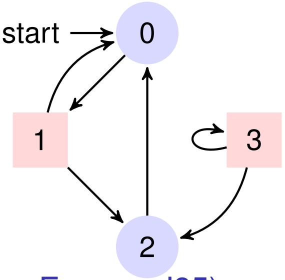
A ***play*** is an infinite sequence $0 = v_0, v_1, v_2, \ldots$ with $v_i \to v_{i+1}$ chosen by the player owning v_i.

The **winner** of a play is the player owning the least vertex which is visited infinitely often in the play.

Strategies

A **(positional) strategy** for A
is a function
$\sigma \colon V_A \to V$ defining A's moves.
(Similar $\tau \colon V_B \to V$ for player B.)

A strategy is a **winning strategy**
if player wins
all plays when using their strategy.



Theorem (Memoryless Determinacy, Emerson'85)

For any simple graph game, one player has a positional winning strategy.

Corollary

Given a simple graph game, deciding whether A has a winning strategy is in NP $\cap$ coNP.

Weak Automatizability and Resolution

$\mathrm{Res}(k)$ proof system

k-DNF: disjunction of conjunctions of literals, each conjunction of size $\leq k$.
Each line in $\mathrm{Res}(k)$-proof is k-DNF, written as list of disjuncts.

$$\text{axiom } \frac{}{a, \neg a} \qquad \wedge\text{-intro } \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}$$

$$\text{weak } \frac{\Gamma}{\Gamma, \Delta} \qquad \text{cut } \frac{\Gamma, a_1 \wedge \ldots \wedge a_m \qquad \Gamma, \neg a_1, \ldots, \neg a_m}{\Gamma}$$

$\mathrm{Res}(k)$ *refutation* of set of disjunctions Γ is sequence of disjunctions ending with the empty disjunction, s.t. each line in proof is either in Γ, or follows from earlier disjunctions by a rule.

$\mathrm{Res}(1)$ is called *resolution*, denoted Res.

Weak Automatizability

Propositional proof system $\mathcal{P}$ is *automatizable* if there is algorithm which, given a tautology, produces proof in time polynomial in size of its smallest proof.

Alekhnovich and Razborov (2008): Resolution not automatizable under reasonable assumption in parameterised complexity theory.

Weak automatizability: proofs of tautologies can be given in an arbitrary proof system, only time of finding proofs restricted to polynomial in size of smallest $\mathcal{P}$ proof. Equivalently:

Definition

$\mathcal{P}$ is *weakly automatizable* if exists polynomial time algorithm which, given formula ϕ and string 1^m, accepts if ϕ satisfiable, and rejects if ϕ has $\mathcal{P}$ refutation of size $\leq m$.

Results on weak automatizability

Theorem (Atserias, Bonet, 2004)

For the following list of proof systems, either all or none are weakly automatizable:

$$\mathrm{Res}, \quad \mathrm{Res}(2), \quad \mathrm{Res}(3), \quad \ldots$$

Open Problem

Is Res *weakly automatizable?*

Result

Theorem (B., Pudlák, Thapen, 2013)

If resolution is weakly automatizable, then parity games can be decided in polynomial time.

Outline of proof

Formalise "σ *is winning strategy for A in G*"
as $\mathrm{Win}_A(n, G, \sigma, \dots)$
"τ *is winning strategy for B in G*"
as $\mathrm{Win}_B(n, G, \tau, \dots)$

Construct, for some k, polynomial size (in n) $\mathrm{Res}(k)$ refutations of $\mathrm{Win}_A(n, G, \sigma, \dots) \wedge \mathrm{Win}_B(n, G, \tau, \dots)$

Result follows by considering

$$G \mapsto (\mathrm{Win}_A(|G|, G, \sigma, \dots), 1^{p(|G|)})$$

where $|G|$ denotes number of vertices in G, and p the polynomial bound in "construct" part of proof outline above.

Bounded Arithmetic

Language

Language L: constant symbols 0 and 1, function and relation symbols. Only restriction: function symbol represent *polynomially bounded functions*.

L^+: Extend L by finitely many new relation symbols $\bar{R}$—will be used to stand for edges in a graph, or strategies in a game, etc.

Bounded Formulas:

$$\begin{array}{ll} \mathrm{U}_1 : & \forall x_1 \le s_1\ \varphi(x_1, y) \\ \mathrm{U}_2 : & \forall x_1 \le s_1\ \exists x_2 \le s_2\ \varphi(x_1, x_2, y) \\ & \vdots \end{array}$$

with quantifier-free φ

Induction:

U_d-Ind : $\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x \varphi(x)$

where $\varphi \in \mathrm{U}_d$

BASIC = a set of true open L-formulas.

Paris-Wilkie Translation

Given assignment α, translate φ into propositional formula $\langle\varphi\rangle_\alpha$:

L^+ formula φ	propositional translation $\langle\varphi\rangle_\alpha$
$R(t)$	propositional variable $p_{\langle t\rangle_\alpha}$
φ in L	$\begin{cases} \top & \text{if } \varphi \text{ is true} \\ \bot & \text{o/w} \end{cases}$
$\neg\varphi$	$\neg\langle\varphi\rangle_\alpha$
$\varphi \vee \psi$	$\langle\varphi\rangle_\alpha \vee \langle\psi\rangle_\alpha$
$(\forall x \leq t)\varphi(x)$	$\bigwedge_{i\leq\langle t\rangle_\alpha}\langle\varphi(i)\rangle_\alpha$

Main Technical Result

Theorem (B., Pudlák, Thapen 2013)

Suppose $\phi_1(x), \dots, \phi_\ell(x)$ are U_2 formulas, with x only free variable, such that U_2-IND proves $\forall x \neg(\phi_1(x) \wedge \cdots \wedge \phi_\ell(x))$. Then for some $k \in \mathbb{N}$ the family

$$\Phi_n := \langle \phi_1(x) \rangle_{[x \mapsto n]} \cup \cdots \cup \langle \phi_\ell(x) \rangle_{[x \mapsto n]}$$

has polynomial size $\mathrm{Res}(k)$ refutations.

Further details on proof

Formalise simple graph game using second order relations V, V_A, V_B, E. Formalise strategies by relations E^σ and E^τ.

Idea: Consider $E^\sigma \cap E^\tau$: no choice, exactly one play possible, winner cannot be both players.
But: reachability in $E^\sigma \cap E^\tau$ cannot be defined or formalised.

Instead: Add further relations $R^\sigma_{\min}(x, y, z)$, intended meaning is *y can be reached from x in E^σ by a path with minimum z* similar $R^\tau_{\min}$.

Consider $R^*(x, y) = \exists z(R^\sigma_{\min}(x, y, z) \wedge R^\tau_{\min}(x, y, z))$. It turns out that this is good enough approximation to $E^\sigma \cap E^\tau$. Argument formalises in U_2-IND.

Conclusion

We have reduced the decision problem for parity games to the question whether resolution is weakly automatizable.
Main technical part was to construct polynomial size refutations of a suitable formalisation of the statement that both players have positional winning strategies.

Further results (not presented): Similar reductions of other games and proof systems (Mean payoff games and Simple Stochastic Games, and PK_1.)
Definition of game for which deciding whether a player has a positional winning strategy is equivalent to weak automatizability for resolution.

Open Problem

Can weak automatizability for resolution be reduced to the decision problem for parity games?