Local Computability and the Ordinal $\omega_1^{CK} \cdot \omega$

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Local Descriptions of Structures

Defn.

A *simple cover* $\mathfrak{A}$ of a structure $\mathcal{S}$ is a set containing all finitely generated substructures of $\mathcal{S}$, up to isomorphism (and nothing else!). Repetitions are allowed.

$\mathfrak{A}$ is *computable* if every $\mathcal{A} \in \mathfrak{A}$ is a computable structure.

$\mathfrak{A}$ is *uniformly computable* if there is a single algorithm listing out all $\mathcal{A}_i$ in $\mathfrak{A}$.

When $\mathcal{S} = \mathcal{L}$ is a linear order, this is trivial: every infinite linear order is locally computable, with a cover containing one copy of each finite linear order.

Embeddings

Let $\mathcal{L}$ be locally computable, with simple cover $\{\mathcal{A}_0, \mathcal{A}_1, \ldots\}$. Suppose $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{L}$ are finite. If

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\subseteq"'] & \mathcal{C} \\ \mathcal{A}_i \arrow[u, "\beta", "\cong"'] \arrow[r, "f"] & \mathcal{A}_j \arrow[u, "\gamma", "\cong"'] \end{tikzcd}$$

commutes, we say that $f : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$ *lifts to the inclusion* $\mathcal{B} \subseteq \mathcal{C}$ via the isomorphisms β and γ.

For linear orders, if $\mathcal{B} = \{a < d < e\} \subseteq \mathcal{C} = \{a < b < c < d < e\}$, and $\mathcal{A}_i = \{x_0 < x_1 < x_2\}$ and $\mathcal{A}_j = \{y_0 < y_1 < y_2 < y_3 < y_4\}$, then take $f(x_0) = y_0$, $f(x_1) = y_3$, $f(x_2) = y_4$.

Computable Covers

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A *cover* of $\mathcal{L}$ comprises a simple cover $\mathfrak{A}$, along with sets $I_{ij}^{\mathfrak{A}}$ of embeddings $\mathcal{A}_i \hookrightarrow \mathcal{A}_j$, such that:

- every $f \in I_{ij}^{\mathfrak{A}}$ lifts to an inclusion $\mathcal{B} \subseteq \mathcal{C}$ within $\mathcal{L}$,
- every inclusion within $\mathcal{L}$ is the lift of some f in some $I_{ij}^{\mathfrak{A}}$,
- and the *Amalgamation Property* holds for all i, j, k, e, and f:

$$\begin{tikzcd} & \mathcal{A}_j \\ \mathcal{A}_i \arrow[ur, "e"] \arrow[dr, "f"'] & \\ & \mathcal{A}_k \end{tikzcd}$$

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$$\begin{tikzcd} & \mathcal{A}_j \arrow[dr, dashed, "g"] & \\ \mathcal{A}_i \arrow[ur, "e"] \arrow[dr, "f"'] & & \mathcal{A}_m. \\ & \mathcal{A}_k \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

The cover is *(uniformly) computable* if all $I_{ij}^{\mathfrak{A}}$ are c.e. uniformly in i and j. In this case, $\mathcal{L}$ is said to be *locally computable*.

For linear orders, use all maps of each smaller $\mathcal{A}_i$ into each larger $\mathcal{A}_j$. So every linear order is locally computable.

Extensionality

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Every embedding from any $\mathcal{A}_i$ into $\mathcal{L}$ is 0-*extensional*. An isomorphism $\beta : \mathcal{A}_i \to \mathcal{B} \subseteq \mathcal{L}$ is $(m+1)$-*extensional* if

- $(\forall j)(\forall f \in I^{\mathfrak{A}}_{ij})(\exists \mathcal{C} \subseteq \mathcal{L})[f$ lifts to $\mathcal{B} \subseteq \mathcal{C}$ via β and some $\gamma]$; and
- $(\forall$ finite $\mathcal{D} \supseteq \mathcal{B})(\exists k)(\exists g \in I^{\mathfrak{A}}_{ik})[f$ lifts to $\mathcal{B} \subseteq \mathcal{C}$ via β and some $\gamma]$

with γ m-extensional in both cases:

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$\mathcal{L}$ is m-extensional if it has a cover $\mathfrak{A}$ s.t. every $\mathcal{A}_i \in \mathfrak{A}$ is the domain of an m-extensional map and every finite $\mathcal{B} \subseteq \mathcal{L}$ is the image of one.

