

# Local Computability and the Ordinal $\omega_1^{CK} \cdot \omega$

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# Local Descriptions of Structures

## Defn.

A *simple cover*  $\mathfrak{A}$  of a structure  $\mathcal{S}$  is a set containing all finitely generated substructures of  $\mathcal{S}$ , up to isomorphism (and nothing else!). Repetitions are allowed.

$\mathfrak{A}$  is *computable* if every  $\mathcal{A} \in \mathfrak{A}$  is a computable structure.

$\mathfrak{A}$  is *uniformly computable* if there is a single algorithm listing out all  $\mathcal{A}_i$  in  $\mathfrak{A}$ .

When  $\mathcal{S} = \mathcal{L}$  is a linear order, this is trivial: every infinite linear order is locally computable, with a cover containing one copy of each finite linear order.

# Embeddings

Let  $\mathcal{L}$  be locally computable, with simple cover  $\{\mathcal{A}_0, \mathcal{A}_1, \dots\}$ . Suppose  $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{L}$  are finite. If

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\subseteq} & \mathcal{C} \\ \beta \uparrow \cong & & \uparrow \cong \gamma \\ \mathcal{A}_i & \xrightarrow{f} & \mathcal{A}_j \end{array}$$

commutes, we say that  $f : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$  *lifts to the inclusion*  $\mathcal{B} \subseteq \mathcal{C}$  via the isomorphisms  $\beta$  and  $\gamma$ .

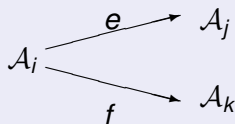
For linear orders, if  $\mathcal{B} = \{a < d < e\} \subseteq \mathcal{C} = \{a < b < c < d < e\}$ , and  $\mathcal{A}_i = \{x_0 < x_1 < x_2\}$  and  $\mathcal{A}_j = \{y_0 < y_1 < y_2 < y_3 < y_4\}$ , then take  $f(x_0) = y_0$ ,  $f(x_1) = y_3$ ,  $f(x_2) = y_4$ .

# Computable Covers

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A *cover* of  $\mathcal{L}$  comprises a simple cover  $\mathfrak{A}$ , along with sets  $I_{ij}^{\mathfrak{A}}$  of embeddings  $\mathcal{A}_i \hookrightarrow \mathcal{A}_j$ , such that:

- every  $f \in I_{ij}^{\mathfrak{A}}$  lifts to an inclusion  $\mathcal{B} \subseteq \mathcal{C}$  within  $\mathcal{L}$ ,
- every inclusion within  $\mathcal{L}$  is the lift of some  $f$  in some  $I_{ij}^{\mathfrak{A}}$ ,
- and the *Amalgamation Property* holds for all  $i, j, k, e$ , and  $f$ :

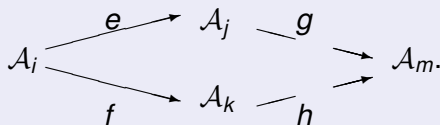


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The cover is (*uniformly*) *computable* if all  $I_{ij}^{\mathfrak{A}}$  are c.e. uniformly in  $i$  and  $j$ . In this case,  $\mathcal{L}$  is said to be *locally computable*.

For linear orders, use all maps of each smaller  $\mathcal{A}_i$  into each larger  $\mathcal{A}_j$ . So every linear order is locally computable.

