Descriptive Set Theory and Computation Theory

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SDF–60 Conference, Vienna, 8.07.2013

Introduction

Descriptive set theory (DST) provides natural tools (hierarchies and reducibilities) to measure topological complexity of subsets and functions on topological spaces.

Computation theory (CT) provides models of computation and concentrates on classifying computational tasks according to their complexity.

From the very beginning, CT was strongly influenced by ideas, methods and tools of DST. Currently there is a deep and fruitful interaction between the both theories. We give a brief historical overview of this interaction and discuss some recent results.

Contents

Borel hierarchy in Polish spaces

This is the family of pointclasses $\mathbf{\Sigma}^0_\alpha(X)$ defined by induction on $\alpha < \omega_1$ as follows: $\mathbf{\Sigma}^0_0(X) = \{\emptyset\}$, $\mathbf{\Sigma}^0_1(X)$ is the collection of the open sets of X, and $\mathbf{\Sigma}^0_\alpha(X) = (\bigcup_{\beta<\alpha}(\mathbf{\Sigma}^0_\beta(X))_c)_\sigma$ (for $\alpha > 2$) is the class of countable unions of sets in $\bigcup_{\beta<\alpha}(\mathbf{\Sigma}^0_\beta(X))_c$.
We also let $\mathbf{\Pi}^0_\beta(X) = (\mathbf{\Sigma}^0_\beta(X))_c$ and $\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$.
The classes $\mathbf{\Sigma}^0_\alpha(X), \mathbf{\Pi}^0_\alpha(X), \mathbf{\Delta}^0_\alpha(X)$ are called the levels of the Borel hierarchy of X.

Luzin hierarchy in Polish spaces

Let $\{\mathbf{\Sigma}^1_n(X)\}_{1\leq n<\omega}$ be Luzin's projective hierarchy in a Polish space X, i.e. $\mathbf{\Sigma}^1_1(X)=(\mathbf{\Pi}^0_1(X))_p$ and $\mathbf{\Sigma}^1_{n+1}(X)=(\mathbf{\Pi}^1_n(X))_p$ for any $n\geq 1$. Let also $\mathbf{\Sigma}^1_0=\mathbf{\Pi}^1_0=\mathbf{\Delta}^1_1$.

Suslin's Theorem:

For any Polish space X, $\mathbf{\Delta}^1_1(X)=\bigcup_{\alpha<\omega_1}\mathbf{\Sigma}^0_\alpha(X)$.

Hausdorff hierarchy in Polish spaces

For any $0<\beta<\omega_1$, let $\{\mathbf{\Sigma}_\alpha^{-1,\beta}(X)\}_{\alpha<\omega_1}$ be Hausdorff's difference hierarchy, i.e. $\mathbf{\Sigma}_1^{-1,\beta}(X)=\mathbf{\Sigma}_\beta^0(X)$, $\mathbf{\Sigma}_2^{-1,\beta}(X)$ is the class of differences of $\mathbf{\Sigma}_\beta^0(X)$ sets, $\mathbf{\Sigma}_3^{-1,\beta}(X)$ is the class of sets $A_0\cup(A_2\setminus A_1)$ where A_i are $\mathbf{\Sigma}_\beta^0(X)$ sets, and so on.
Hausdorff-Kuratowski's Theorem: For any Polish space X and any $0<\beta<\omega_1$, $\mathbf{\Delta}_{\beta+1}^0(X)=\bigcup_{\alpha<\omega_1}\mathbf{\Sigma}_\alpha^{-1,\beta}(X)$.
Non-collapse theorem: all the classical hierarchies do not collapse in any uncountable Polish space, in particular $\mathbf{\Sigma}_\alpha^0(X)\neq\mathbf{\Pi}_\alpha^0(X)$ for each $\alpha<\omega_1$.

Applications of the classical hierarchies

The classical DST was actively developing during the last century by many people including Borel, Lebesgue, Baire, Luzin, Suslin, Hausdorff, Sierpinski, Novikov, Keldysh, Alexandrov, Kolmogorov, Lavrentyev, Kuratowski, Kantorovich, Livenson, Lyapunov, and many others.
Currently, DST is a rich area of mathematics with a wide range of applications including set theory, measure theory, functional analysis and mathematical logic. It also served as an important source of ideas, analogies and notions for CT.

