Viewing λ-terms through Maps

Masahiko Sato

Graduate School of Informatics, Kyoto University

Joint work with Randy Pollack,
Helmut Schwichtenberg and Takafumi Sakurai

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Motivations

- The notion of binding is fundamental in mathematics (not only in proof theory and $\boldsymbol{\lambda}$-calculus.)
- What are lambda terms?
- How are they constructed?
- Can we define $\boldsymbol{\lambda}$-terms without using the notion of equivalence relation?
- A good definition of lambda term will contribute to the design and implementation of proof assistants.

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- A good definition of lambda term will contribute to the design and implementation of proof assistants.

BTW, I am a finitist (in a way).

We have formally verified all the technical results in this work in the proof assistants Minlog and Isabelle.

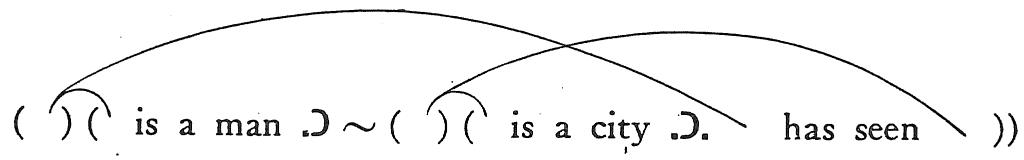
Good points of our approach

- The inductive structure of the terms is nicer compared to other approaches.
- Can define closed lambda terms directly without first defining the lambda terms containing free parameters.
- Can use the same technique to define sentences without first defining formulas containing free parameters.
- A special generic constant $\square$ must be included as a term, however.

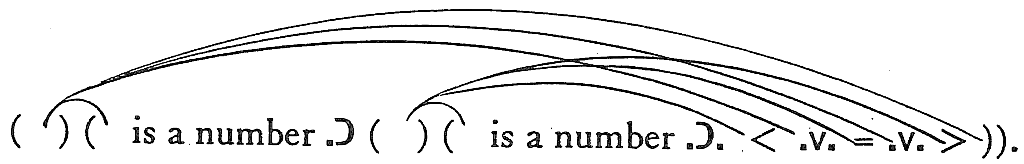
History

- 1930's. Church defined raw lambda terms ($\mathbf{\Lambda}$) and defined α-equivalence relation on them.
- 1940. Quine defined graphical representation of lambda terms. Later, Bourbaki (1954) rediscovered it.
- 1972. de Bruijn defined representation of lambda terms by indices ($\mathbb{D}$).
- 1980. S. defined representation of lambda terms by map and skeleton (precursor of $\mathbb{L}$).
- 2013. This talk clarifies the relationship among the above four representations.

sideration for established usage, the "variation" connoted belongs to a vague metaphor which is best forgotten. The variables have no meaning beyond the pronominal sort of meaning which is reflected in translations such as (20); they serve merely to indicate cross-references to various positions of quantification. Such cross-references could be made instead by curved lines or *bonds*; e.g., we might render (27) thus:



and (26) thus:



But these "quantificational diagrams" are too cumbersome to recommend themselves as a practical notation; hence the use of variables.

$$A$$

$$A'$$

$$A''$$

$$\in AA'$$

$$\in AA''$$

$$\neg \in AA'$$

$$\vee \neg \in AA' \in AA''$$

$$\tau \vee \neg \in \square A' \in \square A''$$

History (cont.)

$$
\begin{tikzcd}
 & \text{Church } (\mathbf{\Lambda}) \arrow[d] & \\
 & \text{Quine-Bourbaki } (\mathbf{\Lambda}/{=_{\boldsymbol{\alpha}}}) \arrow[dl] \arrow[dr] & \\
\text{S. } (\mathbb{L}') \arrow[dr] & & \text{de Bruijn } (\mathbb{D}) \arrow[dl] \\
 & \mathbb{L} \simeq \mathbb{D}_{\mathbf{0}} &
\end{tikzcd}
$$

