Foundations of infinitesimal calculus: surreal numbers and nonstandard analysis

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Abstract

A system of foundations of infinitesimal calculus will be discussed. The system is based on two class-size models, including

1. the surreal numbers, and
2. the K–Shelah set-size-saturated limit ultrapower model.

Some historical remarks will be made, and a few related problems will be discussed, too.
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Section 1.
Extending the real line
Extending the real line

The idea to extend the real line $\mathbb{R}$ by new elements, called initially indivisible, later infinitesimal, and infinite (or infinitely large), emerged in the early centuries of modern mathematics in connection with the initial development of Calculus.
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emerged in the early centuries of modern mathematics in connection with the initial development of Calculus.
A nonarchimedean extension $R_{\text{ext}}$ of the real line is a real-closed ordered field (rcof, for brevity) which properly extends the real number field $R$. Such a nonarchimedean extension $R_{\text{ext}}$ by necessity contains all usual reals: $R \subset R_{\text{ext}}$, along with:

- **Infinitesimals**: $x \in R_{\text{ext}}$ satisfying $0 < |x| < 1/n$ for all $n \in \mathbb{N}$;
- **Infinitely large elements**: $x \in R_{\text{ext}}$ satisfying $|x| > n$ for all $n \in \mathbb{N}$;
- **Various elements of mixed character**: e.g., those of the form $x + \alpha$, where $x \in R$ and $\alpha$ is infinitesimal.
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Nonarchimedean extensions

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and various elements of mixed character, e.g., those of the form $x + \alpha$, where $x \in \mathbb{R}$ and $\alpha$ is infinitesimal.
The problem

Problem of foundations of infinitesimal calculus

Define an extended real line \( \mathbb{R}^{\text{ext}} \) satisfying technical conditions which allow consistent "full-scale" treatment of infinitesimals, and foundational conditions of feasibility, plausibility, etc.

Different solutions have been proposed, and among them the surreal numbers of Conway – Alling.
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Section 2.
The Surreal field
Characterization

Mathematically, the surreal field is defined as the unique modulo isomorphism.

Definition (set-size density)

A total order (or any ordered structure) \( L \) is set-size-dense if for any its subsets \( X, Y \subseteq L \) (of any cardinality, but sets):

- if \( X < Y \), then there is an element \( z \) such that \( X < z < Y \).

Remark

Such an order has to be a proper class (not a set!). Indeed if \( L \) is a set then taking \( X = L \) and \( Y = \emptyset \) leads to an element \( z \in L \) with \( X < z \), which is a contradiction.
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In a more traditional notation, the set-size density is equivalent to being of the order type $\eta^\alpha$ for each ordinal $\alpha$.

**Definition (Hausdorff 1907, 1914)**

A total order (or any ordered structure) $L$ is of type $\eta^\alpha$ if for any subsets $X, Y \subseteq L$ of cardinality $\text{card}(X \cup Y) < \aleph^\alpha$:

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**Digression:** Hausdorff Kanovei (Moscow)
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Digression: Hausdorff
Theorem (the existence thm, Conway 1976, Alling 1985)

There is a set-size-dense $\mathbb{F}_{\infty}$.

Proof (Conway)
Consecutive filling in of all "gaps" $X < Y$, with a suitable (very complex, dozens of pages) definition of the order and the field operations, by transfinite induction.

Proof (Alling)
A far reaching generalization of the Levi–Civita field construction, on the base of Hausdorff's construction of dense ordered sets.
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**Surreals: existence**

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Surreals: conclusion

The extended $\mathbb{F}_\infty$ is:

- rather simply and straightforwardly defined
- set-size-dense
- unique, as the only set-size-dense $\mathbb{F}_{\infty}$ up to isomorphism;
- "smooth", in the sense that the underlying domain consists of sequences of ordinals — at least in the Alling version;
- computable, in the sense that the field operations in $\mathbb{F}_\infty$ are directly computable — at least in the Alling version.
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Surreals: conclusion

This likely solves the Problem of foundations of infinitesimal calculus in Part 2 (foundational conditions) but not yet in Part 1 (technical conditions).

