"The tree property" at the double successor of a measurable cardinal κ with 2^{κ} large

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(joint work with Sy Friedman)

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Definitions

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- A *tree* is a strict partial ordering $(T, <)$ with the property that for each $x \in T$, $\{y : y < x\}$ is well-ordered by $<$.
- The αth *level* of a tree T consists of all x such that $\{y : y < x\}$ has order-type α.
- The *height* of T is the least α such that the αth level of T is empty.
- A *branch* in T is a maximal linearly ordered subset of T.
- We say that a branch is *cofinal* if it hits every level of T.

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An infinite cardinal κ has the *tree property* if every tree of height κ whose levels have size $< \kappa$ has a cofinal branch.

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Theorem (Friedman, H.)

1. Assume that V is a model of ZFC and κ is λ^+-hypermeasurable in V, where λ is the least weakly compact cardinal greater than κ. Then there exists a forcing extension of V in which κ is still measurable, κ^{++} has the tree property and $2^\kappa = \kappa^{+++}$.

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2. If the assumption is strengthened to the existence of a θ-hypermeasurable cardinal (for an arbitrary cardinal $\theta > \lambda$ of cofinality greater than κ) then the proof can be generalized to get $2^\kappa = \theta$.
3. By forcing with the Prikry forcing over the above models one gets $\text{Con}(\mathit{cof}(\kappa) = \omega, \text{TP}(\kappa^{++}), 2^\kappa \text{ large})$.

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On the other hand, $\mathrm{TP}(\aleph_2)$ is consistent with large continuum (a detailed proof was given by Spencer Unger). So, the idea was to prove the analogous result for $\mathrm{TP}(\kappa^{++})$ with κ measurable, using Mitchell's forcing together with a "surgery" argument.

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As in Dobrinen-Friedman paper, the consistency of a cardinal κ of Mitchell order λ^+, where λ is weakly compact and greater than κ, is a lower bound on the consistency strength of $\mathrm{TP}(\kappa^{++})$ with κ measurable and $2^\kappa = \kappa^{+++}$. Therefore our result is in fact almost an equiconsistency result.

The proof: defining the forcing

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Let κ be λ^+-hypermeasurable. Let $j : V \to M$ be an elementary embedding witnessing the hypermeasurability of κ, with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $H(\lambda^+)^V = H(\lambda^+)^M$. We may assume that M is of the form $M = \{j(f)(\alpha) : \alpha < \lambda^+, f : \kappa \to V, f \in V\}$. We first define some forcing notions in order to describe the intended model.

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For a regular cardinal α and an arbitrary cardinal β let $Add(\alpha, \beta)$ denote the forcing for adding β many α-Cohens. The conditions are partial functions from $\alpha \times \beta$ into $\{0, 1\}$ of size $< \alpha$.

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Define a forcing notion P_κ as follows. Let ρ_0 be the first inaccessible cardinal and let λ_0 be the least weakly compact cardinal above ρ_0. For $k < \kappa$, given λ_k, let ρ_{k+1} be the least inaccessible cardinal above λ_k and let λ_{k+1} be the least weakly compact cardinal above ρ_{k+1}. For limit ordinals $k \leq \kappa$, let ρ_k be the least inaccessible cardinal greater than or equal to $\sup_{l<k}\lambda_l$ and let λ_k be the least weakly compact cardinal above ρ_k. Note that $\rho_\kappa = \kappa$ and λ_κ is the least weakly compact cardinal above κ.

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Let P_0 be the trivial forcing. For $i < \kappa$, if $i = \rho_k$ for some $k < \kappa$, let $\dot{Q}_i$ be a P_i-name for the forcing $Add(\rho_k, \lambda_k^+)$. Otherwise let $\dot{Q}_i$ be a P_i-name for the trivial forcing. Let $P_{i+1} = P_i * \dot{Q}_i$. Let P_κ be the iteration $\langle\langle P_i, \dot{Q}_i\rangle : i < \kappa\rangle$ with Easton support.

