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Effective Categoricity of Injection Structures

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joint work with Doug Cenzer and Jeff Remmel

Injection structure $\mathcal{A} = (A, f)$

- $1-1$ function $f : A \to A$.
  For $a \in A$, its *orbit* is

$$\mathcal{O}_f(a) = \{b \in A : (\exists n \in \mathbb{N})[f^n(a) = b \ \vee \ f^n(b) = a]\}$$

- Two types of *infinite orbits*:
  (1) Z-orbits, isomorphic to $(\mathbb{Z}, S)$, hence every element is in $Ran(f)$;

  (2) ω-orbits, isomorphic to $(\mathbb{N}, S)$, hence have the form
  $\mathcal{O}_f(a_0) = \{f^n(a_0) : n \in \mathbb{N}\}$ for some $a_0 \notin Ran(f)$.

- An injection structure is characterized by the number of orbits of size k for each finite $k > 0$, and by the number of orbits of types Z and ω.

- Let C be any infinite, co-infinite c.e. set.
  There is a computable injection structure $\mathcal{A} = (A, f)$
  (having infinitely many orbits of type ω) such that $Ran(f) = C$.

- Let $\mathcal{A} = (A, f)$ be a computable injection structure. Can assume $A = \mathbb{N}$.

  (i) $Inf(\mathcal{A}) = \{a \in A : \mathcal{O}_f(a)$ is infinite$\}$ is a Π^0_1 set.

  $a \in Inf(\mathcal{A}) \Leftrightarrow (\forall n)[f^n(a) \neq a]$

  (ii) $Fin(\mathcal{A})$ is a c.e. set.

  (iii) $\{a : \mathcal{O}_f(a)$ has type $Z\}$ is a Π^0_2 set.

  $(\mathcal{O}_f(a)$ has type $Z) \Leftrightarrow (a \in Inf(\mathcal{A}) \wedge (\forall n)(\exists b)[f^n(b) = a])$

  (iv) $\{a : \mathcal{O}_f(a)$ has type $\omega\}$ is a Σ^0_2 set.

- Let $\mathbf{c}$ be a c.e. degree.

  Let $\mathcal{A}$ be a computable injection structure such that:
  $\mathcal{A}$ has infinitely many orbits of size k for every $k \in \mathbb{N}$, and
  $\mathcal{A}$ has infinitely many infinite orbits.

  Then there is a computable structure $\mathcal{B}$
  isomorphic to $\mathcal{A}$ such that $Fin(\mathcal{B})$ is of degree $\mathbf{c}$.

- $Fin(\mathcal{A})$ cannot be a *simple* c.e. set.

  Each infinite orbit of $\mathcal{A}$ is
  a c.e. subset of the complement of $Fin(\mathcal{A})$.

- No infinite orbit of $\mathcal{A}$ can be a *simple* c.e. set.

  $Fin(\mathcal{A})$ is a c.e. subset of its complement.

- The *character* is defined by

  $\chi(\mathcal{A}) = \{(k,n) : 0 < k, n < \omega \ \wedge \ \mathcal{A}$ has at least n orbits of size $k\}$

  $\chi(\mathcal{A})$ is a c.e. set.

- $K \subseteq (\mathbb{N} - \{0\}) \times (\mathbb{N} - \{0\})$ is a *character* if for all k and $n > 0$:

$$(k, n+1) \in K \Rightarrow (k,n) \in K$$

- For any c.e. character K,
  there is a computable injection structure $\mathcal{A} = (A, f)$
  with character K and with any specified
  finite or countably infinite number of orbits of types ω and Z;
  and with $Fin(\mathcal{A})$ computable and $Ran(f)$ computable.

Let $\mathcal{A}$ be a *computable* structure.

- $\mathcal{A}$ is *computably categorical* if for all computable $\mathcal{B} \cong \mathcal{A}$, there is a computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

- $\mathcal{A}$ is Δ^0_n-*categorical* if for all computable $\mathcal{B} \cong \mathcal{A}$, there is a Δ^0_n isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

- $\mathcal{A}$ is *relatively* Δ^0_n-*categorical* if for all $\mathcal{B} \cong \mathcal{A}$, there is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, which is Δ^0_n relative to the atomic diagram of $\mathcal{B}$.

- $\mathcal{A}$ is relatively Δ^0_n-categorical $\Rightarrow$ $\mathcal{A}$ is Δ^0_n-categorical

- A computable injection structure $\mathcal{A}$ is computably categorical *iff* $\mathcal{A}$ is relatively computably categorical *iff* $\mathcal{A}$ has finitely many infinite orbits.