Technical shortcomings of surreals

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Section 3

Digression:
Hausdorff’s studies on pantachies
Pantachies

Definition (Hausdorff 1907, 1909)

A pantachy is any maximal totally ordered subset \( L \) of a given partially ordered set \( P \), e.g., \( P = \langle R^\omega; \prec \rangle \), where, for \( x, y \in R^\omega \), \( x \prec y \) iff \( x(n) < y(n) \) for all but finite \( n \).

Remark

Any pantachy in \( P = \langle R^\omega; \prec \rangle \) is a set of type \( \eta_1 \).
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**Remark**

Any pantachy in $P = \langle \mathbb{R}^\omega ; \prec \rangle$ is a set of type $\eta_1$. 

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**Definition (Hausdorff 1907, 1909)**
Two pantachy existence theorems

Theorem (Hausdorff 1909)
There is a pantachy in \( \langle \mathbb{R}^\omega; \prec \rangle \) with an \((\omega_1, \omega_1)\)-gap.

Theorem (Hausdorff 1909)
There is a pantachy in \( \langle \mathbb{R}^\omega; \prec \rangle \) which is a rcof in the sense of the eventual coordinate-wise operations — that is, 
\[ x + y = z \text{ iff } \exists n \leq \omega \quad x(n) + y(n) = z(n) \]
for all but finite \( n \), and the same for the product.

Any such a pantachy is a rcof of type \( \eta_1 \).
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The problem of gapless pantachies

Problem (Hausdorff 1907)

Is there a pantachy \((\mathbb{R}^\omega; <)\), containing no \((\omega_1, \omega_1)\)-gaps?

The problem is still open, and it looks like it is the oldest concrete open problem in set theory.

Gödel and Solovay discussed almost the same problem in 1970s.

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The problem of effective existence of pantachies

Problem (Hausdorff 1907)

1. Is the pantachy existence provable without assuming AC?

2. Even assuming AC, is there an individually, effectively defined example of a pantachy?

Solution (K & Lyubetsky 2012)

In the negative (both parts), whenever $P$ is a Borel partial order, in which every countable subset has an upper bound. This result, by no means surprising, is nevertheless based on some pretty nontrivial arguments, including methods related to Stern's absoluteness theorem. But no algebraic structure on $P$ is assumed.

Back to surreals

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Foundations of infinitesimal calculus
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Section 4.
Technical shortcomings of the surreal Field
Shortcomings of the surreal Field

There is no clear way to naturally define sur-integers, most of analytic functions (beginning with $e^x$), accordingly, sur-sequences of surreals, sur-sets of surreals, etc, in $\mathbb{F}_\infty$—so that they satisfy the same internal laws and principles as their counterparts defined over the reals $\mathbb{R}$.

Example

The own system of sur-integers in $\mathbb{F}_\infty$ defined by Conway 1976 has the property that $\sqrt{2}$ is sur-rational, which makes little sense. This crucially limits the role of surreals $\mathbb{F}_\infty$ as a foundational system, in the spirit of the Problem of foundations of infinitesimal calculus.
Observation

There is **no clear way to naturally define** sur-integers, most of analytic functions (beginning with $e^x$), accordingly, sur-sequences of surreals, sur-sets of surreals, *etc*, *etc*, in $F_\infty$. 