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We define the *Mitchell forcing* $M(\kappa, \beta)$ as $Add(\kappa, \beta) * Q$, where

$$Q = \{q \mid q \text{ is a partial function of cardinality} \leq \kappa \text{ on the regular cardinals below } \beta \text{ such that for each } \gamma \text{ in } \mathrm{Dom}(q), \emptyset \Vdash^{Add(\kappa,\gamma)} \text{“}q(\gamma) \in Add(\kappa^+, 1)\text{”}\}.$$

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Since $M(\kappa, \lambda)$ is known to preserve the tree property at λ while making λ into the κ^{++} of the extension, the idea is simply to force with $Add(\kappa, \lambda^+)$ over $V^{M(\kappa,\lambda)}$. However, in order to preserve the measurability of κ, our intended model will be a little different:

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Let $j_0 : V \to M_0$ be the measure ultrapower embedding via the normal measure $U_0 = \{X \subseteq \kappa \mid \kappa \in j(X)\}$ derived from j with critical point κ such that ${}^{\kappa}M_0 \subseteq M_0$ and let λ_0 be the first weakly compact cardinal of M_0 above κ. To prove the theorem we force over V with

$$P_\kappa * Add(\kappa, (\lambda_0^+)^{M_0}) * M(\kappa, \lambda) * Add(\kappa, \lambda^+) * R,$$

where P_κ is the 'preparatory' forcing defined above, and R is the forcing notion defined as follows:

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Let G, g_0 be generic filters on $P_\kappa, Add(\kappa, (\lambda_0^+)^{M_0})$, respectively. In $V[G][g_0]$, we can lift the embedding $j_0 : V \to M_0$ to an embedding $j_0 : V[G] \to M_0[G][g_0][H_0]$, where the generics on the right side correspond to $j_0(P_\kappa)$ factored as $j_0(P_\kappa)_{|\kappa} * j_0(P_\kappa)_\kappa * j_0(P_\kappa)_{\kappa+1, j_0(\kappa)}$.

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The forcing R is defined as $Add(j_0(\kappa), \lambda^+)$ of $M_0[G][g_0][H_0]$. We note here that R is an element of $V[G][g_0]$. Since $j_0(\lambda) = \lambda$, R is actually the image of $Add(\kappa, \lambda^+)$ under j_0.

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More precisely, for an interval I of ordinals let $Add(\kappa, I)_{|\text{even}}$ be the forcing whose conditions are partial functions from $\kappa \times \{\text{even ordinals in } I\}$ into $\{0, 1\}$ of size $< \kappa$. Then, for $q \in Q$ and $\gamma \in \text{Dom}(q)$, $q(\gamma)$ is an $Add(\kappa, [(\lambda_0^+)^{M_0}, \gamma))_{|\text{even}}$-name for a condition in $Add(\kappa^+, 1)$.

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We denote the final model, obtained by forcing over V with $P_\kappa * Add(\kappa, \lambda^+) * Q * R$, as W.

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Definition

Let A and B be two partial orderings. A function $\pi : B \to A$ is called a projection iff the following hold:

1. π is order-preserving and $\pi(B)$ is dense in A.
2. If $\pi(b) = a$ and $a' < a$, then there is $b' \leq b$ such that $\pi(b') \leq a'$.

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Fact

If $\pi : B \to A$ is a projection, then the forcing B is forcing-equivalent to $A * B/A$ for some quotient B/A.

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Since both $Add(\kappa, [(\lambda_0^+)^{M_0}, \lambda))_{|\text{even}}$ and Q exist in the model $V[G][g_0]$, we can also consider the forcing

$$Add(\kappa, [(\lambda_0^+)^{M_0}, \lambda))_{|\text{even}} \times Q,$$

of course, with a different ordering on Q, not depending on $Add(\kappa, [(\lambda_0^+)^{M_0}, \lambda))_{|\text{even}}$.

