Infinitesimals and Computability: A marriage made in Platonic Heaven

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Alexander von Humboldt Stiftung/Foundation

[1]This research is generously supported by the John Templeton Foundation.

Aim and Motivation

AIM: To connect infinitesimals and computability.

WHY: From the scope of CCA 2013: (http://cca-net.de/cca2013/)

The conference is concerned with the theory of computability and complexity over real-valued data. [BECAUSE] Most mathematical models in physics and engineering [...] are based on the real number concept.

The following is more true:

Most mathematical models in physics and engineering [...] are based on the real number concept, via an intuitive calculus with infinitesimals, i.e. informal Nonstandard Analysis.

We present a notion of computability directly based on Nonstandard Analysis. We start with Reverse Mathematics.

Reverse Mathematics

Reverse Mathematics

$=$ finding the minimal axioms $\mathcal{A}$ needed to prove a theorem $\mathcal{T}$

$=$ finding the minimal axioms $\mathcal{A}$ such that RCA_0 proves $(\mathcal{A} \to \mathcal{T})$.

- $\mathcal{T}$ is a theorem of ordinary mathematics (countable/separable)
- The proof takes place in RCA_0 ($\approx$ idealized computer, TM).
- In many cases: RCA_0 proves $(\mathcal{A} \leftrightarrow \mathcal{T})$
- Big Five: RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0

Most theorems of 'ordinary' mathematics are either provable in RCA_0 or equivalent to one of the 'Big Five' theories.

$=$ Main Theme of RM

An example: Reverse Mathematics for WKL_0

Central principle:

Theorem (Weak König's Lemma)

Every infinite binary tree has an infinite path.

Assuming the base theory RCA_0, WKL is equivalent to

1. Heine-Borel: every countable covering of $[0, 1]$ has a finite subcovering.
2. A continuous function on $[0, 1]$ is uniformly continuous.
3. A continuous function on $[0, 1]$ is Riemann integrable.
4. Weierstrass' theorem: a continuous function on $[0, 1]$ attains its maximum.
5. Peano's theorem for differential equations $y' = f(x, y)$.

7. Gödel's completeness/compactness theorem.
8. A countable commutative ring has a prime ideal.
9. A countable formally real field is orderable.
10. A countable formally real field has a (unique) closure.
11. Brouwer's fixed point theorem: A continuous function from $[0,1]^n$ to $[0,1]^n$ has a fixed point.
12. The separable Hahn-Banach theorem.
13. A continuous function on $[0,1]$ can be approximated by (Bernstein) polynomials.
14. And many more. . . > >

The 'Bible' of Reverse Mathematics: *Subsystems of Second-order Arithmetic* (Stephen Simpson)

The Main Theme of RM

= Mathematical theorems seem to 'cluster' around the Big Five, while 'sparse' everywhere else.

Π^1_1-CA_0 $\leftrightarrow$ Cantor-Bendixson $\leftrightarrow$ Silver $\leftrightarrow$ Baire space Det. $\leftrightarrow$ Menger $\leftrightarrow$ ...

ATR_0 $\leftrightarrow$ Ulm $\leftrightarrow$ Lusin $\leftrightarrow$ Perfect Set $\leftrightarrow$ Baire space Ramsey $\leftrightarrow$ ...

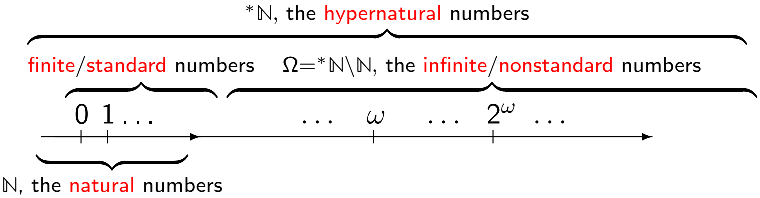
ACA_0 $\leftrightarrow$ Bolzano-Weierstraß $\leftrightarrow$ Ascoli-Arzela $\leftrightarrow$ Köning $\leftrightarrow$ Ramsey ($k \geq 3$)
$\leftrightarrow$ Countable Basis $\leftrightarrow$ Countable Max. Ideal $\leftrightarrow$ ...

WKL_0 $\leftrightarrow$ Peano exist. $\leftrightarrow$ Weierstraß approx. $\leftrightarrow$ Weierstraß max. $\leftrightarrow$ Hahn-Banach $\leftrightarrow$ Heine-Borel $\leftrightarrow$ Brouwer fixp. $\leftrightarrow$ Gödel compl. $\leftrightarrow$ ...