Example: The own system of sur-integers in $F_\infty$ defined by Conway 1976 has the property that $\sqrt{2}$ is sur-rational, which makes little sense. This crucially limits the role of surreals $F_\infty$ as a foundational system, in the spirit of the Problem of foundations of infinitesimal calculus.
There is **no clear way to naturally define** sur-integers, most of analytic functions (beginning with $e^x$), accordingly, sur-sequences of surreals, sur-sets of surreals, *etc, etc*, in $\mathbb{F}_\infty$ — so that they satisfy **the same internal laws and principles** as their counterparts defined over the reals $\mathbb{R}$. 
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The own system of sur-integers in $F_\infty$ defined by Conway 1976 has the property that $\sqrt{2}$ is sur-rational, which makes little sense.
Shortcomings of the surreal Field

Observation

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Example

The own system of sur-integers in $F_\infty$ defined by Conway 1976 has the property that $\sqrt{2}$ is sur-rational, which makes little sense.

This crucially limits the role of surreals $F_\infty$ as a foundational system, in the spirit of the Problem of foundations of infinitesimal calculus.
Problem (upgrade of surreals)

Define a compatible Universe over the surreals $F_{\infty}$, sufficient to technically support "full-scale" treatment of infinitesimals.

Back

To define such a Universe, we employ methods of nonstandard analysis.
The problem of surreals

Problem (upgrade of surreals)

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Back To define such a Universe, we employ methods of nonstandard analysis.
Define a compatible Universe over the surreals $F_\infty$, sufficient to technically support “full-scale” treatment of infinitesimals.
The problem of surreals

**Problem (upgrade of surreals)**

Define a **compatible Universe** over the surreals $F_\infty$, sufficient to technically support “full-scale” treatment of infinitesimals.

To define such a Universe, we employ methods of **nonstandard analysis**.
Section 5.
Nonstandard analysis
Nonstandard analysis

Nonstandard analysis (Robinson) studies elementary extensions $^*\mathbb{V}$ of different structures over the reals $\mathbb{R}$, in particular, elementary extensions $^*\mathbb{V}$ of universes $\mathbb{V}$ over $\mathbb{R}$.

Such an extension $^*\mathbb{V}$ accordingly contains an extension $^*\mathbb{R}$ of $\mathbb{R}$.

Any such an extension $^*\mathbb{R}$ is called hyperreals.

Each $^*\mathbb{R}$ is a rcof (or rcoF) and (except for trivialities) a nonarchimedean one.

$^*\mathbb{V}$ is a compatible universe over $^*\mathbb{R}$. 

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Nonstandard analysis (Robinson) studies elementary extensions $^*\mathbb{V}$ of different structures over the reals $\mathbb{R}$, in particular, elementary extensions $^*\mathbb{V}$ of Universes $\mathbb{V}$ over $\mathbb{R}$. Such an extension $^*\mathbb{R}$ accordingly contains an extension $^*\mathbb{R}$ of $\mathbb{R}$. Any such an extension $^*\mathbb{R}$ is called hyperreals. Each $^*\mathbb{R}$ is a rcof (or rcoF) and (except for trivialities) a nonarchimedean one. $^*\mathbb{V}$ is a compatible Universe over $^*\mathbb{R}$. 
Nonstandard analysis (Robinson) studies elementary extensions \( \star V \) of different structures over the reals \( \mathbb{R} \), in particular, elementary extensions \( \star V \) of Universes \( V \) over \( \mathbb{R} \).

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1. Such an extension $\mathcal{V}$ accordingly contains an extension $\mathcal{R}$ of $\mathbb{R}$.

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Nonstandard analysis (Robinson) studies elementary extensions \( *V \) of different structures over the reals \( \mathbb{R} \), in particular, elementary extensions \( *V \) of Universes \( V \) over \( \mathbb{R} \).

1. Such an extension \( *V \) accordingly contains an extension \( *R \) of \( \mathbb{R} \).

2. Any such an extension \( *R \) is called hyperreals.

3. Each \( *R \) is a rcof (or rcoF) and (except for trivialities) a nonarchimedean one.

4. \( *V \) is a compatible Universe over \( *R \).
Elementary extensions \( \mathcal{V} \) of the ZFC set universe \( V \) can be obtained as ultrapowers or limit ultrapowers of \( V \).