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In order not to confuse it with $Add(\kappa, [(\lambda_0^+)^{M_0}, \lambda))_{|\mathrm{even}} * Q$, which has a different ordering, we will write $Add(\kappa, [(\lambda_0^+)^{M_0}, \lambda))_{|\mathrm{even}} \times Q'$. For the same reason, the conditions (p, q) in the product will be denoted as $(p, (0, q))$.

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It can be shown that Q is κ^+-distributive, and Q' is obviously κ^+-closed in $V[G][g_0]$.

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Lemma

The function π given by $\pi(p,(0,q)) = (p,q)$ is a projection from

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This projection can be naturally extended to a projection from the product

$$Add(\kappa, [(\lambda_0^+)^{M_0}, \lambda^+)) \times Q' \times R$$

onto

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Proof.

The closure follows easily because R is κ^+-closed in $M_0[G][g_0][H_0]$ and $M_0[G][g_0][H_0]$ is closed under κ-sequences in $V[G][g_0]$. Let $\langle p_\alpha : \alpha < \lambda \rangle$ be a sequence of conditions in R, and let p_α be of the form $j_0(f_\alpha)(\kappa)$ for some function $f_\alpha : \kappa \to Add(\kappa, \lambda^+)$, $f_\alpha \in V[G]$. A Δ-system argument shows that λ many of the functions f_α are pointwise compatible. It follows that λ many of the conditions p_α are compatible. $\square$

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The forcings Q', R are closed in the model $V^{P_\kappa * Add(\kappa,(\lambda_0^+)^{M_0})}$ in which they are defined, therefore their product $Q' \times R$ is closed in there as well. By Easton's lemma, after forcing with the κ^+-c.c. forcing $Add(\kappa, [(\lambda_0^+)^{M_0}, \lambda^+))$, the product $Q' \times R$ will remain κ^+-distributive. Since κ^+-distributivity is equivalent to not adding new κ-sequences of ordinals, it follows from the above facts about projections that $Q \times R$ is distributive in $V^{P_\kappa * Add(\kappa,\lambda^+)}$ as well. $\square$

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$\kappa^+ = (\kappa^+)^V$: This follows from the facts that $P_\kappa * Add(\kappa, \lambda^+)$ is κ^+-c.c in V, and $Q * R$ is κ^+-distributive in $V^{P_\kappa * Add(\kappa,\lambda^+)}$.

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$\kappa^{++} = \lambda, \kappa^{+++} = (\lambda^+)^V$: The Mitchell forcing $M(\kappa, \lambda)$ collapses precisely the cardinals between κ^+ and λ. On the other side, in the model $V^{P_\kappa * Add(\kappa, (\lambda_0^+)^{M_0})}$, in which all cardinals are preserved, R has the λ-Knaster property and $M(\kappa, \lambda) * Add(\kappa, \lambda^+)$ satisfies the λ-c.c. It follows that their product also satisfies the λ-c.c., which means that all cardinals above λ are preserved. $\square$

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In the general case where κ is θ-hypermeasurable we can first force to add a function $f : \kappa \to \kappa$ with $j(f)(\kappa) = \theta$. Then θ_0, M_0's version of θ, is less than κ^{++}, because $\theta_0 = j_0(f)(\kappa) < j_0(\kappa) < \kappa^{++}$. It follows that the forcing R still has the λ-Knaster property in $V^{P_\kappa * Add(\kappa,\theta_0)}$, and hence, the above lemmas apply in the general case.

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In order to prove that κ remains measurable in W we extend the elementary embedding $j : V \to M$ to an embedding of W. We have already picked generics G, g_0 for $P_\kappa, Add(\kappa, (\lambda_0^+)^{M_0})$, resp.

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Let g be an $Add(\kappa, [(\lambda_0^+)^{M_0}, \lambda^+))$-generic filter over $V[G][g_0]$. We first use a 'surgery' argument to lift j to an embedding of $V[G][g_0][g]$.