RCA_0 proves Interm. value thm, Soundness thm, Existence of alg. clos. ...

[illegible]
'purely logical' formula, i.e. between subject (math) and formalization (logic).

Nonstandard Analysis: a new way to compute



Standard functions $f : \mathbb{N} \to \mathbb{N}$ are (somehow) generalized to ${}^*f : {}^*\mathbb{N} \to {}^*\mathbb{N}$ such that $(\forall n \in \mathbb{N})(f(n) = {}^*f(n))$.

Definition (Ω-invariance)

For standard $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $\omega \in \Omega$, the function ${}^*f(n, \omega)$ is Ω-invariant if

$$(\forall n \in \mathbb{N})(\forall \omega' \in \Omega)[{}^*f(n, \omega) = {}^*f(n, \omega')].$$

Note that ${}^*f(n, \omega)$ is independent of the choice of $\omega \in \Omega$.

Ω-invariance: A rose by any other name

Definition (Ω-invariance)

For $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $\omega \in \Omega$, the function ${}^*f(n,\omega)$ is Ω-invariant if

$$(\forall n \in \mathbb{N})(\forall \omega' \in \Omega)[{}^*f(n,\omega) = {}^*f(n,\omega')].$$

Principle (Ω-CA)

For all Ω-invariant ${}^*f(n,\omega)$, we have

$$(\exists g : \mathbb{N} \to \mathbb{N})(\forall n \in \mathbb{N})(g(n) = {}^*f(n,\omega)).$$

Theorem (Montalbán-S.)

${}^*\mathrm{RCA}_0 + \Omega\text{-CA} \equiv_{\mathrm{cons}} {}^*\mathrm{RCA}_0 \equiv_{\mathrm{cons}} \mathrm{RCA}_0$

${}^*\mathrm{RCA}_0$ *proves that every Δ^0_1-function is Ω-invariant.*

${}^*\mathrm{RCA}_0 + \Omega\text{-}CA$ *proves that every Ω-invariant function is Δ^0_1.*

Ω-invariance

Principle (Ω-CA)

For all Ω-invariant ${}^*f(n,\omega)$, we have

$$(\exists g : \mathbb{N} \to \mathbb{N})(\forall n \in \mathbb{N})(g(n) = {}^*f(n,\omega)).$$

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${}^*\mathrm{RCA}_0 + \Omega\text{-CA}$ *proves that every Ω-invariant function is Δ^0_1.*

We cannot remove Ω-invariance from Ω-CA, or we obtain WKL.

Principle (implies WKL)

For all (possibly non-Ω-invariant) ${}^*f(n,\omega)$, we have

$$(\exists g : \mathbb{N} \to \mathbb{N})(\forall n \in \mathbb{N})(g(n) = {}^*f(n,\omega)).$$

Ω-invariance and real numbers

Definition

1) For $q_n : \mathbb{N} \to \mathbb{Q}$, $\omega \in \Omega$, ${}^*q_\omega$ is Ω-invariant if $(\forall \omega' \in \Omega)({}^*q_\omega \approx {}^*q_{\omega'})$
2) For $F : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ and $\omega \in \Omega$, ${}^*F(x, \omega)$ is Ω-invariant if

$$(\forall x \in \mathbb{R}, \omega' \in \Omega)({}^*F(x, \omega) \approx {}^*F(x, \omega')). \qquad (**)$$

Theorem (In ${}^*\mathsf{RCA}_0 + \Omega\text{-CA}$)

1) For Ω-invariant ${}^*q_\omega$, there is $x \in \mathbb{R}$ such that $x \approx {}^*q_\omega$.
2) For Ω-invariant ${}^*F(x, \omega)$, there is $G : \mathbb{R} \to \mathbb{R}$ such that $(\forall x \in \mathbb{R})({}^*F(x, \omega) \approx G(x))$.

The standard part map ${}^\circ(x + \varepsilon) = x$ ($x \in \mathbb{R}$ and $\varepsilon \approx 0$) is highly non-computable, but Ω-CA provides a computable alternative for Ω-invariant reals and functions.