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Wadge reducibility

A subset A of the Baire space $\mathcal{N} = \omega^\omega$ is Wadge reducible to a subset B iff $A = f^{-1}(B)$ for some continuous function f on $\mathcal{N}$. The structure of Wadge degrees $(P(\mathcal{N}); \leq_W)$ is fairly well understood and turns our to be rather simple. In particular, $(\mathbf{\Delta}^1_1(\mathcal{N}); \leq_W)$ is semi-well ordered [Wad84], i.e. it has no infinite descending chain and for any $A, B \in \mathbf{\Delta}^1_1(\mathcal{N})$ we have $A \leq_W B$ or $\overline{B} \leq_W A$.

The structure of Wadge degrees subsumes (refines) the classical hierarchies and provides in a sense the finest possible topological classifications of subsets of the Baire space.

Wadge reducibility

Beyond the Borel sets, the structure of Wadge degrees depends on the set-theoretic axioms but under some of these axioms the whole structure remains semi-well ordered. This structure includes the hierarchies from DST and may serve as a nice tool to calibrate the many problems of interest in DST and CA.
The structure of Wadge degrees has similar properties in all zero-dimensional Polish spaces. But it typically becomes much more complicated for non-zero-dimensional spaces. In this case there is still a quest to find adequate topological classifications that refine the classical hierarchies.

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Equivalence relations

A popular topic in DST is the study of some reducibilities on equivalence relations on the Baire space (see e.g. [Ke95] for surveys). Here we note that these reducibilities fit well to our framework and answer a natural question for some of the corresponding degree structures.

The most popular reducibilities on equivalence relations are defined as follows. For equivalence relations E, F on $\mathcal{N}$, E is *continuously* (resp. *Borel*) reducible to F, in symbols $E \leq_c F$ (resp. $E \leq_B F$) if there is a continuous (resp. a Borel) function f on $\mathcal{N}$ such that for all $x, y \in \mathcal{N}$, $E(x, y)$ is equivalent to $F(f(x), f(y))$.

Equivalence relations

The structures $(ER(\mathcal{N}); \leq_c)$ and $(ER(\mathcal{N}); \leq_B)$ where $ER(\mathcal{N})$ is the set of all equivalence relations on $\mathcal{N}$, and especially their substructures on the set of Borel equivalence relations, were intensively studied in DST. In particular, it was shown that both structures are rather rich.

The ongoing active research in this direction is of interest for several fields, including functional analysis, representations of topological groups and model theory.

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Directions of computation theory

Ideas and concepts of DST turned out of principal importance for several branches of CT. Here people develop more or less "effective" versions of hierarchies and reducibilities which are applied to the classification of computational tasks according to their "complexity". Typically, classifications arising in this way have many specifical features related to the "ideology" of a given field of computation theory. Nevertheless, there is always some part of the theory related to DST, hence DST often suggests new ideas and research directions.

We provide a short list of directions where such intersection with DST was already used:

Directions of computation theory

1. **Computability theory**: arithmetical, analytical and Ershov's hierarchy, reducibilities (Kleene, Mostowski, Moschovakis, Turing, Post, Rogers, Ershov and many others).
2. **Computation complexity theory**: polynomial hierarchy and reducibility, difference hierarchy over NP (Meyer, Stockmeyer, Cook, Levin, Hartmanis, Wagner, Wechsung and many others).
3. **Automata on finite words**: Brzozowski, Straubing-Therien, and difference hierarchies, quantifier-free reducibilities (Brzozowski, Thomas, Straubing, Therien, Pin, Wagner, S,. and others).

