On the invariant universality property

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Analytic quasi-orders and Borel reducibility

Definition (Analytic quasi-order)

A *quasi-order* R on a Polish space or, more generally, on a standard Borel space X is called *analytic* if it is an analytic subset of $X \times X$ ($=$ projection of a Borel subset of $X \times X \times {}^{\omega}\omega$).
It is an *analytic equivalence relation* if it is also symmetric.

Definition (Borel (bi-)reducibility and completeness)

Borel reducibility: $R \leq_B S$ iff $\exists$ Borel $f \colon \mathrm{dom}(R) \to \mathrm{dom}(S)$ such that

$$\forall x, y \in \mathrm{dom}(R)\, (x \mathrel{R} y \iff f(x) \mathrel{S} f(y)).$$

Borel bi-reducibility: $R \sim_B S$ iff $R \leq_B S \leq_B R$.

(Borel-)completeness: S is *Borel complete* for a class $\mathcal{C}$ of analytic quasi-orders iff $R \leq_B S$ for all $R \in \mathcal{C}$.

Intuitively: if $R \leq_B S$ then R is not more complicated than S.

Analytic quasi-orders and Borel reducibility

Notation

- $E_R = R \cap R^{-1} =$ the analytic equivalence relation associated to the analytic quasi-order R.
- $\hat{R} =$ the quotient order of R (w.r.t. E_R).

Combinatorially, $\sim_B$ is a (definable) *bi-embeddability*: $R \sim_B S$ iff $\hat{R}$ and $\hat{S}$ are one embeddable into the other via functions admitting Borel liftings.

Definition (Classwise Borel isomorphism)

Classwise Borel isomorphism: $R \simeq_B S$ iff there is an isomorphism f of $\hat{R}$ and $\hat{S}$ such that both f and f^{-1} admit Borel liftings.

Examples: isomorphism and embeddability relations

Notation

- $\mathcal{L}$ = graph language.
- $\mathrm{Mod}^{\omega}_{\mathcal{L}} = {}^{\omega\times\omega}2$ = Polish space of countable $\mathcal{L}$-structures.
- $\mathrm{Mod}^{\omega}_{\varphi} = \{X \in \mathrm{Mod}^{\omega}_{\mathcal{L}} \mid X \models \varphi\}$ for φ an $\mathcal{L}_{\omega_1\omega}$-sentence.

Theorem (Lopez-Escobar)

$B \subseteq \mathrm{Mod}^{\omega}_{\mathcal{L}}$ is Borel and invariant under isomorphism iff $B = \mathrm{Mod}^{\omega}_{\varphi}$ for some $\mathcal{L}_{\omega_1\omega}$-sentence φ.

Therefore the *isomorphism* relation $\cong\restriction \mathrm{Mod}^{\omega}_{\varphi}$ and the *embeddability* relation $\sqsubseteq\restriction \mathrm{Mod}^{\omega}_{\varphi}$ are examples of an *analytic equivalence relation* and an *analytic quasi-order*, respectively. However:

- $\cong\restriction \mathrm{Mod}^{\omega}_{\varphi}$ is a very special kind of equivalence relation and is far from being complete for analytic equivalence relations;
- $\sqsubseteq\restriction \mathrm{Mod}^{\omega}_{\varphi}$ is complete for analytic quasi-orders for suitable φ.

Invariant universality

Theorem (S.Friedman-M.)

For every analytic quasi-order R there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that $R \sim_B \sqsubseteq\restriction \mathrm{Mod}^\omega_\varphi$, and in fact $R \simeq_B \sqsubseteq\restriction \mathrm{Mod}^\omega_\varphi$.

Intuitively: every analytic quasi-order R can be copied in a "faithful" way as an embeddability relation. This is abbreviated with:

$\sqsubseteq\restriction \mathrm{Mod}^\omega_\mathcal{L}$ is *(strongly) invariantly universal*.