Intuition: A 1-extensional map β is a more exact description of $\mathcal{B}$ by $\mathcal{A}_i$: the ways $\mathfrak{A}$ can extend $\mathcal{A}_i$ are exactly the ways of extending $\mathcal{B}$ within $\mathcal{L}$.

Example with Ordinals

For a computable cover of the linear order $\mathcal{L} = (\omega, <)$, we can take all finite linear orders, with all order-embeddings among them.

For a 1-extensional (computable) cover of ω, the suborders $\mathcal{B}_1 = \{0 < 2 < 4\}$ and $\mathcal{B}_2 = \{1 < 5 < 6\}$ of ω cannot both have 1-extensional maps from the same $\mathcal{A}_i$. To cover $\mathcal{B}_1$ 1-extensionally, $\mathcal{A}_i = \{x_0 < x_1 < x_2\}$ would have to have an embedding f into an $\mathcal{A}_j$ which has an element in between $f(x_1)$ and $f(x_2)$. But if this $\mathcal{A}_i$ were also a 1-extensional cover of $\mathcal{B}_2$, say via γ, then this f would indicate that ω contains an element in between $\gamma(x_1) = 5$ and $\gamma(x_2) = 6$.

1-extensionality allows every Σ_1 fact (with parameters) about $\mathcal{L}$ to be expressed using the cover and its embeddings. A 1-extensional cover of ω is not hard to build, but it must have a different $\mathcal{A}_i$ for each finite suborder of ω. (In fact, every computable LO $\mathcal{L}$ has a *canonical cover*, containing all finite suborders of $\mathcal{L}$, with inclusion maps.)

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Let $\mathcal{S}$ be the linear order $\omega + \omega$.

0-extensional cover $\mathfrak{A}$: all finite linear orders $\mathcal{L}$, with all embeddings.

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1-extensional cover $\mathfrak{A}'$: all finite linear orders $\mathcal{L} = \{x_0 < \cdots < x_n\}$, each with a rule saying, for each $i < n$, how many elements may be placed between x_i and x_{i+1}, and how many to the left of x_0. (At most one pair (x_i, x_{i+1}) may have ∞ many.) All embeddings which respect these rules are allowed.

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2-extensional cover $\mathfrak{A}''$: all finite linear orders $\mathcal{L}$, with a computable function giving the order type of each interval (x, y) in each $\mathcal{L}$. Again, embeddings must respect this rule. This cover is m-extensional for every m.

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$\mathfrak{A}_0$ does not distinguish any elements within $\mathcal{S}$; it is a 0-extensional cover of every infinite LO. $\mathfrak{A}_1$ is a 1-extensional cover of $(\omega + \mathbb{Z} \cdot \lambda)$, for every nonempty LO λ. $\mathfrak{A}_2$ is a 2-extensional cover of no LO except $\mathcal{S}$.

Even More Extensionality

For limit θ, a map $\gamma : \mathcal{A}_i \to \mathcal{B}$ is *θ-extensional* if it is ζ-extensional for every ordinal $\zeta < \theta$. For successors $\theta + 1$, use the definition for $m + 1$.

Theorem

Suppose $\mathcal{L}$ has a θ-extensional cover, with $\theta < \omega_{CK}^1$. Then for any finite set P of parameters in $\mathcal{L}$ and $(\forall\zeta \leq \theta)$, the Σ_ζ-theory of $(\mathcal{L}, P)$ is arithmetically Σ_ζ^0, uniformly in i and $\gamma^{-1}(P)$, where $\gamma : \mathcal{A}_i \to \langle P\rangle$ is θ-extensional.