**Theorem (K & Shelah 2004)**

There exists a limit ultrapower \( \mathcal{V} \) of \( V \) such that

1. the corresponding hyperreal line \( \mathcal{R} \in \mathcal{V} \) is set-size-dense,
2. \( \mathcal{V} \) is an elementary extension of the universe \( V \), and
3. \( \mathcal{V} \) is a compatible Universe over \( \mathcal{R} \).

This theorem leads to the following foundational system, solving the Problem of upgrade of the surreals, and the Problem of foundations of infinitesimal calculus.
Set-size-dense nonstandard extensions

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Elementary extensions $^*V$ of the ZFC set universe $V$ can be obtained as ultrapowers or limit ultrapowers of $V$.

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- $^*V$ is set-size-dense,
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This theorem leads to the following foundational system, solving the Problem of upgrade of the surreals, and the Problem of foundations of infinitesimal calculus.
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Elementary extensions $^*V$ of the $\text{ZFC}$ set universe $V$ can be obtained as \textit{ultrapowers} or \textit{limit ultrapowers} of $V$.

\textbf{Theorem (K & Shelah 2004)}

\textit{There exists a limit ultrapower $^*V$ of $V$ such that}

1. the corresponding hyperreal line $^*\mathbb{R} \in ^*V$ is \textit{set-size-dense},

2. $^*V$ is an elementary extension of the universe $V$, and
Elementary extensions $\mathbb{V}$ of the ZFC set universe $\mathbb{V}$ can be obtained as ultrapowers or limit ultrapowers of $\mathbb{V}$.

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There exists a limit ultrapower $\mathbb{V}$ of $\mathbb{V}$ such that

1. the corresponding hyperreal line $\mathbb{R} \in \mathbb{V}$ is set-size-dense,
2. $\mathbb{V}$ is an elementary extension of the universe $\mathbb{V}$, and
3. $\mathbb{V}$ is a compatible Universe over $\mathbb{R}$. 

This theorem leads to the following foundational system, solving the Problem of upgrade of the surreals, and the Problem of foundations of infinitesimal calculus.
Elementary extensions $^\ast V$ of the ZFC set universe $V$ can be obtained as ultrapowers or limit ultrapowers of $V$.

**Theorem (K & Shelah 2004)**

There exists a limit ultrapower $^\ast V$ of $V$ such that

1. the corresponding hyperreal line $^\ast \mathbb{R} \in ^\ast V$ is set-size-dense,
2. $^\ast V$ is an elementary extension of the universe $V$, and
3. $^\ast V$ is a compatible Universe over $^\ast \mathbb{R}$.

This theorem leads to the following foundational system, solving...
Set-size-dense nonstandard extensions

Elementary extensions \( *V \) of the ZFC set universe \( V \) can be obtained as ultrapowers or limit ultrapowers of \( V \).

Theorem (K & Shelah 2004)

There exists a limit ultrapower \( *V \) of \( V \) such that

1. the corresponding hyperreal line \( *\mathbb{R} \in *V \) is set-size-dense,
2. \( *V \) is an elementary extension of the universe \( V \), and
3. \( *V \) is a compatible Universe over \( *\mathbb{R} \).

This theorem leads to the following foundational system, solving the Problem of upgrade of the surreals, and
Elementary extensions $\mathcal{V}$ of the ZFC set universe $\mathcal{V}$ can be obtained as ultrapowers or limit ultrapowers of $\mathcal{V}$.