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The embedding j can be factored as $k \circ j_0$, where $k : M_0 \to M$ is defined by $k([F]_U) := j(F)(\kappa)$. The embedding k is also elementary and its critical point is $(\kappa^{++})^{M_0}$. By elementarity and GCH, $(\kappa^{++})^{M_0} < j_0(\kappa) < \kappa^{++}$. Note also that $k(\lambda_0) = \lambda$.

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It is now possible in $V[G][g_0][g]$ to lift $k : M_0 \to M$ to an embedding $k : M_0[G][g_0][H_0] \to M[G][(g_0 \times g)'][H]$, getting the commutative diagram

$$\begin{array}{ccc} V[G] & \xrightarrow{j} & M[G][(g_0 \times g)'][H] \\ \searrow^{j_0} & & {}^{k}\nearrow \\ & M_0[G][g_0][H_0] & \end{array}$$

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$$\begin{tikzcd} V[G] \arrow[rr, "j"] \arrow[dr, "j_0"'] & & M[G][(g_0 \times g)'][H] \\ & M_0[G][g_0][H_0] \arrow[ur, "k"'] & \end{tikzcd}$$

Next, lift $j : V[G] \to M[G][(g_0 \times g)'][H]$ to an embedding of $V[G][g_0][g]$, as follows:

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We transfer h along k in order to get a generic h^* for $j(Add(\kappa, \lambda^+))$ so that we could lift j to $j : V[G][g_0][g] \to M_0[G][g_0][H_0][h^*]$. Namely, $h^* = \{p \in j(Add(\kappa, \lambda^+)) \mid k(p_0) \leq p \text{ for some } p_0 \in h\}$ is generic for $j(Add(\kappa, \lambda^+))$.

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The fact that h can be transferred to create a generic for $j(Add(\kappa, \lambda^+))$, and the fact that $R = j_0(Add(\kappa, \lambda^+))$ is not a harmful forcing in $V[G][g_0]$, i.e. has κ^+-closure and λ-Knaster property, are the main advantages of factoring j as $k \circ j_0$.

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This lifting argument is called surgery, because we still have to make sure that $j[g_0 \times g] \subseteq h^*$, and that is done by altering the conditions of the generic h^* on small parts of size $< \kappa$.

The proof: measurability of κ

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So far we have proven that in $V[G][g_0][g][h]$ there is a definable elementary embedding $j : V[G][g_0][g] \to M[G][(g_0 \times g)'][H][h^{**}]$.

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We now need to find a generic filter $G_{j(Q)} \times h_{j(R)}$ for $j(Q \times R)$ such that $j[G_Q \times h] \subseteq G_{j(Q)} \times h_{j(R)}$, in order to define our final lifting

$$j : V[G][g_0][g][G_Q][h] \to M[G][(g_0 \times g)'][H][h^{**}][G_{j(Q)}][h_{j(R)}].$$

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This last step is, however, just another transferring argument since, by one of our lemmas, $Q \times R$ is κ^+-distributive over $V[G][g_0][g]$, that is, $\{(q, r) \mid j(q_0, r_0) \leq (q, r) \text{ for some } (q_0, r_0) \in G_Q \times h\}$ is an appropriate generic.

The proof: measurability of κ

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This completes the proof of measurability of κ. $\square$

The proof: "the tree property" at κ^{++}

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Recall that W can be written as $V^{P_\kappa * Add(\kappa,(\lambda_0^+)^{M_0}) * M(\kappa,\lambda) * Add(\kappa,\lambda^+) * R}$.

Let V_1 denote the model $V^{P_\kappa * Add(\kappa,(\lambda_0^+)^{M_0}) * M(\kappa,\lambda)}$ and let $R' = R_{|\lambda}$ be the forcing $Add(j_0(\kappa),\lambda)$ of $M_0[G][g_0][H_0]$.