Possible generalizations:

1. Replace $\sqsubseteq$ with other morphism relations, e.g.:
   - epimorphisms between countable structures,
   - continuous embeddability between continua (dendrites),
   - isometric embeddability between Polish metric spaces,
   - linear isometric embeddability between separable Banach spaces...
2. Replace "analytic" ($= \mathbf{\Sigma}^1_1$) with: $\mathbf{\Sigma}^1_{n+2}$, projective, κ-Souslin, $\mathrm{L}(\mathbb{R})$ ...
3. Replace countable structures with structures of size $\kappa > \omega$.

Other morphism relations

Using the Lopez-Escobar therorem, $\sqsubseteq \restriction \mathrm{Mod}^{\omega}_{\mathcal{L}}$ is *strongly invariantly universal* iff for every analytic quasi-order R there is a *Borel* and *invariant under isomorphism* $B \subseteq \mathrm{Mod}^{\omega}_{\mathcal{L}}$ such that $R \simeq_B \sqsubseteq \restriction B$.

Definition

S analytic quasi-order, E analytic equivalence relation s.t. $E \subseteq S$. The pair (S, E) is *strongly invariantly universal* iff for every analytic quasi-order R there is a *Borel* and *E-invariant* $B \subseteq dom(S)$ such that $R \simeq_B (S \restriction B)$.

Intuitively: (S, E) is strongly invariantly universal iff S contains in a "natural" (= sufficiently closed and simply definable) way a "faithful" copy of all other analytic quasi-orders.

Usually we take (S, E) to be pairs of the form (morphism, isomorphism):

- (continuous embeddability, homeomorphism) between compacta,
- (isometric embeddability, isometry) between Polish metric spaces,
- (linear isometric embeddability, linear isometry) between separable Banach spaces, and so on.

Other morphism relations

Theorem (Camerlo-Marcone-M.)

The following (complete) analytic quasi-orders are in fact invariantly universal when paired with the associated "isomorphism" relation E:

1. *embeddability/epimorphisms for countable graphs ($E = \cong$),*
2. *color preserving embeddings between countable colored linear orders ($E =$ color preserving isomorphism),*
3. *continuous embeddability/open cont. surjections between dendrites, cont. surjections between compacta ($E =$ homeomorphism),*
4. *isometric embeddability between (discrete/ultrametric) Polish metric spaces ($E =$ isometry),*
5. *linear isometric embeddability between separable Banach spaces ($E =$ linear isometry),*
6. *and so on...*

Remark: Each proof is different and requires some specific work.

More complicated quasi-orders

Notation

- Given a cardinal κ, consider the *generalized Baire space* ${}^{\kappa}\kappa$ endowed with the bounded topology (i.e. with the topology generated by basic clopen sets of the form $\mathbf{N}_s = \{x \in {}^{\kappa}\kappa \mid x \supseteq s\}$ for $s \in {}^{<\kappa}\kappa$).
- $\mathrm{Mod}^{\kappa}_{\mathcal{L}}$ = space of models of size κ (homeomorphic to ${}^{\kappa}2 \subseteq {}^{\kappa}\kappa$).
- $\mathrm{Mod}^{\kappa}_{\varphi} = \{X \in \mathrm{Mod}^{\kappa}_{\mathcal{L}} \mid X \models \varphi\}$ for φ an $\mathcal{L}_{\kappa^+\kappa}$-sentence.

Theorem (Andretta-M.)

Work in ZF *and let κ be an uncountable cardinal. For every κ-Souslin quasi-order R there is an $\mathcal{L}_{\kappa^+\kappa}$-sentence φ such that R is bi-reducible with (in fact: classwise isomorphic to) $\sqsubseteq \restriction \mathrm{Mod}^{\kappa}_{\varphi}$.*

Remark: The maps involved in the bi-reducibility are κ^+-Borel, absolutely definable, and so on...

This rather technical result may be applied in a great variety of situations...