Directions of computation theory

4. **Automata on infinite words**: Borel, difference and Wagner hierarchies, atomatic reducibilities (Büchi, Trachtenbrot, Rabin, Wagner, S., Perrin, Pin, Duparc, Finkel and many others).
5. **Domain theory**: Borel and difference hierarchies, Wadge reducibilities (Scott, Tang, S., Becher, Grigorieff and others).
6. **Computable analysis**: Classical hierarchies and their effective versions, Wadge and Weihrauch reducibilities (Weihrauch, Hertling, Bratka, S. and others).

Contents

Introduction

Computable analysis (CA) deals with computations on continuous structures, hence it is natural to search for a class of topological spaces which admit a reasonable computation theory and include all spaces of interest for analysis and numeric mathematics. Starting with the well-known class of admissibly represented spaces we try to identify spaces with the specified properties. The leading idea is to look for spaces which have a reasonable DST.

Quasi-Polish Spaces

Recall that a space X is *Polish* if it is countably based and metrizable with a metric d such that (X, d) is a complete metric space. Polish spaces are used in many fields of mathematics and are fairly well understood.

A space X is *quasi-Polish* [Br11] if it is countably based and quasi-metrizable with a quasi-metric d such that (X, d) is a complete quasi-metric space. A *quasi-metric* on X is a function from $X \times X$ to the nonnegative reals such that $x = y$ iff $d(x, y) = d(y, x) = 0$, and $d(x, y) \leq d(x, z) + d(z, y)$. Since for the quasi-metric spaces different notions of completeness and of a Cauchy sequence are considered, the definition of quasi-Polish spaces should be made more precise (see [Br11] for additional details).

Quasi-Polish spaces include the Polish spaces and the ω-continuous domains [Br11].

Borel hierarchy in arbitrary spaces

Definition

For $\alpha < \omega_1$, define the family of pointclasses $\mathbf{\Sigma}^0_\alpha = \{\mathbf{\Sigma}^0_\alpha(X)\}$ by induction on α as follows: $\mathbf{\Sigma}^0_0(X) = \{\emptyset\}$, $\mathbf{\Sigma}^0_1(X) = \tau_X$ is the collection of the open sets of X, $\mathbf{\Sigma}^0_2(X) = ((\mathbf{\Sigma}^0_1(X))_d)_\sigma$ is the collection of all countable unions of differences of open sets, and $\mathbf{\Sigma}^0_\alpha(X) = (\bigcup_{\beta<\alpha}(\mathbf{\Sigma}^0_\beta(X))_c)_\sigma$ (for $\alpha > 2$) is the class of countable unions of sets in $\bigcup_{\beta<\alpha}(\mathbf{\Sigma}^0_\beta(X))_c$.
We also let $\mathbf{\Pi}^0_\beta(X) = (\mathbf{\Sigma}^0_\beta(X))_c$ and $\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$.

Luzin hierarchy in arbitrary spaces

Let $\{\mathbf{\Sigma}^1_n(X)\}_{1\leq n<\omega}$ be Luzin's projective hierarchy in X, i.e. $\mathbf{\Sigma}^1_1(X)=(\mathbf{\Pi}^0_2(X))_p$ and $\mathbf{\Sigma}^1_{n+1}(X)=(\mathbf{\Pi}^1_n(X))_p$ for any $n\geq 1$. Let also $\mathbf{\Sigma}^1_0=\mathbf{\Pi}^1_0=\mathbf{\Delta}^1_1$. The reason why the definition of the first level of the Luzin hierarchy is distinct from the classical definition $\mathbf{\Sigma}^1_1(X)=(\mathbf{\Pi}^0_1(X))_p$ for Polish spaces is that the inclusion $\mathbf{\Sigma}^0_1(X)\subseteq(\mathbf{\Pi}^0_1(X))_p$ may fail in general.
Definitions apply to all spaces X and differ from the classical definition for Polish spaces only for the level 2, and for the case of Polish spaces our definitions are equivalent to the classical ones. Notice that the classical definition cannot be applied in general to non metrizable spaces precisely because the inclusion $\mathbf{\Sigma}^0_1\subseteq\mathbf{\Sigma}^0_2$ may fail.