# Extensionality

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Every embedding from any  $\mathcal{A}_i$  into  $\mathcal{L}$  is 0-extensional. An isomorphism  $\beta : \mathcal{A}_i \rightarrow \mathcal{B} \subseteq \mathcal{L}$  is  $(m+1)$ -extensional if

- $(\forall j)(\forall f \in I_{ij}^{\alpha}) (\exists \mathcal{C} \subseteq \mathcal{L}) [f \text{ lifts to } \mathcal{B} \subseteq \mathcal{C} \text{ via } \beta \text{ and some } \gamma]$ ; and
- $(\forall \text{ finite } \mathcal{D} \supseteq \mathcal{B}) (\exists k) (\exists g \in I_{ik}^{\alpha}) [f \text{ lifts to } \mathcal{B} \subseteq \mathcal{C} \text{ via } \beta \text{ and some } \gamma]$

with  $\gamma$   $m$ -extensional in both cases:

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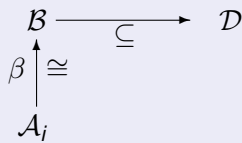
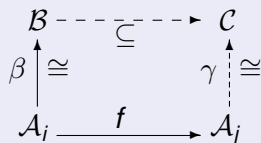
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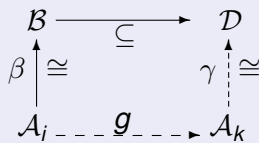
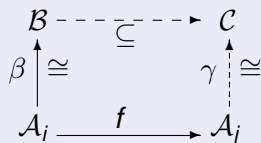
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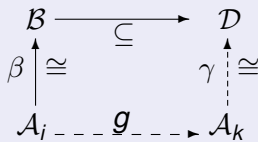
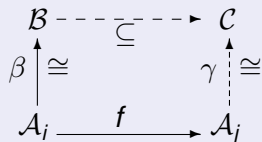
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$\mathcal{L}$  is  $m$ -extensional if it has a cover  $\mathfrak{A}$  s.t. every  $\mathcal{A}_i \in \mathfrak{A}$  is the domain of an  $m$ -extensional map and every finite  $\mathcal{B} \subseteq \mathcal{L}$  is the image of one.

Intuition: A 1-extensional map  $\beta$  is a more exact description of  $\mathcal{B}$  by  $\mathcal{A}_i$ : the ways  $\mathfrak{A}$  can extend  $\mathcal{A}_i$  are exactly the ways of extending  $\mathcal{B}$  within  $\mathcal{L}$ .

## Example with Ordinals

For a computable cover of the linear order  $\mathcal{L} = (\omega, <)$ , we can take all finite linear orders, with all order-embeddings among them.

For a 1-extensional (computable) cover of  $\omega$ , the suborders  $\mathcal{B}_1 = \{0 < 2 < 4\}$  and  $\mathcal{B}_2 = \{1 < 5 < 6\}$  of  $\omega$  cannot both have 1-extensional maps from the same  $\mathcal{A}_i$ . To cover  $\mathcal{B}_1$  1-extensionally,  $\mathcal{A}_i = \{x_0 < x_1 < x_2\}$  would have to have an embedding  $f$  into an  $\mathcal{A}_i$  which has an element in between  $f(x_1)$  and  $f(x_2)$ . But if this  $\mathcal{A}_i$  were also a 1-extensional cover of  $\mathcal{B}_2$ , say via  $\gamma$ , then this  $f$  would indicate that  $\omega$  contains an element in between  $\gamma(x_1) = 5$  and  $\gamma(x_2) = 6$ .

1-extensionality allows every  $\Sigma_1$  fact (with parameters) about  $\mathcal{L}$  to be expressed using the cover and its embeddings. A 1-extensional cover of  $\omega$  is not hard to build, but it must have a different  $\mathcal{A}_i$  for each finite suborder of  $\omega$ . (In fact, every computable LO  $\mathcal{L}$  has a *canonical cover*, containing all finite suborders of  $\mathcal{L}$ , with inclusion maps.)

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Let  $\mathcal{S}$  be the linear order  $\omega + \omega$ .

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2-extensional cover  $\mathfrak{A}''$ : all finite linear orders  $\mathcal{L}$ , with a computable function giving the order type of each interval  $(x, y)$  in each  $\mathcal{L}$ . Again, embeddings must respect this rule. This cover is  $m$ -extensional for every  $m$ .

