Foundations of infinitesimal calculus: surreal numbers and nonstandard analysis

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Abstract

A system of foundations of infinitesimal calculus will be discussed. The system is based on two class-size models, including

1. the surreal numbers, and
2. the K – Shelah set-size-saturated limit ultrapower model.

Some **historical remarks** will be made, and a few **related problems** will be discussed, too.

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Section 1.
Extending the real line

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Extending the real line

Extending the real line

The idea to extend the real line $\mathbb{R}$ by new elements, called initially

- indivisible,

later

- infinitesimal, and
- infinite (or infinitely large),

emerged in the early centuries of modern mathematics in connection with the initial development of Calculus.

Nonarchimedean extensions

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and various elements of mixed character, *e. g.*, those of the form $x + \alpha$, where $x \in \mathbb{R}$ and α is infinitesimal.

The problem

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Different solutions have been proposed, and among them

the surreal numbers of Conway – Alling.

Section 2.
The Surreal field

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Digression: Hausdorff

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- unique , as the only set-size-dense **rcoF** up to isomorphism;
- "smooth" , in the sense that the underlying domain consists of sequences of ordinals — at least in the Alling version;
- computable , in the sense that the field operations in $\mathbf{F}_\infty$ are directly computable — at least in the Alling version.

Surreals: conclusion

This likely solves the Problem of foundations of infinitesimal calculus in **Part 2** (foundational conditions) but not yet in **Part 1** (technical conditions).

Technical shortcomings of surreals

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Section 3

Digression: Hausdorff's studies on pantachies

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Remark

Any pantachy in $P = \langle \mathbb{R}^\omega ; \prec \rangle$ is a set of type η_1.

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Gödel and Solovay discussed almost the same problem in 1970s.

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This result, by no means surprising, is nevertheless based on some pretty nontrivial arguments, including methods related to Stern's absoluteness theorem. But no algebraic structure on P is assumed.

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Section 4.
Technical shortcomings of the surreal Field

Shortcomings of the surreal Field

Observation

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This crucially limits the role of surreals $\mathbf{F}_\infty$ as a foundational system, in the spirit of the Problem of foundations of infinitesimal calculus.

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The problem of surreals

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To define such a Universe, we employ methods of **nonstandard analysis**.

Section 5.
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Superstructure over the surreals

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$$\xrightarrow{\mathbf{F}_\infty}$$

Superstructure over the surreals

$\xrightarrow{\mathbf{F}_\infty}$

surreals

Superstructure over the surreals

$\xrightarrow{}$

$\mathbf{F}_{\infty}$

surreals

a nicely
defined rcoF

Back

Superstructure over the surreals

$\xrightarrow[\mathbf{F}_\infty]{}$ $\xrightarrow[{}^*\mathbb{R}]{}$

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$\xrightarrow{}$ $\mathbf{F}_\infty$

$\xrightarrow{}$ $^*\mathbb{R}$

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set-size-dense hyperreals

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**set-size-dense
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admit a compatible
Universe

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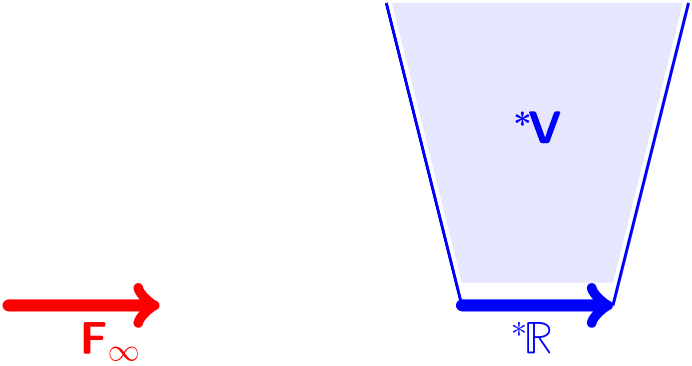

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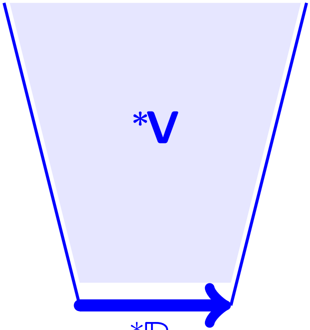

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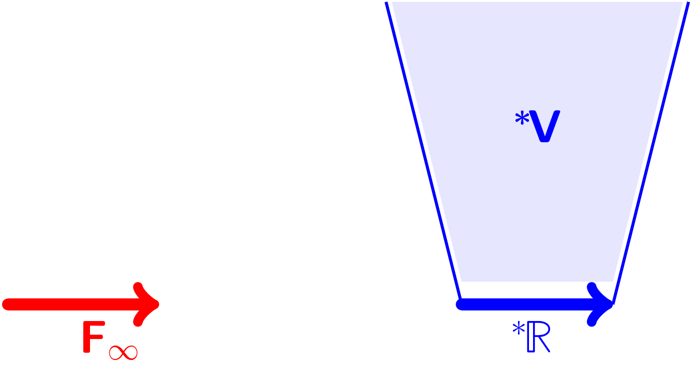

surreals ⟷ **set-size-dense hyperreals**

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isomorphic under Global Choice as two set-size-dense **rcoF**