Hierarchies in quasi-Polish spaces and beyond

Based on our previous work on DST for domains [Se05], M. de Brecht [Br11] has developed a reasonable descriptive set theory for the quasi-Polish spaces. In particular, the Suslin and Hausdorff-Kuratowski theorems hold true in arbitrary quasi-Polish space.

Further we discuss some recently developed hierarchies of topological spaces and explore a basic DST in such spaces. We hope these considerations are of some interest to CA (because one obtains several natural complexity measures on the corresponding spaces) and to DST (because in this way one extends the classical DST to a much wider class of spaces).

Hierarchies in quasi-Polish spaces and beyond

A basic notion of Computable Analysis (CA) [Wei00] is the notion of an *admissible representation* of a topological space X. This is a partial continuous surjection δ from the Baire space $\mathcal{N}$ onto X satisfying a certain universality property). The class of admissibly represented spaces is wide enough to include most spaces of interest for Analysis or Numerical Mathematics.
As shown in [Sch03], the class of admissibly represented spaces coincides with the class of the so-called QCB_0-spaces, i.e. T_0-spaces which are quotients of countably based spaces, and it forms a cartesian closed category (with the continuous functions as morphisms).

Admissibly represented spaces

Thus, among QCB_0-spaces one meets many important function spaces including the continuous functionals of finite types [Kl59, Kr59] interesting for several branches of logic and computability theory.

Along with these nice properties of QCB_0-spaces, this class seems to be too broad to admit a deep understanding. Hence, it makes sense to search for natural subclasses of this class which still include "practically" important spaces but are (hopefully) easier to study. Interesting examples of such subclasses are obtained if we consider, for each level Γ of the classical Borel or Luzin hierarchies , the class of spaces which have an admissible representation of the complexity Γ (below we make this precise).

Countably based T_0-spaces

Along with the hierarchies of QCB_0-spaces, we will consider Borel and Luzin hierarchies of countably based T_0-spaces (CB_0-spaces for short) which are induced by the well-known fact that any CB_0-space may be embedded in the algebraic domain $P\omega$ of all subsets of ω. Hierarchies of spaces obtained in this way turn out to be closely related to the corresponding hierarchies of QCB_0-spaces. Moreover, among the first levels of the Borel hierarchy of CB_0-spaces we meet some classes of spaces which attracted attention of several researches in the field of quasi-metric spaces, in particular the class of quasi-Polish spaces [Br11].

Hierarchies of CB_0-Spaces

Definition

1. Let Γ be a family of pointclasses. A topological space X is called a *Γ-space* if X is homeomorphic to a subspace $A \subseteq P\omega$ with $A \in \Gamma(P\omega)$. The class of all Γ-spaces is denoted $CB_0(\Gamma)$.
2. By the *Borel hierarchy* of CB_0-spaces we mean the sequence $\{CB_0(\mathbf{\Sigma}^0_\alpha)\}_{\alpha<\omega_1}$. By *levels* of this hierarchy we mean the classes $CB_0(\mathbf{\Sigma}^0_\alpha)$ as well as the classes $CB_0(\mathbf{\Pi}^0_\alpha)$ and $CB_0(\mathbf{\Delta}^0_\alpha)$.
3. The *Luzin hierarchy* of CB_0-spaces is defined similarly.

Hierarchies of CB_0-Spaces

Obviously, we have the natural inclusions for levels of the introduced hierarchies.

Proposition

Let $\Gamma \in \{\mathbf{\Pi}^0_2, \mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n \mid 3 \leq \alpha < \omega_1, 1 \leq n < \omega\}$*. Then any retract of a* Γ*-space is a* Γ*-space.*

Proposition

For any countable ordinal $\alpha \geq 2$, $\mathrm{CB}_0(\mathbf{\Sigma}^0_\alpha) \cap \mathrm{CB}_0(\mathbf{\Pi}^0_\alpha) = \mathrm{CB}_0(\mathbf{\Delta}^0_\alpha)$.
For any positive integer n, $\mathrm{CB}_0(\mathbf{\Sigma}^1_n) \cap \mathrm{CB}_0(\mathbf{\Pi}^1_n) = \mathrm{CB}_0(\mathbf{\Delta}^1_n)$.
Therefore, the introduced hierarchies of spaces does not collapse.