More complicated quasi-orders

Theorem (ZF)

For every $\boldsymbol{\Sigma}^1_2$ *quasi-order* R *there is an* $\mathcal{L}_{\aleph_2\aleph_1}$*-sentence* φ *such that* R *and* $\sqsubseteq \restriction \mathrm{Mod}^{\aleph_1}_\varphi$ *are classwise* $\aleph_2$*-Borel isomorphic.*

Theorem (ZFC + $\forall x \in \mathbb{R}\,(x^\#$ exist))

For every $\boldsymbol{\Sigma}^1_3$ *quasi-order* R *there is an* $\mathcal{L}_{\aleph_3\aleph_2}$*-sentence* φ *such that* R *and* $\sqsubseteq \restriction \mathrm{Mod}^{\aleph_2}_\varphi$ *are classwise* $\aleph_3$*-Borel isomorphic.*

Remark: These results make (more) sense when the continuum is large!

Theorem (ZF + AD)

For every $\boldsymbol{\Sigma}^1_3$ *quasi-order* R *there is an* $\mathcal{L}_{\aleph_{\omega+1}\aleph_\omega}$*-sentence* φ *such that* R *and* $\sqsubseteq \restriction \mathrm{Mod}^{\aleph_\omega}_\varphi$ *are classwise* $\aleph_{\omega+1}$*-Borel isomorphic.*

More complicated quasi-orders

Define $j \colon \omega \to \omega$ by $j(n) = \begin{cases} 2^{k+1} - 1 & \text{if } n = 2k+1 \\ 2^{k+1} & \text{if } n = 2k+2. \end{cases}$

Theorem (ZFC + $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$)

For every $\boldsymbol{\Sigma}^1_n$ quasi-order R there is an $\mathcal{L}_{\aleph_{j(n)+1}\aleph_{j(n)}}$-sentence φ such that R and $\sqsubseteq \restriction \mathrm{Mod}^{\aleph_{j(n)}}_{\varphi}$ are classwise $\aleph_{j(n)+1}$-Borel isomorphic.

In particular, the above theorem holds if we assume e.g. ZFC+ "there are ω-many Woodin cardinals with a measurable above", or ZFC + PFA, ...

Theorem (ZF + AD)

For $n > 0$, let κ_n be such that $\boldsymbol{\delta}^1_n = \kappa_n^+$ (that is $\kappa_n = \boldsymbol{\delta}^1_{n-1}$ if n is even and $\kappa_n = \lambda^1_n$ if n is odd). For every $\boldsymbol{\Sigma}^1_n$ quasi-order R there is an $\mathcal{L}_{\boldsymbol{\delta}^1_n \kappa_n}$-sentence φ such that $R \simeq_{\mathsf{L}(\mathbb{R})} \sqsubseteq \restriction \mathrm{Mod}^{\kappa_n}_{\varphi}$.

More complicated quasi-orders

Theorem ($\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$)

For every $\mathbf{\Sigma}^2_1$ quasi-order R there is an $\mathcal{L}_{(\boldsymbol{\delta}^2_1)^+\boldsymbol{\delta}^2_1}$-sentence φ such that $R \simeq_{\mathsf{L}(\mathbb{R})} \sqsubseteq\restriction \mathrm{Mod}^{\boldsymbol{\delta}^2_1}_\varphi$.

Theorem ($\mathsf{ZF} + \mathsf{AD}(\mathbb{R})$)

For every quasi-order R on a standard Borel space there is $\kappa < \Theta$ and an $\mathcal{L}_{\kappa^+\kappa}$-sentence φ such that R and $\sqsubseteq\restriction \mathrm{Mod}^\kappa_\varphi$ are classwise κ^+-Borel isomorphic.

Theorem

Work in Solovay's model. For every real-ordinal definable (briefly: $\mathsf{OD}(\mathbb{R})$) quasi-order R defined on a standard Borel space, there is an $\mathcal{L}_{\aleph_2\aleph_1}$-sentence φ such that $R \simeq_{\mathsf{OD}(\mathbb{R})} \sqsubseteq\restriction \mathrm{Mod}^{\aleph_1}_\varphi$.