Logical View

$$\begin{tikzcd}[column sep=small]
 & \text{Church } (\mathbf{\Lambda}) \arrow[dl] \arrow[dr] & \\
\text{S. } (\mathbb{L}') \arrow[dr] & & \text{de Bruijn } (\mathbb{D}) \arrow[dl] \\
 & \mathbb{L} \simeq \mathbb{D}_{\mathbf{0}} \arrow[d] & \\
 & \text{Quine-Bourbaki } (\mathbf{\Lambda}/{=_\alpha}) &
\end{tikzcd}$$

example

Summary of the talk

Three datatypes

We will relate the three datatypes $(\boldsymbol{\Lambda}, \mathbb{L}, \mathbb{D})$ of expressions introduced by Church, S. and de Bruijn.

$\boldsymbol{\Lambda}$ = The datatype of raw $\boldsymbol{\lambda}$-terms.

$\mathbb{L}$ = The datatype of lambda-expressions.

$\mathbb{D}$ = The datatype of de Bruijn expressions.

Three types of abstractions

$\boldsymbol{\Lambda}$:abstraction by parameters $\boldsymbol{x} \in \mathbb{X}$.

$\mathbb{L}$:abstraction by maps $\boldsymbol{m} \in \mathbb{M}$.

$\mathbb{D}$:abstraction by indices $\boldsymbol{i} \in \mathbb{I}$.

Summary of the talk (cont.)

$$
\begin{aligned}
K, L \in \Lambda &::= x \mid \square \mid \mathsf{app}(K, L) \mid \mathsf{lam}(x, K).\\
M, N \in \mathbb{L} &::= x \mid \square \mid \mathsf{app}(M, N) \mid \mathsf{mask}(m, M) \;\; (m \mid M).\\
D, E \in \mathbb{D} &::= x \mid \square \mid \mathsf{app}(D, E) \mid i \mid \mathsf{bind}(D).\\
x &\in \mathbb{X}.\\
m &\in \mathbb{M}.\\
i &\in \mathbb{I}.
\end{aligned}
$$

$\square$ (called a box) is a special constant denoting a hole to be filled with lambda expressions.

Datatypes and constructors

1. Every object of a datatype is uniquely constructed by an application of a constructor function c to already constructed objects:
$$d = c(d_1, \ldots, d_k).$$
2. Every construtor has a unique type:
$$c : D_1 \times \cdots \times D_k \to D,$$
where c can be a partial function.
3. Given $d_1 \in D_1, \ldots, d_k \in D_k$, it is primitive recursively decidable if c can be successfully applied to these objects.

Hence, every object can be represented by a finite tree and we can naturally associate an induction principle with every datatatype.

The notion of map

The notion of map is a generalization of the notion of occurrence of a symbol in syntactic expressions such as formulas or lambda terms.

Plan of the talk

- Part 1. $\mathbb{L}$.
- Part 2. $\mathbf{\Lambda}$. Will show $\mathbb{L} \simeq \mathbf{\Lambda}/ \equiv_{\boldsymbol{\alpha}}$.
- Part 3. $\mathbb{D}$. Will show $\mathbb{L} \simeq \mathbb{D}_{\mathbf{0}}$.

map/skeleton functions will play important roles in all the 3 parts.

Part 1

$\mathbb{L}$

The Datatype of Lambda-exressions

The Datatype $\mathbb{M}$ of Maps

$$\frac{}{0 \in \mathbb{M}} \qquad \frac{}{1 \in \mathbb{M}}$$

$$\frac{\boldsymbol{m} \in \mathbb{M} \quad \boldsymbol{n} \in \mathbb{M} \quad \boldsymbol{m} \neq 0 \text{ or } \boldsymbol{n} \neq 0}{\mathsf{cons}(\boldsymbol{m}, \boldsymbol{n}) \in \mathbb{M}}$$

Note that

$$\mathsf{cons} : \mathbb{M} \times \mathbb{M} \to \mathbb{M}$$

is a partial function.
We will write $(\boldsymbol{m}\ \boldsymbol{n})$ or $\boldsymbol{mn}$ for $\mathsf{cons}(\boldsymbol{m}, \boldsymbol{n})$.