Hierarchies of CB_0-Spaces

For a representation δ,
$EQ(\delta) := \{(p,q) \in \mathcal{N}^2 \mid p,q \in \mathrm{dom}(\delta) \wedge \delta(p) = \delta(q)\}$.

Definition

1. Let Γ be a family of pointclasses. A topological space X is called Γ-representable if X has an admissible representation δ with $EQ(\delta) \in \Gamma(\mathcal{N} \times \mathcal{N})$. The class of all Γ-representable spaces is denoted $\mathsf{QCB}_0(\Gamma)$.
2. By the *Borel hierarchy* of QCB_0-spaces we mean the sequence $\{\mathsf{QCB}_0(\mathbf{\Sigma}^0_\alpha)\}_{\alpha<\omega_1}$ By *levels* of this hierarchy we mean the classes $\mathsf{QCB}_0(\mathbf{\Sigma}^0_\alpha)$ as well as the classes $\mathsf{QCB}_0(\mathbf{\Pi}^0_\alpha)$ and $\mathsf{QCB}_0(\mathbf{\Delta}^0_\alpha)$.
3. The *Luzin hierarchy* of QCB_0-spaces is defined similarly.

Hierarchies of QCB_0-Spaces

Proposition

Let $\Gamma \in \{\boldsymbol{\Sigma}^0_\alpha, \boldsymbol{\Pi}^0_\alpha, \boldsymbol{\Sigma}^1_n, \boldsymbol{\Pi}^1_n \mid 1 \leq \alpha < \omega_1, 1 \leq n < \omega\}$. Then any retract of a Γ-representable space is a Γ-representable space.

Theorem

The Borel hierarchy and the Luzin hierarchy of QCB_0-spaces do not collapse. More precisely, $QCB_0(\boldsymbol{\Sigma}^0_\alpha) \not\subseteq QCB_0(\boldsymbol{\Pi}^0_\alpha)$ for each countable ordinal $\alpha \geq 2$, and $QCB_0(\boldsymbol{\Sigma}^1_n) \not\subseteq QCB_0(\boldsymbol{\Pi}^1_n)$ for each positive integer n.

Relating the Hierarchies

Here we establish close relationships of the hierarchies of CB_0-spaces with the corresponding hierarchies of QCB_0-spaces.

Proposition

Let $\Gamma \in \{\mathbf{\Pi}^0_2, \mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n \mid 3 \leq \alpha < \omega_1, 1 \leq n < \omega\}$*. Then* $CB_0(\Gamma) = QCB_0(\Gamma) \cap CB_0$*.*

Luzin Hierarchy and Continuous Functionals

Define the Kleene-Kreisel continuous functionals by induction on k: $\mathbb{N}\langle 0\rangle := \omega$ and $\mathbb{N}\langle k+1\rangle := \omega^{\mathbb{N}\langle k\rangle}$.

Theorem

Let k be a positive integer and B a non-empty subset of $\mathcal{N}$. Then $B \in \mathbf{\Sigma}^1_k(\mathcal{N})$ iff there is a continuous function $f : \mathbb{N}\langle k\rangle \to \mathcal{N}$ with $rng(f) = B$.

Theorem

For any positive integer k, $\mathbb{N}\langle k+1\rangle \in \mathrm{QCB}_0(\mathbf{\Pi}^1_k) \setminus \mathrm{QCB}_0(\mathbf{\Sigma}^1_k)$.

The category of projective QCB_0-spaces

Theorem

The category $QCB_0(\mathbf{P})$ of projective qcb-spaces is cartesian closed.

It turns out that $QCB_0(\mathbf{P})$ is in a sense the smallest cartesian closed subcategory of QCB_0 containing ω.

Theorem

There is no full cartesian closed subcategory C of QCB_0 such that C inherits binary products from QCB_0, contains the discrete space ω of natural numbers and is contained itself in $QCB_0(\mathbf{\Sigma}^1_n)$ for some $1 \leq n < \omega$.