- Let $\mathbf{d}$ be a c.e. Turing degree.

  Let computable injection structure $\mathcal{A} = (\mathbb{N}, f)$ have infinitely many orbits of type ω. There is computable $\mathcal{B} = (\mathbb{N}, g) \cong \mathcal{A}$ such that $Ran(g)$ is a c.e. set of degree $\mathbf{d}$, there is $x \in \mathbb{N}$ such that $\mathcal{O}_g(x)$ is of type ω and is a c.e. set of degree $\mathbf{d}$, and for all $y \in \mathbb{N} - \mathcal{O}_g(x)$, if $\mathcal{O}_g(y)$ is of type ω, then $\mathcal{O}_g(y)$ is computable.

  Let computable injection structure $\mathcal{A} = (\mathbb{N}, f)$ have infinitely many infinite orbits of type Z. There is a computable $\mathcal{B} = (\mathbb{N}, g) \cong \mathcal{A}$ such that there is $x \in \mathbb{N}$ for which $\mathcal{O}_g(x)$ is of type Z and is a c.e. set of degree $\mathbf{d}$, and for all $y \in \mathbb{N} - \mathcal{O}_g(x)$, if $\mathcal{O}_g(y)$ is of type Z, then $\mathcal{O}_g(y)$ is computable.

- A computable injection structure $\mathcal{A}$ is $\mathbf{\Delta}^0_2$-categorical *iff*
  $\mathcal{A}$ is relatively $\mathbf{\Delta}^0_2$-categorical *iff*
  $\mathcal{A}$ has finitely many orbits of type ω or finitely many orbits of type Z.

- Assume computable $\mathcal{A} = (A, f)$ has finitely many orbits of type ω or finitely many orbits of type Z.

  Then $\{a : \mathcal{O}_f(a)$ has type $\omega\}$ and $\{a : \mathcal{O}_f(a)$ has type $Z\}$ are $\mathbf{\Delta}^0_2$ sets.
  Given computable $\mathcal{B} = (B, g) \cong \mathcal{A}$,
  can use oracle $\mathbf{0}'$ to partition A and B into three sets each:
  the orbits of finite type, the orbits of type ω, and the orbits of type Z.

- Let C be a $\mathbf{\Sigma}^0_2$ set.
  There is a computable injection structure $\mathcal{B} = (B, g) \cong \mathcal{A}$,
  in which $\{b \in B : \mathcal{O}_g(b)$ has type $\omega\}$ is
  a $\mathbf{\Sigma}^0_2$ set with Turing degree equal to $deg(C)$.

- Any computable injection structure $\mathcal{A}$ is relatively $\mathbf{\Delta}_3^0$-categorical.

  Let $\mathcal{B}$ be isomorphic to $\mathcal{A}$.
  Using an oracle for $(deg(\mathcal{B}))''$,
  we can partition B into three $\mathbf{\Delta}_3^0$ sets:
  the orbits of finite type, the orbits of type ω, and the orbits of type Z.

- Definition (Fokina, Kalimullin, and R. Miller)

  Let $\mathbf{d}$ be a Turing degree.
  A computable structure $\mathcal{A}$ is $\mathbf{d}$-*computably categorical* if
  for every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$,
  there exists a $\mathbf{d}$-computable isomorphism from $\mathcal{B}$ onto $\mathcal{A}$.

  The *degree of categoricity* of a computable structure $\mathcal{A}$ is
  the least Turing degree $\mathbf{d}$ for which $\mathcal{A}$ is $\mathbf{d}$-*computably categorical*.

  $\mathbf{0}$-computably categorical $=$ computably categorical

- Let $\mathcal{M}$ be a (computable) Δ^0_2-categorical injection structure, which is not computably categorical.

  Then the degree of categoricity of $\mathcal{M}$ is $\mathbf{0}'$.

- Let $\mathcal{M}$ be a computable injection structure, which is not Δ^0_2-categorical.

  Then the degree of categoricity of $\mathcal{M}$ is $\mathbf{0}''$.

- An enumeration of structures $\mathcal{A}_e = (\mathbb{N}, \phi_e)$, where ϕ_e is the usual eth partial computable function.

  Includes every computable injection structure with universe $\mathbb{N}$.

- $Inj = \{e : \mathcal{A}_e$ is a computable injection structure$\}$ is a Π^0_2-complete set.

  Inj is Π^0_2 : ($e \in Inj \Leftrightarrow \phi_e$ is total and $1-1$)

  For the completeness of Inj, consider a Π^0_2-complete set $Inf = \{e : W_e$ is infinite$\}$ and define a reduction of Inf to Inj.