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We first notice that there must be a κ^{++}-Aronszajn tree already in $V_1^{Add(\kappa,\lambda)\times R'}$ because $Add(\kappa,\lambda^+)\times R$ has the λ-c.c. in V_1 and the tree is of size κ^{++}.

Proof continued

The proof: "the tree property" at κ^{++}

Proof continued

Similarly as before, we can rewrite

$$P_\kappa * Add(\kappa, (\lambda_0^+)^{M_0}) * M(\kappa, \lambda) * Add(\kappa, \lambda) \times R'$$

as

$$P_\kappa * Add(\kappa, (\lambda_0^+)^{M_0}) * Add(\kappa, \lambda) * Q \times R',$$

where Q is defined only using the even components of $Add(\kappa, \lambda)$.

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where Q is defined only using the even components of $Add(\kappa, \lambda)$.

Hence, in terms of our chosen generics, the above means that there is a κ^{++}-Aronszajn tree T in $V[G][g_0][g_{|\lambda}][G_Q][h_{|\lambda}]$. Let $\dot{T}$ be an $Add(\kappa, \lambda) * Q \times R'$-name in $V[G][g_0]$ for T.

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Recall that λ is a weakly compact cardinal in $V[G][g_0]$. Therefore, there exist in $V[G][g_0]$ transitive ZF^--models N_0, N_1 of size λ and an elementary embedding $k : N_0 \to N_1$ with critical point λ, such that $N_0 \supseteq H(\lambda)^{V[G][g_0]}$ and $G, g_0, \dot{T} \in N_0$.

The proof: "the tree property" at κ^{++}

Proof continued

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Note that $g_{|\lambda} * G_Q * h_{|\lambda}$ is also $Add(\kappa, \lambda) * Q \times R'$-generic over N_0.

The proof: "the tree property" at κ^{++}

Proof continued

Note that $g_{|\lambda} * G_Q * h_{|\lambda}$ is also $Add(\kappa, \lambda) * Q \times R'$-generic over N_0.

Since crit(k)=λ, we can factor $k(Add(\kappa, \lambda) * Q \times R')$ as

$$Add(\kappa, \lambda) * Add(\kappa, [\lambda, k(\lambda))) * Q * Q^* \times R' \times R^*$$

where Q^* and R^* denote the tail forcings $k(Q)/Q$ and $k(R')/R'$, respectively, with components indexed from the interval $[\lambda, k(\lambda))$.

The proof: "the tree property" at κ^{++}

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where Q^* and R^* denote the tail forcings $k(Q)/Q$ and $k(R')/R'$, respectively, with components indexed from the interval $[\lambda, k(\lambda))$.

Since k is the identity on $g_{|\lambda} * G_Q * h_{|\lambda}$ we can extend the embedding $k : N_0 \to N_1$ in some large universe U to an embedding

$$k : N_0[g_{|\lambda}][G_Q][h_{|\lambda}] \to N_1[g_{|\lambda}][g^*][G_Q][G_{Q^*}][h_{|\lambda}][h^*]$$

where g^*, G_{Q^*}, h^* are arbitrary generics for $Add(\kappa, [\lambda, k(\lambda)))$, Q^*, R^*, respectively, picked in U.

The proof: "the tree property" at κ^{++}

Proof continued

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Since $T \in N_0[g_{|\lambda}][G_Q][h_{|\lambda}]$ is a λ-Aronszajn tree, by elementarity $k(T)$ is a $k(\lambda)$-Aronszajn tree in $N_1[g_{|\lambda}][g^*][G_Q][G_{Q^*}][h_{|\lambda}][h^*]$ which coincides with T up to level λ. Hence T has a cofinal branch b in $N_1[g_{|\lambda}][g^*][G_Q][G_{Q^*}][h_{|\lambda}][h^*]$. We will show that b must actually belong to $N_1[g_{|\lambda}][G_Q][h_{|\lambda}]$ (i.e. the tail generics g^*, G_{Q^*}, h^* can not add a new branch), and thereby reach the desired contradiction to the assumption that T has no cofinal branches in $V[G][g_0][g][G_Q][h]$!