Finally, a cover $\mathfrak{A}$ of $\mathcal{L}$ is *∞-extensional* if there is a set M of maps $\gamma : \mathcal{A}_i \to \mathcal{L}$, from each $\mathcal{A}_i$ into $\mathcal{L}$ and with each finite $\mathcal{C} \subseteq \mathcal{L}$ as a range, such that each f with domain $\mathcal{A}_i$ lifts to an inclusion lifts via γ and some other $\delta \in M$, and each inclusion of rg(γ) lifts some f via γ and some other $\delta \in M$.

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Theorem

Suppose $\mathcal{L}$ has a θ-extensional cover, with $\theta < \omega_{CK}^1$. Then for any finite set P of parameters in $\mathcal{L}$ and $(\forall \zeta \leq \theta)$, the Σ_ζ-theory of $(\mathcal{L}, P)$ is arithmetically Σ_ζ^0, uniformly in i and $\gamma^{-1}(P)$, where $\gamma : \mathcal{A}_i \to \langle P \rangle$ is θ-extensional.

Finally, a cover $\mathfrak{A}$ of $\mathcal{L}$ is *∞-extensional* if there is a set M of maps $\gamma : \mathcal{A}_i \to \mathcal{L}$, from each $\mathcal{A}_i$ into $\mathcal{L}$ and with each finite $\mathcal{C} \subseteq \mathcal{L}$ as a range, such that each f with domain $\mathcal{A}_i$ lifts to an inclusion lifts via γ and some other $\delta \in M$, and each inclusion of rg(γ) lifts some f via γ and some other $\delta \in M$.

$$\begin{tikzcd}
\mathcal{B} \arrow[r, dashed, "\subseteq"'] & \mathcal{C} \arrow[r, "\subseteq"'] & \mathcal{D} \arrow[r, dashed, "\subseteq"'] & \mathcal{E} \arrow[r, "\subseteq"'] & \mathcal{F} \\
\mathcal{A}_i \arrow[u, "\gamma", "\cong"'] \arrow[r, "f"] & \mathcal{A}_j \arrow[u, dashed, "\delta_0", "\cong"'] \arrow[r, dashed, "g"] & \mathcal{A}_k \arrow[u, dashed, "\delta_1", "\cong"'] \arrow[r, "h"] & \mathcal{A}_l \arrow[u, dashed, "\delta_2", "\cong"'] & \cdots\cdots
\end{tikzcd}$$

The Actual Question

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For an ordinal α, when is there a θ-extensional (computable) cover of the linear order α?

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Theorem (Miller-Mulcahey, 2008)

Let $\mathcal{S}$ be a countable structure. Then TFAE:

- $\mathcal{S}$ has an ∞-extensional computable cover (with the AP);
- $\mathcal{S}$ has an θ-extensional computable cover for some $\theta >$ the Scott rank of $\mathcal{S}$;
- $\mathcal{S}$ is computably presentable ($\equiv$ there exists a Turing-computable structure isomorphic to $\mathcal{S}$).

Corollary

Every $\alpha < \omega_1^{CK}$ has an ∞-extensional cover.

Negative Results

Theorem

ω_1^{CK} has no ω_1^{CK}-extensional computable cover.

Proof: we give a construction which would use an ω_1^{CK}-extensional cover to build a computable presentation of ω_1^{CK}. (Indeed, this shows that ω_1^{CK} cannot have any ω_1^{CK}-extensional hyperarithmetical cover.)

Corollary

No ordinal $\alpha > \omega_1^{CK}$ has an $(\omega_1^{CK} + 1)$-extensional cover.

From a such a cover, fix a singleton $\mathcal{A}_{i_0} = \{x_0\}$ with an $(\omega_1^{CK} + 1)$-extensional map onto the suborder $\{\omega_1^{CK}\}$. Then we could build a cover of ω_1^{CK} by considering those $\mathcal{A}_j$ for which

$$(\exists k)(\exists f : \mathcal{A}_{i_0} \to \mathcal{A}_k)(\exists g : \mathcal{A}_j \to \mathcal{A}_k)[(\forall y \in \mathcal{A}_j)\ g(y) < f(x_0)].$$

This would be an ω_1^{CK}-extensional cover of ω_1^{CK}.

The Remaining Questions

When does α have a θ-extensional computable cover? Answers so far...