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Superstructure over the surreals

consider an isomorphism

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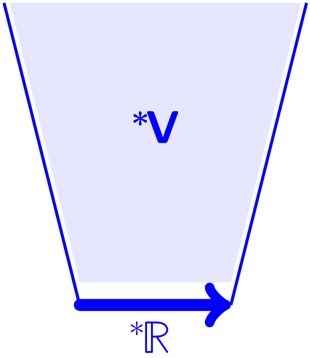

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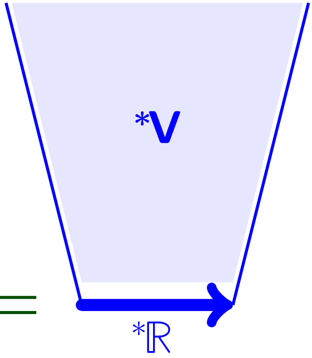

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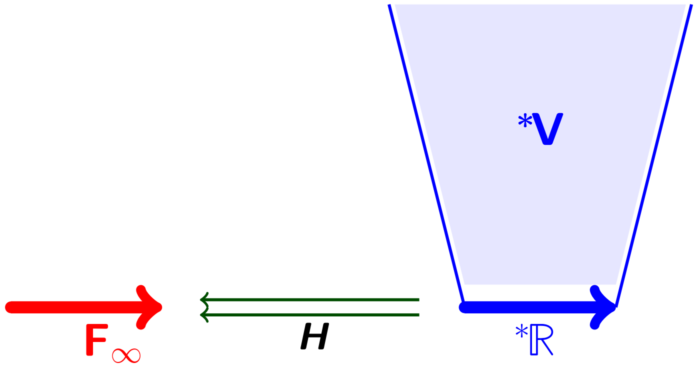
Superstructure over the surreals

isomorphism ***H*** induces a Universe over $\mathbf{F}_\infty$



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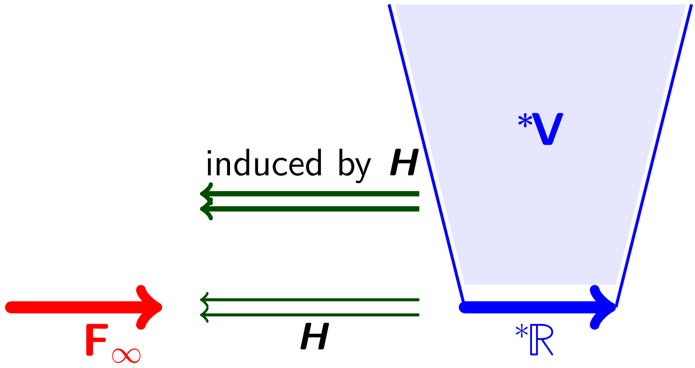

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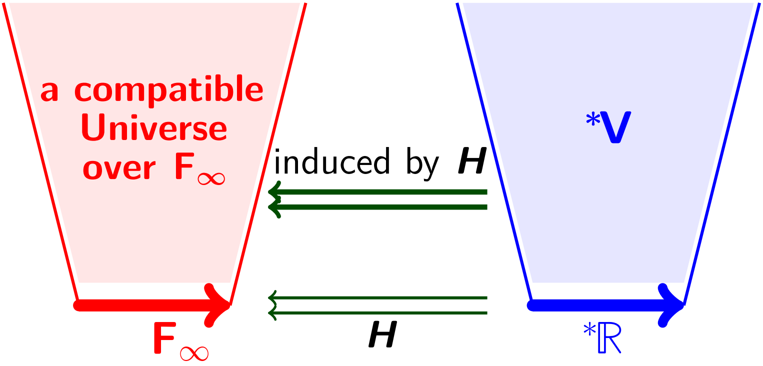

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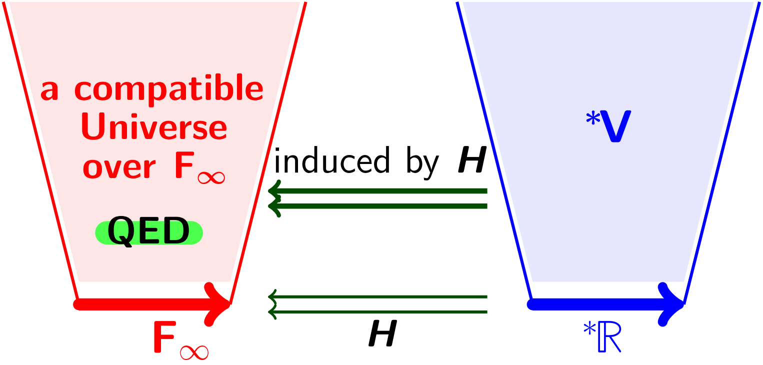