Embeddability between larger structures

So far we considered only the embeddability relation between uncountable structures of "small" size, i.e. of cardinality smaller than (or incomparable to) the continuum $|\mathbb{R}|$... what about structures of larger cardinality?

Definition (Standard Borel κ-spaces and analytic quasi-orders)

- A topological space is called *standard Borel κ-space* if it is κ^+-Borel isomorphic to a κ^+-Borel subset of ${}^\kappa\kappa$.
- A *quasi-order R* on a standard Borel κ-space X is called *analytic* if it is the projection of a κ^+-Borel subset of $X \times X \times {}^\kappa\kappa$.

Theorem (Mildenberger-M.)

Let $\kappa = \kappa^{<\kappa} > \omega$. Then for every analytic quasi-order R defined on a standard Borel κ-space, there is an $\mathcal{L}_{\kappa^+\kappa}$-sentence φ such that R and $\sqsubseteq \restriction \mathrm{Mod}^\kappa_\varphi$ are classwise κ^+-Borel isomorphic.

Remark: The condition $\kappa^{<\kappa} = \kappa$ is nearly optimal.

Back to the countable case: an application of our method

Example

- If $\varphi_{\mathsf{LO}} \in \mathcal{L}_{\omega_1\omega}$ axiomatizes *linear orders*, then $\cong \restriction \mathrm{Mod}^{\omega}_{\varphi_{\mathsf{LO}}}$ is S_∞-complete and $\sqsubseteq \restriction \mathrm{Mod}^{\omega}_{\varphi_{\mathsf{LO}}}$ is a bqo (hence the bi-embeddability relation on $\mathrm{Mod}^{\omega}_{\varphi_{\mathsf{LO}}}$ has only $\aleph_1$-many classes);
- If $\varphi_{\mathsf{CT}} \in \mathcal{L}_{\omega_1\omega}$ axiomatizes *combinatorial trees*, then $\cong \restriction \mathrm{Mod}^{\omega}_{\varphi_{\mathsf{CT}}}$ is S_∞-complete and $\sqsubseteq \restriction \mathrm{Mod}^{\omega}_{\varphi_{\mathsf{CT}}}$ is complete for analytic quasi-orders.

Louveau and Rosendal asked whether it is possible to increase the gap between the complexities of $\sqsubseteq$ and $\cong$ on the same $\mathrm{Mod}^{\omega}_{\varphi}$ (for φ an $\mathcal{L}_{\omega_1\omega}$-sentence). More generally:

Question

Given a $\cong$-like analytic equivalence relation E and an analytic quasi-order R, can we find an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that $E \sim_B \cong \restriction \mathrm{Mod}^{\omega}_{\varphi}$ and $R \sim_B \sqsubseteq \restriction \mathrm{Mod}^{\omega}_{\varphi}$?

"$\cong$-like" means that E is Borel bireducible with *some* isomorphism relation.

Back to the countable case: an application of our method

Since bi-embeddability is obviously refined by the isomorphism relation, we get a negative answer for the pair E, R if one of the following holds:

- E has at most countably many classes but E_R has more classes than E (that is $E \leq_B \mathrm{id}(\omega)$ but $E_R \not\leq_B E$);
- E_R has perfectly many classes while E has at most $\aleph_1$-many classes (that is $\mathrm{id}(\mathbb{R}) \leq_B E_R$ but $\mathrm{id}(\mathbb{R}) \not\leq_B E$).

Quite surprisingly, these are (almost) the unique limitations:

Theorem (M.)

Let E be a $\cong$-like analytic equivalence relation and let R be an analytic quasi-order. Assume that either

- $\mathrm{id}(\mathbb{R}) \leq_B E$, *or*
- $E_R \leq_B E, \mathrm{id}(\omega)$

Then there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that $E \sim_B \cong \restriction \mathrm{Mod}^\omega_\varphi$ and $R \sim_B \sqsubseteq \restriction \mathrm{Mod}^\omega_\varphi$.

The end

Thank you for your attention!