However, (**) is implied by $(\forall x \in \mathbb{R})(\forall \varepsilon, \varepsilon' \approx 0)({}^*F(x, \varepsilon) \approx {}^*F(x, \varepsilon'))$.
'Computable' physics

Ω-invariance and Continuity

Theorem (In ${}^*\mathrm{RCA}_0 + \Omega\text{-CA}$)

For $F : \mathbb{R} \to \mathbb{R}$, NS-continuity implies 'continuity with modulus':

$$(\forall x \in \mathbb{R}, y \in {}^*\mathbb{R})(x \approx y \to {}^*F(x) \approx {}^*F(y))$$

implies

$$(\exists g : \mathbb{N} \to \mathbb{Q})(\forall x, y \in \mathbb{R})(\forall \varepsilon > 0)(|x-y| < g(\varepsilon, x) \to |F(x)-F(y)| < \varepsilon).$$

Computable modulus of continuity. (Same for uniform continuity.)

About that coding in Reverse Mathematics. . .

Coding and RM

II.6. CONTINUOUS FUNCTIONS 85

DEFINITION II.6.1 (continuous functions). Within RCA_0, let $\widehat{A}$ and $\widehat{B}$ be complete separable metric spaces. A (code for a) *continuous partial function* ϕ from $\widehat{A}$ to $\widehat{B}$ is a set of quintuples $\Phi \subseteq \mathbb{N} \times A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$ which is required to have certain properties. We write $(a,r)\Phi(b,s)$ as an abbreviation for $\exists n\,((n,a,r,b,s) \in \Phi)$. The properties which we require are:

1. if $(a,r)\Phi(b,s)$ and $(a,r)\Phi(b',s')$, then $d(b,b') \leq s+s'$;
2. if $(a,r)\Phi(b,s)$ and $(a',r') < (a,r)$, then $(a',r')\Phi(b,s)$;
3. if $(a,r)\Phi(b,s)$ and $(b,s) < (b',s')$, then $(a,r)\Phi(b',s')$;

where the notation $(a',r') < (a,r)$ means that $d(a,a') + r' < r$.

Ω-invariance and continuity

Reverse Mathematics Without Coding, given the 'right' definitions.

NOT: $x = (q_n)_{n\in\mathbb{N}}$ is a real IF $(\forall n, i \in \mathbb{N})(|q_n - q_{n+i})| < \frac{1}{2^n}$.

BUT: $x = (q_n)_{n\in\mathbb{N}}$ is a real IF $(\forall \omega, \omega' \in \Omega)({}^*q_\omega \approx {}^*q_{\omega'})$.

Represent a continuous function $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ via $G : \mathbb{Q} \to \mathbb{Q}$ such that

$$(\forall x \in \mathbb{R})(\forall z, y \in {}^*\mathbb{Q})(x \approx y \approx z \to {}^*G(y) \approx {}^*G(z)),$$

and define $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ as $\mathcal{G}(x) = \mathcal{G}(q_n) := {}^*G(q_\omega)$.

Then $\mathcal{G}$ is pointwise NS-continuous and Ω-invariant. I.e. $\mathcal{G}(x)$ is a real number for $x \in \mathbb{R}$.

Even discontinuous functions $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ can be represented by Ω-invariant (nonstandard) $H : {}^*\mathbb{Q} \to {}^*\mathbb{Q}$

All this works because ${}^*\mathbb{Q} \approx \mathbb{R}$.

Higher-order RM

Ulrich Kohlenbach's system RCA_0^ω extends RCA_0 with all finite types.

Equivalences between classical principles in RCA_0^ω.

1. $(\exists^2) \equiv (\exists\varphi^2)(\forall f^1)(\varphi f =_0 0 \leftrightarrow (\exists x^0)f(x_0) = 0)$.
2. There exists standard $F^{1\to 0}$ such that for all $x \in \mathbb{R}$ we have
$$F(x) = \begin{cases} 0 & x \leq_{\mathbb{R}} 0 \\ 1 & x >_{\mathbb{R}} 0 \end{cases}.$$
3. $\mathsf{UWKL} \equiv (\exists\Phi^{1\to 1})(\forall f^1)(T_\infty(f) \to (\forall x^0)(f([\overline{\Phi f}]x) =_0 0))$.
4. $\mathsf{UIVT} \equiv (\exists\Phi)(\forall F \in \overline{C})(F(\Phi(F)) =_{\mathbb{R}} 0)$. (BHK)

But also intuitionistic principles can be studied in RCA_0^ω:

$$(\exists\Psi^3)(\forall\varphi^2)(\forall f^1, g^1 \leq_1 1)[\overline{f}(\Psi(\varphi)) = \overline{g}(\Psi(\varphi)) \to \varphi(f) =_0 \varphi(g)].$$

The 'fan functional' Ψ implies that all type 2 functions are (uniformly) continuous.