The proof: "the tree property" at κ^{++}

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Since $T \in N_0[g_{|\lambda}][G_Q][h_{|\lambda}]$ is a λ-Aronszajn tree, by elementarity $k(T)$ is a $k(\lambda)$-Aronszajn tree in $N_1[g_{|\lambda}][g^*][G_Q][G_{Q^*}][h_{|\lambda}][h^*]$ which coincides with T up to level λ. Hence T has a cofinal branch b in $N_1[g_{|\lambda}][g^*][G_Q][G_{Q^*}][h_{|\lambda}][h^*]$. We will show that b must actually belong to $N_1[g_{|\lambda}][G_Q][h_{|\lambda}]$ (i.e. the tail generics g^*, G_{Q^*}, h^* can not add a new branch), and thereby reach the desired contradiction to the assumption that T has no cofinal branches in $V[G][g_0][g][G_Q][h]$!

Similarly as above, in N_1 there is a projection from the product

$$Add(\kappa,\lambda) \times Add(\kappa,[\lambda,k(\lambda))) \times Q' \times Q^{*\prime} \times R' \times R^*$$

onto

$$Add(\kappa,\lambda) * Add(\kappa,[\lambda,k(\lambda))) * Q * Q^* \times R' \times R^*,$$

where $Q', Q^{*\prime}$ are κ^+-closed forcings defined in N_1. Let $G_{Q'} \times G_{Q^{*\prime}}$ be $Q' \times Q^{*\prime}$-generic over $N_1[g_{|\lambda}][g^*]$.

The proof: "the tree property" at κ^{++}

Proof continued

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If we can show that the bigger generic $g^* * G_{Q^{*\prime}} * h^*$ doesn't add the branch b through T over the bigger model $N_1[g_{|\lambda}][G_{Q'}][h_{|\lambda}]$, then in particular the smaller generic $g^* * G_{Q^*} * h^*$ doesn't add b over the smaller model $N_1[g_{|\lambda}][G_Q][h_{|\lambda}]$, and we are done.

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Since all the forcings are defined in N_1, we can write

$$N_1[g_{|\lambda}][g^*][G_{Q'}][G_{Q^{*\prime}}][h_{|\lambda}][h^*]$$

as

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Note that in $N_1[G_{Q'}][h_{|\lambda}]$, $Q^{*\prime} \times R^*$ is κ^+-closed forcing and $Add(\kappa, k(\lambda))$ is κ^+-c.c. Therefore, it can be shown that $Q^{*\prime} \times R^*$ doesn't add any branches to T over the model $N_1[G_{Q'}][h_{|\lambda}][g_{|\lambda}][g^*]$.

The proof: "the tree property" at κ^{++}

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If we can show that the bigger generic $g^* * G_{Q^{*\prime}} * h^*$ doesn't add the branch b through T over the bigger model $N_1[g_{|\lambda}][G_{Q'}][h_{|\lambda}]$, then in particular the smaller generic $g^* * G_{Q^*} * h^*$ doesn't add b over the smaller model $N_1[g_{|\lambda}][G_Q][h_{|\lambda}]$, and we are done.
Since all the forcings are defined in N_1, we can write

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Note that in $N_1[G_{Q'}][h_{|\lambda}]$, $Q^{*\prime} \times R^*$ is κ^+-closed forcing and $Add(\kappa, k(\lambda))$ is κ^+-c.c. Therefore, it can be shown that $Q^{*\prime} \times R^*$ doesn't add any branches to T over the model $N_1[G_{Q'}][h_{|\lambda}][g_{|\lambda}][g^*]$. Finally, $Add(\kappa, [\lambda, k(\lambda)))$ has the κ^{++}-Knaster property, which means that it couldn't have added the branch b over the model $N_1[G_{Q'}][h_{|\lambda}][g_{|\lambda}]$ either. This proves TP(κ^{++}). $\square$

Siesta time in Bucharest