Orthogonality relation on maps

$$\frac{}{m \perp 0} \qquad \frac{}{0 \perp n} \qquad \frac{m \perp n \quad m' \perp n'}{mm' \perp nn'}$$

Example: $(1\ 0) \perp (0\ 1)$ but not $(1\ 1) \perp (0\ 1)$.

The Datatype $\mathbb{X}$ of Parameters

We assume a countably infinite set $\mathbb{X}$ of parameters.
We will write x, y, z for parameters.
We assume that equality relation on $\mathbb{X}$ is decidable.

The Datatype $\mathbb{L}$ and the Divisibility Relation

$$\frac{}{\boldsymbol{x} \in \mathbb{L}}\ \textit{par} \qquad \frac{}{\Box \in \mathbb{L}}\ \textit{box}$$

$$\frac{\boldsymbol{M} \in \mathbb{L} \quad \boldsymbol{N} \in \mathbb{L}}{\mathsf{app}(\boldsymbol{M}, \boldsymbol{N}) \in \mathbb{L}}\ \mathsf{app} \qquad \frac{\boldsymbol{m} \in \mathbb{M} \quad \boldsymbol{M} \in \mathbb{L} \quad \boldsymbol{m} \mid \boldsymbol{M}}{\mathsf{mask}(\boldsymbol{m}, \boldsymbol{M}) \in \mathbb{L}}\ \mathsf{mask}$$

$$\frac{}{0 \mid \boldsymbol{x}} \qquad \frac{}{0 \mid \Box} \qquad \frac{}{1 \mid \Box}$$

$$\frac{\boldsymbol{m} \mid \boldsymbol{M} \quad \boldsymbol{n} \mid \boldsymbol{N}}{\mathsf{mapp}(\boldsymbol{m}, \boldsymbol{n}) \mid \mathsf{app}(\boldsymbol{M}, \boldsymbol{N})} \qquad \frac{\boldsymbol{m} \mid \boldsymbol{N} \quad \boldsymbol{n} \mid \boldsymbol{N} \quad \boldsymbol{m} \perp \boldsymbol{n}}{\boldsymbol{m} \mid \mathsf{mask}(\boldsymbol{n}, \boldsymbol{N})}$$

The Datatype $\mathbb{L}$ of lambda-expressions (cont.)

Notational Convention

- We use M, N, P as metavariables ranging over lambda-expressions.
- We write $(M\ N)$ and also MN for $\mathsf{app}(M, N)$.
- We write $m\backslash M$ for $\mathsf{mask}(m, M)$.
- A lambda-expression of the form $m\backslash M$ is called an abstract.
- We use A, B as metavariables ranging over abstarcts, and write $\mathbb{A}$ for the subset of $\mathbb{L}$ consisting of all the abstracts.

Map and Skeleton

We write M_x for map(x, M), and M^x for skel(x, M).

$$\begin{aligned}
\mathsf{map} &: \mathbb{X} \times \mathbb{L} \to \mathbb{M} \\
y_x &:= \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases} \\
(M\ N)_x &:= (M_x\ N_x). \\
(m \backslash M)_x &:= M_x.
\end{aligned}$$

$$\begin{aligned}
\mathsf{skel} &: \mathbb{X} \times \mathbb{L} \to \mathbb{L} \\
y^x &:= \begin{cases} \square & \text{if } x = y, \\ y & \text{if } x \neq y. \end{cases} \\
(M\ N)^x &:= (M^x\ N^x). \\
(m \backslash M)^x &:= m \backslash M^x.
\end{aligned}$$

Lambda Abstraction in $\mathbb{L}$

We define lam $: \mathbb{X} \times \mathbb{L} \to \mathbb{L}$ by:

$$\mathsf{lam}(x, M) := M_x \backslash M^x.$$

Examples. We assume that x, y and z are distinct parameters.