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This theorem leads to the following foundational system, solving

- the Problem of upgrade of the surreals, and
- the Problem of foundations of infinitesimal calculus.
Superstructure over the surreals

\[ F_\infty \]
Superstructure over the surreals

\[ F_\infty \]

surreals
Superstructure over the surreals

\[ F_\infty \]

surreals

a nicely defined rcoF

Back
Superstructure over the surreals

A problem

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Superstructure over the surreals

surreals

F_∞

a nicely defined rcoF

set-size-dense hyperreals

*R

QED

A problem

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Superstructure over the surreals

- $F_\infty$
- $\text{surreals}$
- A nicely defined $rcoF$
- $*\mathbb{R}$
- $\text{set-size-dense hyperreals}$
- admit a compatible Universe

QED

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Superstructure over the surreals

$F_{\infty}$

surreals

a nicely defined rcoF

set-size-dense hyperreals

admit a compatible Universe

QED
Superstructure over the surreals

F_\infty

surreals

a nicely defined rcoF

Back

set-size-dense hyperreals

admit a compatible Universe

*V

*\mathbb{R}

QED

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Superstructure over the surreals

- \( F_{\infty} \)
- \( \text{surreals} \)
- A nicely defined \( \text{rcoF} \)
- \( \text{Back} \)
- \( \text{set-size-dense hyperreals} \)
- \( \text{induced by } H \)
- \( \text{isomorphic under } \text{Global Choice} \)
- \( \text{as two } \text{set-size-dense } \text{rcoF} \)
- \( \text{admit a compatible Universe} \)

QED
Superstructure over the surreals

consider an isomorphism

\[ H : \ast \mathbb{R} \to F_\infty \]

surreals

\( \ast \mathbb{R} \)

\( \ast \mathbb{V} \)

set-size-dense hyperreals

isomorphic under Global Choice as two set-size-dense \( \text{rcoF} \)

admit a compatible Universe

QED
consider an isomorphism

\[ H : \mathcal{F}_\infty \rightarrow F_\infty \]

surreals

a nicely defined $rcoF$

Back

set-size-dense hyperreals

isomorphic under Global Choice as two set-size-dense $rcoF$

admit a compatible Universe

QED
Superstructure over the surreals

isomorphism $H$ induces a Universe over $F_\infty$

set-size-dense hyperreals

a nicely defined $rcoF$

isomorphic under Global Choice as two set-size-dense $rcoF$

admit a compatible Universe
Superstructure over the surreals

Isomorphism $H$ induces a Universe over $F_\infty$

Induced by $H$

Surreals

Set-size-dense hyperreals

A problem definable non-definable

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Superstructure over the surreals

isomorphism $H$ induces a Universe over $F_\infty$

**a compatible Universe over $F_\infty$**

**surreals**

isomorphic under Global Choice as two set-size-dense rcoF

**set-size-dense hyperreals**

admit a compatible Universe

\[ \text{sdf60 2013 25 / 35} \]
Superstructure over the surreals

A problem...
Superstructure over the surreals

A problem

A compatible Universe over $F_\infty$

QED

induced by $H$

$\ast V$

$\ast \mathbb{R}$

set-size-dense
hyperreals

surreals

a nicely defined $rcoF$

Back

isomorphic under Global Choice as two set-size-dense $rcoF$

admit a compatible Universe
Superstructure over the surreals

A problem

definable

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isomorphic under Global Choice as two set-size-dense $rcoF$

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surreals

a nicely defined $rcoF$

QED

Back

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Superstructure over the surreals

A problem

A problem

non-definable

definable

\[ a \text{ compatible Universe over } F_{\infty} \]

\[ \text{surreals} \]

\[ \text{a nicely defined } rcoF \]

\[ \text{set-size-dense hyperreals} \]

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\[ \text{induced by } H \]

\[ \text{induced by } H \]

\[ \text{QED} \]

\[ F_{\infty} \]

\[ \text{Back} \]
Problems

Observation
At the moment, the isomorphism \( H \) between \( F_\infty \) and \( \ast R \) can be obtained only using the Global Choice axiom GC. Accordingly, both the isomorphism \( H \), and the induced Universe over the surreals \( F_\infty \) are non-definable.

Problem 1
Is there a direct construction of \( H \), w/o appeal to GC?

