Strongly minimal theories with computable models

Uri Andrews and Julia F. Knight

July 10, 2013

Motivation

In computable model theory, there is work on the algorithmic complexity of the models of a given elementary first order theory.

Guiding principle. For a theory that is well-behaved from the point of view of model theory, it should be easier to understand the complexity of the models.

Computable models. We consider models with universe a subset of ω. We identify a model $\mathcal{M}$ with its atomic diagram. So, $\mathcal{M}$ is *computable* if $D(\mathcal{M})$, identified with a subset of ω, is computable.

$\aleph_0$-categorical theories

Theorem (Lerman-Schmerl). If T is an arithmetical $\aleph_0$-categorical theory and for $n \geq 1$, $T \cap \exists_{n+1}$ is Σ^0_n, then T has a computable model.

Theorem (K). If T is an $\aleph_0$-categorical theory and $T \cap \exists_{n+1}$ is Σ^0_n uniformly in n, then T has a computable model.

Complicated $\aleph_0$-categorical theories

Theorem (Khoussainov-Montalbán). There is a non-arithmetical $\aleph_0$-categorical theory with a computable model.

(The proof uses "Marker extensions", which makes the language infinite.)

Theorem (Andrews). There is such a theory in a finite language.

Strongly minimal theories

Definition. A theory T is *strongly minimal* if for every model $\mathcal{M}$, and every formula $\varphi(\overline{a}, x)$ with parameters $\overline{a}$ in $\mathcal{M}$, exactly one of $\varphi^{\mathcal{M}}(\overline{a}, x)$, $\neg\varphi^{\mathcal{M}}(\overline{a}, x)$ is infinite.

Familiar examples.

1. the theory of $(\mathbb{Z}, S)$
2. the theory of infinite $\mathbb{Q}$-vector spaces
3. the theory of the field $\mathbb{C}$ of complex numbers

Algebraic closure and dimension

Definition. Let T be a strongly minimal theory, and let $\mathcal{M}$ be a model.

1. The *algebraic closure* of S, denoted by $acl_{\mathcal{M}}(S)$, is the union of the finite sets $\varphi^{\mathcal{M}}(\overline{c}, x)$ definable in $\mathcal{M}$ with parameters $\overline{c}$ in S.
2. A set $I \subseteq \mathcal{M}$ is *algebraically independent* if for all $i \in I$, $i \notin acl_{\mathcal{M}}(\{I - \{i\})$.

Remark. Algebraic closure gives a well-defined notion of dimension. Each model of T is determined, up to isomorphism, by its dimension.

Trivial strongly minimal theories

Definition. A strongly minimal theory is *trivial* if for all models $\mathcal{M}$ and $S \subseteq \mathcal{M}$, $acl_{\mathcal{M}}(S) = \cup_{s \in S} acl_{\mathcal{M}}(\{s\})$.

Theorem (Goncharov-Harizanov-Laskowski-Lempp-McCoy). Every trivial strongly minimal theory with a computable model is Δ^0_3.

(It follows that all models have Δ^0_3 copies.)

Complicated theories with computable models

Goncharov-Khoussainov, Fokina. For each n, there is an $\aleph_1$-categorical theory T s.t. T has a computable model, and T is not Δ^0_n.

(The proof uses Marker extensions, so the language is infinite and the theory is not strongly minimal.)

Andrews. There is a non-arithmetical strongly minimal theory with a computable model.

New results

Main Theorem (Andrews-K). Let T be a strongly minimal theory in a relational language. If $T \cap \exists_{n+3}$ is Δ^0_n, uniformly in n, then every model of T has a computable copy.

Relativizing to $\emptyset^{(4)}$, we get the following.

Corollary. If T has a computable model $\mathcal{M}$, then every model has a copy computable in $\emptyset^{(4)}$ (i.e., Δ^0_5).

Proof.

Since there is a computable model, $T \cap \exists_n$ is Δ^0_{n+1}, uniformly.
Then $T \cap \exists_{n+3}$ is Δ^0_{n+4}, which is Δ^0_n relative to $\emptyset^{(4)}$. □

Cases

The proof of the Main Theorem splits into cases, according to whether the theory T is arithmetical, and whether the model $\mathcal{M}$ that we are copying is saturated, or has a "bounded" saturation property.

1. T is arithmetical, and $\mathcal{M}$ is boundedly saturated.
2. T is not arithmetical and $\mathcal{M}$ is saturated.
3. T is not arithmetical and $\mathcal{M}$ has dimension k for some finite k, but is boundedly saturated.
4. T is arithmetical, and $\mathcal{M}$ is not boundedly saturated
5. T is not arithmetical, and $\mathcal{M}$ is not boundedly saturated

Bounded types and bounded saturation

Definition.

1. An *n-formula* is a Boolean combination of $\exists_n$-formulas.
2. An *n-type* is the set of n-formulas in a complete type.
3. A model $\mathcal{A}$ is *n-saturated* if for all $\overline{a}$, every n-type $p(\overline{a}, x)$ consistent with the type of $\overline{a}$ is realized.