$$\begin{aligned}
\mathsf{lam}(x, x) &= 1\backslash\Box. \\
\mathsf{lam}(x, y) &= 0\backslash y. \\
\mathsf{lam}(x, \mathsf{lam}(y, x)) &= \mathsf{lam}(x, 0\backslash x) \\
&= 1\backslash 0\backslash\Box. \\
\mathsf{lam}(x, \mathsf{lam}(y, y)) &= \mathsf{lam}(x, \mathsf{lam}(1, \Box)) \\
&= 0\backslash 1\backslash\Box. \\
\mathsf{lam}(x, \mathsf{lam}(y, \mathsf{lam}(z, (xz\ yz)))) &= \\
&\quad (10\ 00)\backslash(00\ 10)\backslash(01\ 01)\backslash(\Box\Box\ \Box\Box)
\end{aligned}$$

Hole Filling and Instantiation

We write $M_m[P]$ for the result of filling boxes (holes) in M specified by map m with P. $M_m[P]$ is defined only if $m \mid M$. We write $A \blacktriangledown P$ for the result of instantiating abstract A with P.

$$\begin{aligned}
\text{fill}: &\quad \mathbb{L} \times \mathbb{M} \times \mathbb{L} \to \mathbb{L} \\
\square_1[P] &:= P. \\
\square_0[P] &:= \square. \\
x_0[P] &:= x. \\
(M\ \ N)_{(m\ \ n)}[P] &:= (M_m[P]\ \ N_n[P]). \\
(n\backslash N)_m[P] &:= n\backslash N_m[P].
\end{aligned}$$

$$\begin{aligned}
\blacktriangledown: &\quad \mathbb{A} \times \mathbb{L} \to \mathbb{L}_{\Lambda} \\
(m\backslash M) \blacktriangledown P &:= M_m[P].
\end{aligned}$$

Substitution

We can now define substitution operation:
subst : $\mathbb{L} \times \mathbb{X} \times \mathbb{L} \to \mathbb{L}$ as follows.

$$[P/x]M := \mathsf{lam}(x, M) \blacktriangledown P.$$

subst enjoys the following properties.

$$[P/x]y = \begin{cases} P & \text{if } x = y, \\ y & \text{if } x \neq y. \end{cases}$$

$$[P/x]\square = \square.$$

$$[P/x](M\ N) = ([P/x]M\ [P/x]N).$$

$$[P/x](m \backslash M) = (m \backslash [P/x]M).$$

Substitution (cont.)

Example.

$$
\begin{aligned}
[y/x]\mathsf{lam}(y, yx) &= [y/x](10\backslash\Box x) \\
&= 10\backslash[y/x](\Box\ x) \\
&= 10\backslash([y/x]\Box\ [y/x]x) \\
&= 10\backslash\Box y \\
&= \mathsf{lam}(z, zy)
\end{aligned}
$$

Remark. By internalizing the substitution operation, we can easily get an explicit substitution calculus.

Substitution Lemma

If $x \neq y$ and $x \notin \mathsf{FP}(P)$, then

$$[P/y][N/x]M = [[P/y]N/x][P/y]M.$$

Proof. By induction on $M \in \mathbb{L}$. Here, we only treat the case where $M = m\backslash M'$.

$$\begin{aligned}
&[P/y][N/x]M \\
&= [P/y][N/x](m\backslash M') \\
&= m\backslash[P/y][N/x]M' \\
&= m\backslash[[P/y]N/x][P/y]M' && \text{(by IH)} \\
&= [[P/y]N/x][P/y](m\backslash M') \\
&= [[P/y]N/x][P/y]M.
\end{aligned}$$

The $\mathbb{L}_\beta$-calculus

$$\frac{}{AM \to_\beta A\blacktriangledown M}\ \beta$$

$$\frac{M \to_\beta M'}{MN \to_\beta M'N}\ \mathsf{appl} \qquad \frac{M \in \mathbb{L}}{MN \to_\beta MN'}\ \mathsf{appr}$$

$$\frac{M \to_\beta N}{\mathsf{lam}(x, M) \to_\beta \mathsf{lam}(x, N)}\ \xi$$

Remark. Traditional way of formulating β-conversion rule is:

$$(\mathsf{lam}(x, M)\ N) \to_\beta [N/x]M.$$

Part 2

$\mathbf{\Lambda}$

The Datatype of Raw Lambda-terms

The Datatype Λ of Raw λ-terms

$$\frac{}{\boldsymbol{x} \in \boldsymbol{\Lambda}}\ \textit{par} \qquad \frac{}{\square \in \boldsymbol{\Lambda}}\ \textit{box}$$

$$\frac{\boldsymbol{K} \in \boldsymbol{\Lambda} \quad \boldsymbol{L} \in \boldsymbol{\Lambda}}{\mathsf{app}(\boldsymbol{K}, \boldsymbol{L}) \in \boldsymbol{\Lambda}}\ \mathsf{app} \qquad \frac{\boldsymbol{x} \in \mathbb{X} \quad \boldsymbol{K} \in \boldsymbol{\Lambda}}{\mathsf{lam}(\boldsymbol{x}, \boldsymbol{K}) \in \boldsymbol{\Lambda}}\ \mathsf{lam}$$

$$\boldsymbol{K}, \boldsymbol{L} \in \boldsymbol{\Lambda} ::= \boldsymbol{x} \mid \square \mid \mathsf{app}(\boldsymbol{K}, \boldsymbol{L}) \mid \mathsf{lam}(\boldsymbol{x}, \boldsymbol{K}).$$

Remark. lam binds parameter $\boldsymbol{x}$ in $\boldsymbol{M}$.

Map and Skeleton

We define map : $\mathbb{X} \times \boldsymbol{\Lambda} \to \mathbb{M}$ and skel : $\mathbb{X} \times \boldsymbol{\Lambda} \to \boldsymbol{\Lambda}$.

$$
\begin{aligned}
y_x &:= \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases} \\
\square_x &:= 0. \\
(K\ L)_x &:= (K_x\ L_x). \\
\mathsf{lam}(y, K)_x &:= \begin{cases} 0 & \text{if } x = y, \\ K_x & \text{if } x \neq y. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
y^x &:= \begin{cases} \square & \text{if } x = y, \\ y & \text{if } x \neq y. \end{cases} \\
\square^x &:= \square. \\
(K\ L)^x &:= (K^x\ L^x). \\
\mathsf{lam}(y, K)^x &:= \begin{cases} \mathsf{lam}(y, K) & \text{if } x = y, \\ \mathsf{lam}(y, K^x) & \text{if } x \neq y. \end{cases}
\end{aligned}
$$

Map and Skeleton (cont.)

$$
\begin{aligned}
& \boldsymbol{x} \text{ does not occur free in } \boldsymbol{K} \\
\iff & \boldsymbol{K}_{\boldsymbol{x}} = 0 \\
\iff & \boldsymbol{K}^{\boldsymbol{x}} = \boldsymbol{K}
\end{aligned}
$$

Remark. This shows that the notion of map is a generalization of the notion of occurrence.

α-equivalence Relation

We define the α-equivalence relation, $=_\alpha$, using the map/skeleton functions.

$$\frac{}{x =_\alpha x} \qquad \frac{}{\Box =_\alpha \Box}$$

$$\frac{K =_\alpha K' \quad L =_\alpha L'}{KL =_\alpha K'L'} \qquad \frac{K_x = L_y \quad K^x =_\alpha L^y}{\mathsf{lam}(x, K) =_\alpha \mathsf{lam}(y, L)}$$

Remark. No renaming is needed in this definition, and it is easy to verify that this is indeed a decidable equivalence relation.

α-equivalence Relation

We can show that $\mathsf{lam}(x, \mathsf{lam}(y, yx)) =_\alpha \mathsf{lam}(y, \mathsf{lam}(x, xy))$ as follows.

$$\dfrac{01 = 01 \qquad \dfrac{10 = 10 \qquad \dfrac{\dfrac{}{\Box =_\alpha \Box} \quad \dfrac{}{\Box =_\alpha \Box}}{\Box\Box =_\alpha \Box\Box}}{\mathsf{lam}(y, y\Box) =_\alpha \mathsf{lam}(x, x\Box)}}{\mathsf{lam}(x, \mathsf{lam}(y, yx)) =_\alpha \mathsf{lam}(y, \mathsf{lam}(x, xy))}$$

Substitution

We think that it is more natural to define substitution as a relation (which is invariant under α-equivalence) than to define it as a function (using a choice function which chooses a fresh parameter).