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$\mathfrak{A}_0$  does not distinguish any elements within  $\mathcal{S}$ ; it is a 0-extensional cover of every infinite LO.  $\mathfrak{A}_1$  is a 1-extensional cover of  $(\omega + \mathbb{Z} \cdot \lambda)$ , for every nonempty LO  $\lambda$ .  $\mathfrak{A}_2$  is a 2-extensional cover of no LO except  $\mathcal{S}$ .

## Even More Extensionality

For limit  $\theta$ , a map  $\gamma : \mathcal{A}_i \rightarrow \mathcal{B}$  is  $\theta$ -*extensional* if it is  $\zeta$ -extensional for every ordinal  $\zeta < \theta$ . For successors  $\theta + 1$ , use the definition for  $m + 1$ .

### Theorem

Suppose  $\mathcal{L}$  has a  $\theta$ -extensional cover, with  $\theta < \omega_{CK}^1$ . Then for any finite set  $P$  of parameters in  $\mathcal{L}$  and  $(\forall \zeta \leq \theta)$ , the  $\Sigma_\zeta$ -theory of  $(\mathcal{L}, P)$  is arithmetically  $\Sigma_\zeta^0$ , uniformly in  $i$  and  $\gamma^{-1}(P)$ , where  $\gamma : \mathcal{A}_i \rightarrow \langle P \rangle$  is  $\theta$ -extensional.

Finally, a cover  $\mathfrak{A}$  of  $\mathcal{L}$  is  $\infty$ -*extensional* if there is a set  $M$  of maps  $\gamma : \mathcal{A}_i \rightarrow \mathcal{L}$ , from each  $\mathcal{A}_i$  into  $\mathcal{L}$  and with each finite  $\mathcal{C} \subseteq \mathcal{L}$  as a range, such that each  $f$  with domain  $\mathcal{A}_i$  lifts to an inclusion lifts via  $\gamma$  and some other  $\delta \in M$ , and each inclusion of  $\text{rg}(\gamma)$  lifts some  $f$  via  $\gamma$  and some other  $\delta \in M$ .



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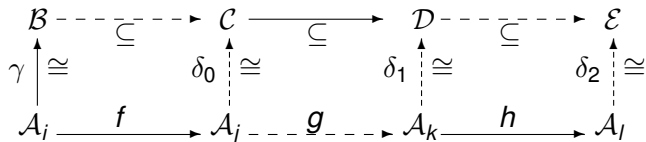
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For limit  $\theta$ , a map  $\gamma : \mathcal{A}_i \rightarrow \mathcal{B}$  is  $\theta$ -*extensional* if it is  $\zeta$ -*extensional* for every ordinal  $\zeta < \theta$ . For successors  $\theta + 1$ , use the definition for  $m + 1$ .

### Theorem

Suppose  $\mathcal{L}$  has a  $\theta$ -*extensional* cover, with  $\theta < \omega_{CK}^1$ . Then for any finite set  $P$  of parameters in  $\mathcal{L}$  and  $(\forall \zeta \leq \theta)$ , the  $\Sigma_\zeta$ -theory of  $(\mathcal{L}, P)$  is arithmetically  $\Sigma_\zeta^0$ , uniformly in  $i$  and  $\gamma^{-1}(P)$ , where  $\gamma : \mathcal{A}_i \rightarrow \langle P \rangle$  is  $\theta$ -*extensional*.

Finally, a cover  $\mathfrak{A}$  of  $\mathcal{L}$  is  $\infty$ -*extensional* if there is a set  $M$  of maps  $\gamma : \mathcal{A}_i \rightarrow \mathcal{L}$ , from each  $\mathcal{A}_i$  into  $\mathcal{L}$  and with each finite  $\mathcal{C} \subseteq \mathcal{L}$  as a range, such that each  $f$  with domain  $\mathcal{A}_i$  lifts to an inclusion lifts via  $\gamma$  and some other  $\delta \in M$ , and each inclusion of  $\text{rg}(\gamma)$  lifts some  $f$  via  $\gamma$  and some other  $\delta \in M$ .