Problem 2
Is there a definable (OD) compatible Universe over \( F_\infty \)?
Problems

Observation

At the moment, the isomorphism $H$ between $F_\infty$ and $_*R$ can be obtained only using the Global Choice axiom $GC$. Accordingly, both the isomorphism $H$, and the induced Universe over the surreals $F_\infty$ are non-definable.

Problem 1
Is there a direct construction of $H$, w/o appeal to $GC$?

Problem 2
Is there a definable (OD) compatible Universe over $F_\infty$?
At the moment, the isomorphism $H$ between $F_{\infty}$ and $\ast \mathbb{R}$ can be obtained only using the Global Choice axiom $\text{GC}$. Accordingly,
Observation

At the moment, the isomorphism $H$ between $\mathbb{F}_\infty$ and $\ast \mathbb{R}$ can be obtained only using the Global Choice axiom $\text{GC}$. Accordingly,

- both the isomorphism $H$, 

---

Problems
At the moment, the isomorphism $H$ between $F_\infty$ and $^*(-\mathbb{R})$ can be obtained only using the Global Choice axiom $GC$. Accordingly,

- both the isomorphism $H$, and
- the induced Universe over the surreals $F_\infty$
Problems

Observation
At the moment, the isomorphism $H$ between $F_\infty$ and $^*\mathbb{R}$ can be obtained only using the \textbf{Global Choice axiom} GC. Accordingly,

- both the \textbf{isomorphism $H$}, and
- the \textbf{induced Universe} over the surreals $F_\infty$

are \textbf{non-definable}.
Observation

At the moment, the isomorphism $H$ between $F_\infty$ and $^\ast\mathbb{R}$ can be obtained only using the Global Choice axiom GC. Accordingly,

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Problem
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1. Is there a direct construction of $H$, w/o appeal to $\text{GC}$? A
2. Is there a definable (OD) compatible Universe over $F_\infty$?
Problems

Problem

Is there an OD isomorphism between the Conway and the Alling surreals?


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Problem

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Uniqueness of set-size-dense rcoF modulo isomorphism

**Theorem (Alling 1961, 1985, on the base of Hausdorff 1907)**

*Assuming the Global Choice axiom, any two set-size-dense rcoF are isomorphic, and hence a set-size-dense rcoF is unique (mod isomorphism) if exists.*
Theorem (Alling 1961, 1985, on the base of Hausdorff 1907)

Assuming the Global Choice axiom, any two set-size-dense \( rcoF \) are isomorphic, and hence a set-size-dense \( rcoF \) is unique (mod isomorphism) if exists.

Proof

Use a back-and-forth type argument.
Theorem (Alling 1961, 1985, on the base of Hausdorff 1907)

Assuming the Global Choice axiom, any two set-size-dense rcoF are isomorphic, and hence

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Proof

Use a back-and-forth type argument.
Digression: classes

A field (a group, order, etc.) is a field (resp., group, ordered domain, etc.) whose underlying domain is a proper class.

A rcoF is a rcof whose underlying domain is a proper class.
A **Field** (a **Group**, **Order**, *etc.*) is a field (resp., group, ordered domain, *etc.*) whose underlying domain is a proper class.
Definition (capitalization of classes)

1. A Field (a Group, Order, etc.) is a field (resp., group, ordered domain, etc.) whose underlying domain is a proper class.

2. A rcoF is a rcof whose underlying domain is a proper class.

Back to Surreals
Universes

A Universe \( F \) over a Structure (set or class) \( V \) is a Model (set or class) of ZFC, containing \( F \) as a set.

A Universe \( V \) over a set \( F \) is compatible, iff it is true in \( V \) that \( F \) is an archimedean set.

Remark: The universe of all sets \( V \) is a compatible Universe over the reals \( R \).

But it is not clear at all how to define a compatible Universe over a non-archimedean set \( F \).