Enumerations of n-types

We need enumerations R^n of the n-types.

Lemma. There is a family $(R^n)_{n \geq 1}$ of enumerations of the n-types s.t. R^1 is computable, and for $n \geq 2$, R^n is Δ^0_{n-1}, uniformly in n.

Morley rank

Definition.

1. The Morley *rank of a formula* is the maximum dimension of a tuple satisfying the formula.

2. The *Morley rank of a type* is the minimum of the ranks of the formulas in the type.

Definability

Remarks. For each formula $\varphi(\overline{u}, x)$, there is some k s.t. for any model $\mathcal{A}$ and any $\overline{a}$ in $\mathcal{A}$, only one of $\varphi^{\mathcal{A}}(\overline{a}, x)$, $\neg\varphi^{\mathcal{A}}(\overline{a}, x)$ has size $\geq k$. Then

$$\mathcal{A} \models (\exists^{\geq k} x)\varphi(\overline{a}, x) \rightarrow (\exists^{\infty} x)\varphi(\overline{a}, x)$$

For an n-formula $\varphi(\overline{u}, x)$, we find the appropriate k as above using $T \cap \exists_{n+1}$. For a $\exists_{n+1}$ formula, we can find the appropriate k using $T \cap \exists_{n+2}$.

Rank and the enumerations of types

Lemma. For an n-formula $\varphi(\overline{x})$, using $T \cap \exists_{n+2}$, we can find $\exists_{n+1}$ formulas saying that $\varphi(\overline{x})$ has rank at least k. Then using $T \cap \exists_{n+2}$, we can determine the rank.

For the enumeration R^n in Lemma 1, for each $\overline{x}$, we list the type of full rank first. When we see a split, with one side having lower rank, the current index stays with the type of higher rank, and we add a new index for the type of lower rank.

Labeled models

Definition. Let $\mathcal{M}$ be a model of T with universe ω. The R^n-*labeling* for $\mathcal{M}$ is the function taking each tuple $\overline{a}$ in $\mathcal{M}$ to the R^n-index for the type.

Case 1—T is arithmetical and $\mathcal{M}$ is boundedly saturated

Lemma 1 (Harrington, Khisamiev). Suppose T is strongly minimal. If T is Δ^0_N, then every model of T has a copy whose complete diagram is Δ^0_N.

Lemma 2. Suppose R^N is Δ^0_N. If $\mathcal{M}$ is a model whose complete diagram is Δ^0_N, then the R^N-labeling of $\mathcal{M}$ is Δ^0_{N+1}.

Working our way down

Lemma 3. For $n \geq 2$, if $\mathcal{M}$ is an n-saturated model with a Δ^0_{n+1} R^n-labeling, then there is a copy with a Δ^0_n R^{n-1}-labeling.

Lemma 4. If $\mathcal{M}$ is a 1-saturated model of T with a Δ^0_2 R^1-labeling, then there is a computable copy.

In the proofs of Lemmas 3 and 4, bounded saturation helps us map elements of the copy we are building to elements of the given model.

Putting the pieces together for Case 1

Suppose T is Δ^0_N, and $\mathcal{M}$ is N-saturated. First, we apply Lemmas 1 and 2 to get a copy of $\mathcal{M}$ with a Δ^0_{N+1} R^N-labeling. Then we work our way down, applying Lemma 3 until we have a copy with a Δ^0_2 R^1-labeling. Finally, we apply Lemma 4 to get a computable copy.

Case 2—T is not arithmetical and $\mathcal{M}$ is saturated

We build a copy $\mathcal{A}$ of the saturated model "on the diagonal"; i.e., the Δ_3^0 worker carries out the first step, assigning a 3-type to a first element, the Δ_4^0 worker carries out the second step, assigning a 4-type to the first two elements, etc. At even steps, the new element is designated as generic—helping to build the saturated model, and at odd steps, the new element is a witness as in the standerd Henkin construction.

For each $n \geq 3$, after contributing to the diagonal, the Δ_n^0 worker goes into guessing mode, giving an R^{n-1}-labeling for a structure $\mathcal{B}_n$, based on guesses at the R^n-labeling produced by the Δ_{n+1}^0 worker. It takes effort to show that the $\mathcal{B}_n$ are all isomorphic, and that they are isomorphic to $\mathcal{A}$.

Models that are not boundedly saturated

If $\mathcal{M}$ is not n-saturated, we have a tuple $\overline{a}$ with an n-type $p(\overline{a}, x)$ that is consistent with the type of $\overline{a}$ but is not realized in $\mathcal{M}$. The type $p(\overline{a}, x)$ is the type of an n-generic over $\overline{a}$. Every element b satisfies some algebraic n-formula $\psi(\overline{a}, x)$. We can use this, for Cases 4 and 5, in the same way that we used bounded saturation.