But we skip the discussion here.

Interpretation of Λ in $\mathbb{L}$

We define the interpretation function $[\![-]\!] : \boldsymbol{\Lambda} \to \mathbb{L}$ as follows.

$$
\begin{aligned}
[\![x]\!] &:= x. \\
[\![\square]\!] &:= \square. \\
[\![KL]\!] &:= [\![K]\!][\![L]\!]. \\
[\![\mathsf{lam}(x, K)]\!] &:= \mathsf{lam}(x, [\![K]\!]).
\end{aligned}
$$

Remark. Two raw $\boldsymbol{\lambda}$-terms K and L are α-equivalent iff $[\![M]\!] = [\![N]\!]$.

Part 3

$\mathbb{D}$

The Datatype of de Bruijn-expressions

The Datatype $\mathbb{I}$ of Indices

We use natural numbers as indices.

$$\boldsymbol{i, j} \in \mathbb{I} ::= \mathbf{0} \mid \mathbf{1} \mid \mathbf{2} \mid \mathbf{3} \mid \cdots$$

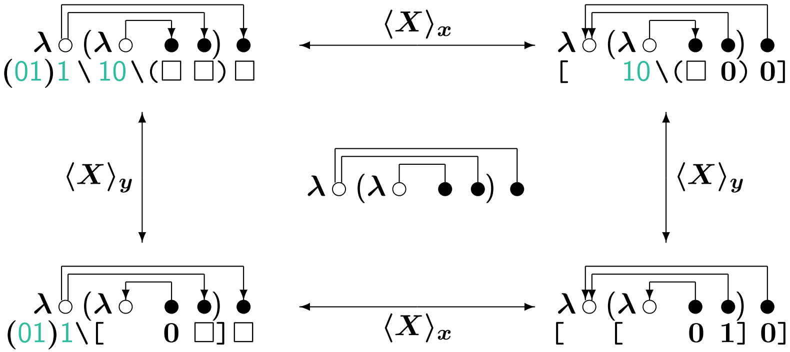
The Datatype $\mathbb{D}$ of de Bruijn-expressions

$$\frac{}{\boldsymbol{x} \in \mathbb{D}}\ \textit{par} \qquad \frac{}{\Box \in \mathbb{D}}\ \textit{box} \qquad \frac{}{\boldsymbol{i} \in \mathbb{D}}\ \textit{idx}$$

$$\frac{\boldsymbol{D} \in \mathbb{D} \quad \boldsymbol{E} \in \mathbb{D}}{\mathsf{app}(\boldsymbol{D}, \boldsymbol{E}) \in \mathbb{D}}\ \mathsf{app} \qquad \frac{\boldsymbol{D} \in \mathbb{D}}{\mathsf{bind}(\boldsymbol{D}) \in \mathbb{D}}\ \mathsf{bind}$$

We write $[\boldsymbol{D}]$ for $\mathsf{bind}(\boldsymbol{D})$.

Example: $\lambda x.\,(\lambda y.\,yx)x$



summary

Conclusion

- We have introduced a new datatype ($\mathbb{L}$) whose elements canonically represent the lambda terms.
- In particular, abstracts are represented as pairs of the map parts and the skeleteton parts of the abstracts.
- Substitution operation on $\mathbb{L}$ is a homomorphism, and can be computed by first-order term rewriting (explicit substitution without renaming).
- Induction principle on $\mathbb{L}$ is structural and follow the pattern of the inductive definition of the datatype.
- $\mathbb{L}$ is isomorphic to the datatype of raw $\boldsymbol{\lambda}$-terms modulo α-equivalence, and also is isomprphic to the datatype of de Bruijn-expressions. These isomorphisms respect substitution operations.
- We are almost finishing formal verifications of all the technical results in the proof assistants Isabelle/{HOL, Nominal} and Minlog.