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$$\begin{array}{ccccccc}
 \mathcal{B} & \overset{\subseteq}{\dashrightarrow} & \mathcal{C} & \xrightarrow{\subseteq} & \mathcal{D} & \overset{\subseteq}{\dashrightarrow} & \mathcal{E} & \xrightarrow{\subseteq} & \mathcal{F} \\
 \uparrow \cong & & \uparrow \cong & & \uparrow \cong & & \uparrow \cong & & \\
 \gamma & & \delta_0 & & \delta_1 & & \delta_2 & & \\
 \mathcal{A}_i & \xrightarrow{f} & \mathcal{A}_j & \overset{\subseteq}{\dashrightarrow} & \mathcal{A}_k & \xrightarrow{h} & \mathcal{A}_l & & 
 \end{array}$$



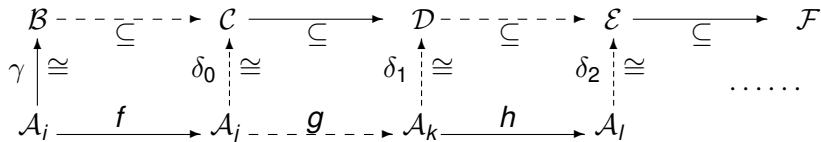
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# The Actual Question

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## Theorem (Miller-Mulcahey, 2008)

Let  $\mathcal{S}$  be a countable structure. Then TFAE:

- $\mathcal{S}$  has an  $\infty$ -extensional computable cover (with the AP);
- $\mathcal{S}$  has an  $\theta$ -extensional computable cover for some  $\theta >$  the Scott rank of  $\mathcal{S}$ ;
- $\mathcal{S}$  is computably presentable ( $\equiv$  there exists a Turing-computable structure isomorphic to  $\mathcal{S}$ ).

## Corollary

Every  $\alpha < \omega_1^{CK}$  has an  $\infty$ -extensional cover.

# Negative Results

## Theorem

$\omega_1^{CK}$  has no  $\omega_1^{CK}$ -extensional computable cover.

Proof: we give a construction which would use an  $\omega_1^{CK}$ -extensional cover to build a computable presentation of  $\omega_1^{CK}$ . (Indeed, this shows that  $\omega_1^{CK}$  cannot have any  $\omega_1^{CK}$ -extensional hyperarithmetical cover.)

## Corollary

No ordinal  $\alpha > \omega_1^{CK}$  has an  $(\omega_1^{CK} + 1)$ -extensional cover.

From a such a cover, fix a singleton  $\mathcal{A}_{i_0} = \{x_0\}$  with an  $(\omega_1^{CK} + 1)$ -extensional map onto the suborder  $\{\omega_1^{CK}\}$ . Then we could build a cover of  $\omega_1^{CK}$  by considering those  $\mathcal{A}_j$  for which

$$(\exists k)(\exists f : \mathcal{A}_{i_0} \rightarrow \mathcal{A}_k)(\exists g : \mathcal{A}_j \rightarrow \mathcal{A}_k)[(\forall y \in \mathcal{A}_j) g(y) < f(x_0)].$$

This would be an  $\omega_1^{CK}$ -extensional cover of  $\omega_1^{CK}$ .

## The Remaining Questions

When does  $\alpha$  have a  $\theta$ -extensional computable cover? Answers so far...

$\theta$					
$\infty$	Y	Y	N	N	N
$\vdots$	Y	Y	N	N	N
$\omega_1^{CK} + 1$	Y	Y	N	N	N
$\omega_1^{CK}$	Y	Y	N	??	??
$\vdots$	Y	Y	??	??	??
0	Y	Y	Y	Y	Y
	0	...	$\omega_1^{CK}$	$\omega_1^{CK} + 1$	... : $\alpha$

Useful observation: every  $\beta < \omega^{\theta+1}$  can be identified by a Boolean combination of computable  $\Sigma_{2,\theta+1}$ -formulas, but  $\omega^{\theta+1}$  itself cannot be.