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A problem

•

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QED

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*V

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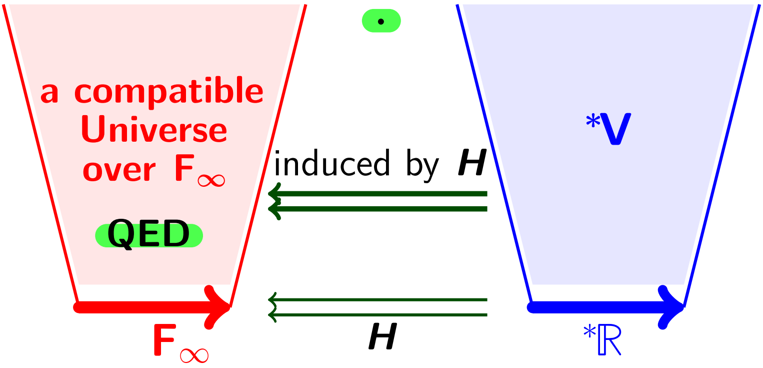
Superstructure over the surreals

A problem **definable**



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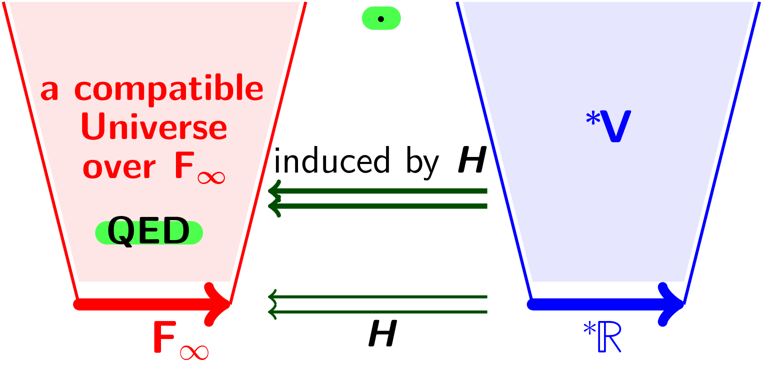
non-definable | A problem | definable



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Problems

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TOC

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Titlepage TOC

Theorem (Alling 1961, 1985, on the base of Hausdorff 1907)

Assuming the Global Choice *axiom, any two set-size-dense* **rcoF** *are* **isomorphic**, *and hence*

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Return to Surreals

Digression: classes

Definition (capitalization of classes)

1. A **Field** (a **Group, Order,** *etc*.) is a field (resp., group, ordered domain, *etc*.) whose underlying domain is a proper class.

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Back to Surreals

Back

Universes

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The universe of all sets $\mathbf{V}$ is a compatible Universe over the reals $\mathbb{R}$.

But it is not clear at all how to define a compatible Universe over a non-archimedean **rcoF** F.

Back

Back to the surreals problem

Definition

The Global Choice axiom **GC** asserts that there is a Function (a proper class!) G such that

- the domain $\mathtt{dom}\, G$ consists of all sets, and
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Global Choice

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GC definitely exceeds the capacities of the ordinary set theory **ZFC**. However, **GC** is **rather innocuous**, in the sense that any theorem provable in **ZFC** + **GC** and saying something only on sets (not on classes) is provable in **ZFC** alone.

Back

Answer

This question answers **in the negative**, by the following theorem.

Theorem

1. *There is no definable* **ZFC***-provable even* *bijection* *between:*
   - *the underlying domain of* $\mathbf{F}_\infty$ *(in the Alling version), and*
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2. *But, there is a definable* **ZFC***-provable* *injection* *from the underlying domain of* $\mathbf{F}_\infty$ *to the underlying domain of* ${}^*\mathbf{V}$.

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Back to problems

Hausdorff's early papers

1 . F. Hausdorff, Untersuchungen über Ordnungstypen IV, V. *Ber. über die Verhandlungen der Königlich Sächsische Gesellschaft der Wissenschaften zu Leipzig, Math.-phys. Kl.*, **1907**, 59, pp. 84–159.

2 . F. Hausdorff, Die Graduierung nach dem Endverlauf. *Abhandlungen der Königlich Sächsische Gesellschaft der Wissenschaften zu Leipzig, Math.-phys. Kl.*, **1909**, 31, pp. 295–334.

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And translated and commented in

4. F. Hausdorff, *Hausdorff on ordered sets*, Translated, edited, and commented by J. M. Plotkin. AMS and LMS, 2005.

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Density and saturation

Remark

For orders and **rcof** of type η_0 ($=$ simply dense) **being η_α** is equivalent to $\aleph_\alpha$-saturation.

Back