Higher-order RM and NSA

(Joint work with Erik Palmgren)

Nelson's internal approach: st(x) is a new predicate with axioms **ST** guaranteeing the basic properties of 'x is standard'

RCA_0^Ω is $\mathrm{RCA}_0^\omega + \mathbf{ST} + \mathbf{\Omega}$-CA and plus:

$$(\forall^{st} x^\tau)[F(x) = 1 \leftrightarrow A^{st}(x)] \rightarrow (\forall^{st} x^\tau)[A^{st}(x) \leftrightarrow A(x)]. \qquad (\bigstar)$$

Here be functionals

In RCA_0^Ω, we have

1. $(\exists^2)^{st} \equiv (\exists^{st}\varphi^2)(\forall^{st} f^1)(\varphi f =_0 0 \leftrightarrow (\exists^{st} x^0) f(x_0) = 0)$.
2. $\Pi_1^0\text{-TRANS} \equiv (\forall^{st} F^1)[(\forall^{st} x^0) F(x) = 0 \to (\forall x) F(x^0) = 0]$
3. $\mathsf{UWKL} \equiv (\exists^{st}\Phi^{1\to 1})(\forall^{st} f^1)(T_\infty^{st}(f) \to (\forall^{st} x^0)(f([\overline{\Phi f}]x) =_0 0))$.
4. $\mathsf{WKL}^* \equiv (\forall^{st} f^1)(T_\infty^{st}(f) \to (\exists^{st}\alpha^1)(\forall x^0)(f(\overline{\alpha}x) =_0 0))$.
5. $\mathsf{UIVT} \equiv (\exists^{st}\Phi)(\forall^{st} F \in \overline{C})(F(\Phi(F)) =_{\mathbb{R}} 0)$.
6. $\mathsf{IVT}^* \equiv (\forall^{st} F \in \overline{C})(\exists^{st} x \in [0,1])(F(x) =_{{}^*\mathbb{R}} 0)$.
7. $\mathsf{UWEI}^{st} \equiv (\exists^{st}\Phi)(\forall^{st} F \in \overline{C})(\forall^{st} y \in [0,1])(F(y) \leq_{\mathbb{R}} F(\Phi(F)))$.
8. $\mathsf{WEI}^* \equiv (\forall^{st} F \in \overline{C})(\exists^{st} x^1 \in [0,1])(\forall y^1 \in [0,1])(F(y) \leq_{{}^*\mathbb{R}} F(x))$.
9. $\mathsf{WEI}^{**}(\approx) \equiv (\forall^{st} F \in \overline{C})(\exists q^0 \in [0,1])(\forall r^0 \in [0,1])(F(r) \lessapprox F(q))$.
10. Decidability of weak Π_1^1-form. in $\mathcal{L}_{\mathbb{R}} = \{0, 1, +, \times, \leq_{\mathbb{R}}, x^1, F \in C\}$.

General Theme I $(\exists^2)^{st} \leftrightarrow UT \leftrightarrow T^* \leftrightarrow T^{**}(\approx) \leftrightarrow \Pi_1^0\text{-TRANS}$

WHERE: theorem T provable from $(\exists^2)$, UT is uniform/functional version, T^* and $T^{**}(\approx)$ are nonstandard versions.

Kijk omhoog...

General theme I connects WKL_0 and ACA_0 $(\approx (\exists^2))$.

$$\frac{\mathsf{WKL}_0}{\mathsf{ACA}_0} = \frac{\mathsf{ATR}_0}{\Pi_1^1\mathsf{CA}_0} \qquad \text{(Analogy by Simpson)}$$

The Suslin functional S^2 is essentially Π_1^1-CA_0.

General Theme II $(S^2)^{st} \leftrightarrow UT \leftrightarrow T^* \leftrightarrow \Pi_1^1$-TRANS

WHERE: theorem T provable from ATR_0 but not from $(\exists^2)$, UT is uniform/functional version, T^* is a special nonstandard version.

'Down under': BD-N, WLPO, Fan Functional, ...

Final Thoughts

The two eyes of exact science are mathematics and logic, the mathematical sect puts out the logical eye, the logical sect puts out the mathematical eye; each believing that it sees better with one eye than with two.

Augustus De Morgan

...there are good reasons to believe that Nonstandard Analysis, in some version or other, will be the analysis of the future.

Kurt Gödel

This research is generously sponsored by the John Templeton Foundation.

Thank you for your attention!

Any questions?

A joke about $\mathbb{R}$, courtesy of SMBC