$\underline{\theta}$						
∞	Y	Y	N	N	N	
$\vdots$	Y	Y	N	N	N	
$\omega_1^{CK}+1$	Y	Y	N	N	N	
ω_1^{CK}	Y	Y	N	??	??	
$\vdots$	Y	Y	??	??	??	
0	Y	Y	Y	Y	Y	
	0	$\cdots$	ω_1^{CK}	$\omega_1^{CK}+1$	$\cdots$	$: \alpha$

Useful observation: every $\beta < \omega^{\theta+1}$ can be identified by a Boolean combination of computable $\Sigma_{2\cdot\theta+1}$-formulas, but $\omega^{\theta+1}$ itself cannot be.

A Further Answer

Theorem

For every computable ordinal θ, every ordinal α has a θ-extensional computable cover.

This is not surprising. Roughly speaking, with a θ-extensional cover, we cannot distinguish powers ω^ζ for $\zeta > \theta$, so just write $\alpha = \omega^{(\theta+1)} \cdot \mu + \nu$, with $\nu < \omega^{(\theta+1)}$.

$\underline{\theta}$						
$\vdots$	*Y*	*Y*	*N*	*N*	*N*	
$\omega_1^{CK}+1$	*Y*	*Y*	*N*	*N*	*N*	
ω_1^{CK}	*Y*	*Y*	*N*	??	??	
$\vdots$	*Y*	*Y*	**Y**	**Y**	**Y**	
0	*Y*	*Y*	*Y*	*Y*	*Y*	
	0	$\cdots$	ω_1^{CK}	$\omega_1^{CK}+1$	$\cdots$	$: \alpha$

Another Answer

Theorem

If $\omega_1^{CK} \leq \alpha < \omega_1^{CK} \cdot \omega$, then α has no ω_1^{CK}-extensional cover.

Such an α contains finitely many multiples of ω_1^{CK}. In a cover, we could pick a single $\mathcal{A}_i$ which ω_1^{CK}-extensionally covers all of those multiples. A delicate construction then uses this cover to build a computable presentation of ω_1^{CK}, which is impossible.

$\underline{\theta}$								
$\vdots$	Y	Y	N	N	N	N	N	
$\omega_1^{CK}+1$	Y	Y	N	N	N	N	N	
ω_1^{CK}	Y	Y	N	**N**	**N**	??	??	
$\vdots$	Y	Y	Y	Y	Y	Y	Y	
0	Y	Y	Y	Y	Y	Y	Y	
	0	$\cdots$	ω_1^{CK}	$\omega_1^{CK}+1$	$\cdots$	$\omega_1^{CK} \cdot \omega$	$\cdots$	$: \alpha$

The Last Answer

Surprise Theorem (joint with Julia Knight)

All ordinals $\alpha \geq \omega_1^{CK}\cdot\omega$ have ω_1^{CK}-extensional computable covers.

In fact, the canonical cover of a computable presentation of the Harrison ordering $\omega_1^{CK}\cdot(1+\eta)$ is an ω_1^{CK}-extensional cover of each order $\omega_1^{CK}\cdot(1+\rho)$ with ρ infinite. Likewise, each such ordinal has an ω_1^{CK}-back-and-forth with the Harrison ordering.

θ								
$\vdots$	Y	Y	N	N	N	N	N	
$\omega_1^{CK}+1$	Y	Y	N	N	N	N	N	
ω_1^{CK}	Y	Y	N	N	N	**Y**	**Y**	
$\vdots$	Y	Y	Y	Y	Y	Y	Y	
0	Y	Y	Y	Y	Y	Y	Y	
	0	$\cdots$	ω_1^{CK}	$\omega_1^{CK}+1$	$\cdots$	$\omega_1^{CK}\cdot\omega$	$\cdots$	$:\alpha$

What More Could We Ask?

....Well, we could ask the same question about κ-recursion theory and κ-local computability, in which we list all substructures generated by subsets of size $< \kappa$. Does the same phenomenon occur? At $(\kappa^+)^{CK} \cdot \kappa$, or where? Is there a Harrison ordering for κ? (Is there Barwise compactness for κ?) Do we need to use $L_{\kappa\kappa}$ formulas to make this happen?

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Happy Birthday, Sy!