  Let $s_0 < s_1 < \cdots$ enumerate the (possibly finite) set of stages at which a new element appears in the standard enumeration of W_e.

  To define a structure $\mathcal{A}_{f(e)} = (\mathbb{N}, \phi_{f(e)} = \phi)$: wait until s_0 appears and let $\phi(0) = 1, \phi(1) = 2, \ldots, \phi(s_0 - 1) = s_0$ and $\phi(s_0) = 0$;
  then wait until s_1 appears and
  let $\phi(s_0 + 1) = s_0 + 2, \ldots, \phi(s_1 - 1) = s_1$ and $\phi(s_1) = s_0 + 1; \ldots$

  If W_e is finite and s_k is the last stage at which an element enters W_e, then $\phi_{f(e)}(s_k + 1)$ is undefined, so $f(e) \notin Inj$.

  If W_e is infinite, then $\phi_{f(e)}$ is total and $Fin(\mathcal{A}_{f(e)}) = \mathbb{N}$.

- The set Inj_0^0 of indices of injection structures with no infinite orbits is Π_2^0-complete.

- Inj_0^0 is also Π_2^0-complete within Inj.

- A set I is Π_2^0 *within* B if I is the intersection of B with a Π_2^0 set.

  I is Π_2^0-*complete within* B if for any Π_2^0 set C,
  there is a computable function f that for every e,
  $f(e) \in B \wedge (e \in C \Leftrightarrow f(e) \in I)$.

- Let $m \geq 0$.

  (i) The set $Inj_{\leq m}$ of indices of computable injection structures with $\leq m$ orbits of type ω is Π^0_2-complete.

  (ii) The set $Inj_{>m}$ of indices of computable injection structures with $> m$ orbits of type ω is D^0_2-complete.

  A set is D^0_2 if it is the difference of two Σ^0_2 sets.

  $Inj_{>m}$ is Σ^0_2 *within* Inj,
  the intersection of the Σ^0_2 set S with Inj, where

  S is the set of indices e such that
  there are at least $m+1$ elements x with $x \notin Ran(\phi_e)$.

  $e \in S$ *iff* ($\mathcal{A}_e$ has $> m$ orbits of type ω).

- For the Π^0_2-completeness of $Inj_{\leq m}$, define a computable function g so that:

  $e \in Inf$ iff $\mathcal{A}_{g(e)}$ has all orbits finite;

  $e \in Fin$ iff $\mathcal{A}_{g(e)}$ has one orbit of type ω, and all other (finitely many) orbits finite.

  We then define $g_k(e)$ to be the computable function such that $\mathcal{A}_{g_k(e)}$ is the disjoint union of k computable copies of $\mathcal{A}_{g(e)}$.

  $e \in Inf \Leftrightarrow (g_{m+1}(e) \in Inj_{\leq m})$

- Let $m > 0$.

  (iii) The set Inj_m of indices of computable injection structures with exactly m orbits of type ω is D_2^0-complete.

  We reduce D_2^0-complete set
  $D = \{\langle a, b\rangle : a \in Fin \ \wedge \ b \in Inf\}$ to Inj_m.

  Define a computable reduction function h such that
  $\mathcal{A}_{h(\langle a,b\rangle)} = \mathcal{A}_{g_m(a)} \oplus (\mathcal{A}_{g_m(b)} \oplus \mathcal{A}_{g_m(b)})$.

  If $\langle a, b\rangle \in D$, then $\mathcal{A}_{g_m(a)}$ has exactly m orbits of type ω and
  $\mathcal{A}_{g(b)}$ has no infinite orbits, so
  $\mathcal{A}_{h(\langle a,b\rangle)}$ has exactly m orbits of type ω.

  If $a \in Inf$, then $\mathcal{A}_{h(\langle a,b\rangle)}$ has no infinite orbits if $b \in Inf$, and has $2m$ orbits of type ω if $b \in Fin$.

- Let $n \geq 0$.
  (i) The set $Inj^{\leq n}$ of indices of computable injection structures with $\leq n$ orbits of type Z is Π^0_3-complete.

  (ii) The set $Inj^{>n}$ of indices of computable injection structures with $> n$ orbits of type Z is Σ^0_3-complete.

  $\mathcal{A}_e$ has $> n$ orbits of type Z *iff* there exist $n+1$ elements $x_0, \dots, x_n$, each having an orbit of type Z, and no two being in the same orbit.