Back to the surreals problem.
### Definition (universes)

A **universe** over a structure $F$ is a model $V$ of ZFC, containing $F$ as a set.

A universe $V$ over a structure $F$ is compatible, iff it is true in $V$ that $F$ is an archimedean structure.

Remark: The universe of all sets $V$ is a compatible universe over the reals $R$.

But it is not clear at all how to define a compatible universe over a non-archimedean structure $F$.
A **Universe** over a Structure (set or class) $F$ is a Model (set or class) $V$ of ZFC, containing $F$ as a set.
Universes

Definition (universes)

- A **Universe** over a Structure (set or class) $F$ is a Model (set or class) $V$ of ZFC, containing $F$ as a set.

- A Universe $V$ over a rco$F$ $F$ is **compatible**,
A **Universe** over a Structure (set or class) $F$ is a Model (set or class) $V$ of $\text{ZFC}$, containing $F$ as a set.

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Definition (universes)

- A **Universe** over a Structure (set or class) $F$ is a Model (set or class) $V$ of ZFC, containing $F$ as a set.

- A Universe $V$ over a $rcoF$ $F$ is **compatible**, iff it is true in $V$ that $F$ is an archimedean $rcof$.

Remark

The universe of all sets $V$ is a compatible Universe over the reals $\mathbb{R}$.
Definition (universes)

- A **Universe** over a Structure (set or class) $F$ is a Model (set or class) $V$ of ZFC, containing $F$ as a set.

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Remark

The universe of all sets $V$ is a compatible Universe over the reals $\mathbb{R}$. But it is not clear at all how to define a compatible Universe over a non-archimedean $\text{rcoF} F$. 
The **Global Choice axiom** $\text{GC}$ asserts that there is a Function (a proper class!) $G$ such that
- the domain $\text{dom } G$ consists of all sets, and
- $G(x) \in x$ for all $x \neq \emptyset$. 

**Remark**

$\text{GC}$ definitely exceeds the capacities of the ordinary set theory $\text{ZFC}$. However, $\text{GC}$ is rather innocuous, in the sense that any theorem provable in $\text{ZFC} + \text{GC}$ and saying something only on sets (not on classes) is provable in $\text{ZFC}$ alone.
Global Choice

Definition

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This question answers **in the negative**, by the following theorem.

**Theorem**

1. There is no definable \( \text{ZFC} \)-provable even **bijection** between:
   - the underlying domain of \( F_\infty \) (in the Alling version), and
   - the underlying domain of the Universe \( {}^*V \) of the K-Shelah theorem.

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This question answers in the negative, by the following theorem.

**Theorem**

1. There is no definable ZFC-provable even bijection between:
   - the underlying domain of $F_\infty$ (in the Alling version), and
   - the underlying domain of the Universe $^*V$ of the K-Shelah theorem.

2. But, there is a definable ZFC-provable injection from the underlying domain of $F_\infty$ to the underlying domain of $^*V$.

Back to problems
Hausdorff’s early papers


Hausdorff’s early papers


The early papers of Hausdorff have been reprinted and commented in

Hausdorff’s early papers

1. F. Hausdorff, Untersuchungen über Ordnungstypen IV, V. 
   Ber. über die Verhandlungen der Königlich Sächsische Gesellschaft der 

2. F. Hausdorff, Die Graduierung nach dem Endverlauf. 
   Abhandlungen der Königlich Sächsische Gesellschaft der 

The early papers of Hausdorff have been reprinted and commented in

3. F. Hausdorff, Gesammelte Werke, Band IA: Allgemeine 

And translated and commented in

4. F. Hausdorff, Hausdorff on ordered sets, Translated, edited, and 
   commented by J. M. Plotkin. AMS and LMS, 2005.
Density and saturation

Remark
For orders and $\text{rcof}$ of type $\eta_0$ (= simply dense) being $\eta_\alpha$ is equivalent to $\aleph_\alpha$-saturation.