# A Further Answer

## Theorem

For every computable ordinal  $\theta$ , every ordinal  $\alpha$  has a  $\theta$ -extensional computable cover.

This is not surprising. Roughly speaking, with a  $\theta$ -extensional cover, we cannot distinguish powers  $\omega^\zeta$  for  $\zeta > \theta$ , so just write  $\alpha = \omega^{(\theta+1)} \cdot \mu + \nu$ , with  $\nu < \omega^{(\theta+1)}$ .

$\underline{\theta}$					
$\vdots$	Y	Y	N	N	N
$\omega_1^{CK} + 1$	Y	Y	N	N	N
$\omega_1^{CK}$	Y	Y	N	??	??
$\vdots$	Y	Y	<b>Y</b>	<b>Y</b>	<b>Y</b>
0	Y	Y	Y	Y	Y
	0	...	$\omega_1^{CK}$	$\omega_1^{CK} + 1$	... : $\alpha$

# Another Answer

## Theorem

If  $\omega_1^{CK} \leq \alpha < \omega_1^{CK} \cdot \omega$ , then  $\alpha$  has no  $\omega_1^{CK}$ -extensional cover.

Such an  $\alpha$  contains finitely many multiples of  $\omega_1^{CK}$ . In a cover, we could pick a single  $\mathcal{A}_i$  which  $\omega_1^{CK}$ -extensionally covers all of those multiples. A delicate construction then uses this cover to build a computable presentation of  $\omega_1^{CK}$ , which is impossible.

$\underline{\theta}$							
$\vdots$	Y	Y	N	N	N	N	N
$\omega_1^{CK+1}$	Y	Y	N	N	N	N	N
$\omega_1^{CK}$	Y	Y	N	<b>N</b>	<b>N</b>	??	??
$\vdots$	Y	Y	Y	Y	Y	Y	Y
0	Y	Y	Y	Y	Y	Y	Y
	0	...	$\omega_1^{CK}$	$\omega_1^{CK+1}$	...	$\omega_1^{CK} \cdot \omega$	... : $\alpha$

# The Last Answer

## Surprise Theorem (joint with Julia Knight)

All ordinals  $\alpha \geq \omega_1^{CK} \cdot \omega$  have  $\omega_1^{CK}$ -extensional computable covers.

In fact, the canonical cover of a computable presentation of the Harrison ordering  $\omega_1^{CK} \cdot (1 + \eta)$  is an  $\omega_1^{CK}$ -extensional cover of each order  $\omega_1^{CK} \cdot (1 + \rho)$  with  $\rho$  infinite. Likewise, each such ordinal has an  $\omega_1^{CK}$ -back-and-forth with the Harrison ordering.

$\underline{\theta}$							
$\vdots$	Y	Y	N	N	N	N	N
$\omega_1^{CK} + 1$	Y	Y	N	N	N	N	N
$\omega_1^{CK}$	Y	Y	N	N	N	<b>Y</b>	<b>Y</b>
$\vdots$	Y	Y	Y	Y	Y	Y	Y
0	Y	Y	Y	Y	Y	Y	Y
	0	...	$\omega_1^{CK}$	$\omega_1^{CK} + 1$	...	$\omega_1^{CK} \cdot \omega$	... : $\alpha$



# What More Could We Ask?

....Well, we could ask the same question about  $\kappa$ -recursion theory and  $\kappa$ -local computability, in which we list all substructures generated by subsets of size  $< \kappa$ . Does the same phenomenon occur? At  $(\kappa^+)^{CK \cdot \kappa}$ , or where? Is there a Harrison ordering for  $\kappa$ ? (Is there Barwise compactness for  $\kappa$ ?) Do we need to use  $L_{\kappa\kappa}$  formulas to make this happen?

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Happy Birthday, Sy!