  To show that $Inj^{>n}$ is Σ^0_3-complete
  we define a reduction from the Σ^0_3-complete set
  $Cof = \{e : W_e \text{ is co-finite}\}$.

- Let $n > 0$.
  (iii) The set Inj^n of indices of computable injection structures with exactly n orbits of type Z is D^0_3-complete.

- The property of computable categoricity for computable injection structures (over $\mathbb{N}$) is $\mathbf{\Sigma}_3^0$-complete.

  $\{e : \mathcal{A}_e$ is an injection structure with finitely many infinite orbits$\}$ is a $\mathbf{\Sigma}_3^0$-complete set.

- $\mathcal{A}_e$ has finitely many infinite orbits iff there exists a finite sequence $a_0, \ldots, a_{k-1}$ such that for every b, if $b \notin \mathcal{O}(a_i)$ for all $i < k$, then $\mathcal{O}(b)$ is finite.

  Define a reduction f such that for every e, $\mathcal{A}_{f(e)}$ has finitely many infinite orbits iff W_e is co-finite.

  The orbits of $\mathcal{A}_{f(e)}$ will be exactly the orbits $\mathcal{O}(2i+1)$ for $i \in \mathbb{N}$, and the even numbers will be used to fill out the orbits.

  $\mathcal{O}_{f(e)}(2i+1)$ is finite *iff* $i \in W_e$.

  The function $\phi_{f(e)}$ is total and $1-1$.

- The property of $\mathbf{\Delta}^0_2$-categoricity for computable injection structures is $\mathbf{\Sigma}^0_4$-complete.

- $I_{\mathbf{\Delta}^0_2} = \{e : \mathcal{A}_e$ is an injection structure with finitely many orbits of type ω or finitely many orbits of type $Z\}$ is a $\mathbf{\Sigma}^0_4$-complete set.

- $\mathbf{\Sigma}^0_3$ condition: There exists a finite sequence $a_0, \ldots, a_{k-1}$ such that for every a, if $a \notin \mathcal{O}(a_i)$ for all $i < k$, then $\mathcal{O}(a)$ does not have type ω.

  $\mathbf{\Sigma}^0_4$ condition: There exists a finite sequence $b_0, \ldots, b_{l-1}$ such that for every b, if $b \notin \mathcal{O}(b_i)$ for all $i < l$, then $\mathcal{O}(b)$ does not have type Z.

- For $\mathbf{\Sigma}^0_4$-completeness of $I_{\mathbf{\Delta}^0_2}$, let C be any $\mathbf{\Sigma}^0_4$ set.

Then there is a Π^0_2 relation Q such that for every e:
$e \in \overline{C}$ iff $\{n : Q(e,n)\}$ is infinite.

Now, there is a computable relation R such that for every n:
$Q(e,n)$ iff $\{r : R(e,n,r)\}$ is infinite.

Hence, $e \in \overline{C}$ iff
there are infinitely many n, such that there are infinitely many r,
for which $R(e,n,r)$ holds.

Define a reduction f such that for every e, $\mathcal{A}_{f(e)}$ has only infinite orbits, and $\mathcal{A}_{f(e)}$ has infinitely many orbits of type Z iff $e \in \overline{C}$.

The orbits of $\mathcal{A}_{f(e)}$ will be exactly the orbits $\mathcal{O}(2n+1)$ for $n \in \omega$, and the even numbers will be used to fill out the orbits.

- We may assume that R is enumerated in stages.

  If (e, n, r) enters R at stage $s + 1$,
  then we add a new element in front of $\mathcal{O}(2n + 1)$.

  For every n, $\mathcal{O}_{f(e)}(2n + 1)$ is infinite, and
  $\mathcal{O}_{f(e)}(2n + 1)$ has type Z iff $\{r : R(e, n, r)\}$ is infinite.

  The function $\phi_{f(e)}$ is total and $1 - 1$.

  Finally, let $\mathcal{B}$ be a computable structure consisting of an infinite number of orbits, each of type ω, and let $\mathcal{A}_{g(e)}$ be the disjoint union of $\mathcal{B}$ with $\mathcal{A}_{f(e)}$. It follows that

$$e \in C \Leftrightarrow g(e) \in I_{\Delta^0_2}$$

The *Isomorphism Problem*
for infinite computable injection structures

- $\{(i,j)$: $\mathcal{A}_i$ is isomorphic to $\mathcal{A}_j\}$ is a Π^0_4-complete set.

- $\{(i,j)$: $\mathcal{A}_i$ is computably isomorphic to $\mathcal{A}_j\}$ is Σ^0_3-complete set.

HAPPY BIRTHDAY, SY!