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Introduction

In this part of the talk we present results of a recent joint work with Luca Motto Ros and Philipp Schlicht [RSS12] on some weaker versions of the Wadge reductions in the class of quasi-Polish spaces recently identified by Matthew de Brecht as a natural class of spaces for DST and CA.
There are several reasons and several directions to generalize the Wadge reducibility $\leq_W$ on the Baire space including the following:

Possible variations of the Wadge reducibility

1. one can consider another natural class of reducing functions in place of the continuous functions;
2. one can consider more complicated topological spaces instead of the Baire space (the notion of the Wadge reducibility makes sense for arbitrary topological space);
3. one can consider reducibility of functions on topological spaces rather than the reducibility of sets (the sets may be identified with their characteristic functions);
4. one can consider more complicated reductions than many-one reductions.

What is already achieved

In any of the mentioned directions, a certain progress is already achieved, although the situation typically becomes more complicated than in the classical case.

E.g., if we consider direction 3) for the simplest possible generalization — partitions of the Baire space to $k \geq 3$ subsets, we obtain a rather complicated degree structure, but it is still a well partial order, hence it can serve as a scale to measure the topological complexity of k-partitions of the Baire space.

In direction 4), the so called Weihrauch reductions became popular and useful to characterize topological complexity of some important computation problems in CA.

Δ^0_α-Reductions in $\mathcal{N}$

For a space X and a pointclass $C \subseteq P(X)$, *C-reducibility* is the preorder on $P(X)$ corresponding to many-one reductions by functions on X such that the preimage of any set in C is again in C. In a series of papers, A. Andretta, D. Martin and L. Motto Ros [An06, MR09] have shown that, under suitable set-theoretic assumptions, the structure of C-degrees in the Baire space is isomorphic to the structure of Wadge degrees, where C is the class of Borel sets or is a level of the Borel hierarchy $\mathbf{\Sigma}^0_\alpha$, $1 \leq \alpha < \omega_1$. Note that in fact the $\mathbf{\Sigma}^0_\alpha$- $\mathbf{\Pi}^0_\alpha$- and $\mathbf{\Delta}^0_\alpha$-reducibilities coincide for each $1 \leq \alpha < \omega_1$.
Thus, we obtain a series of natural weaker (than the Wadge reducibility) classifications of subsets of the Baire space.

Wadge reducibility in non-zero-dimensional spaces

P. Hertling [Her96]has shown that the structure of Wadge degrees in the space of reals is much more complicated than the structure of Wadge degrees in the Baire space. In particular, there are infinite antichains and infinite descending chains in the structure of Wadge degrees of $\mathbf{\Delta}^0_2$-sets.

Also for many other non-zero-dimensional spaces the structure of Wadge degrees turns out more complicated than the structure of Wadge degrees in zero-dimensional spaces [Se05, IST12, Sc12]. To my knowledge, currently there is no good understanding of the structure of Wadge degrees in non-zero-dimensional spaces.

But maybe, the structure of $\mathbf{\Delta}^0_\alpha$-degrees in such spaces for $\alpha > 1$ is easier? We show that this is really the case, at least for some natural spaces.

$\mathbf{\Delta}^0_\alpha$-isomorphisms of Uncountable QP-Spaces

Theorem

[RSS12] Let X be an uncountable quasi-Polish space.

(1) $\mathcal{N} \simeq_\omega X$;

(2) *if* $\dim(X) \neq \infty$ *then* $\mathcal{N} \simeq_3 X$;

(3) *if* $\dim(X) = \infty$ *and X is Polish then* $\mathcal{N} \not\simeq_n X$ *for every* $n < \omega$;

(4) $P\omega \not\simeq_n \mathcal{N}$ *for every* $n < \omega$. *The same result holds when replacing $P\omega$ with any other quasi-Polish space which is universal for (compact) Polish spaces.*

$\mathbf{\Delta}^0_\alpha$-isomorphisms of Uncountable QP-Spaces

Definition

The empty set $\emptyset$ is the only space with *dimension* -1, in symbols $\dim(\emptyset) = -1$.
Let α be an ordinal and $\emptyset \neq X$. We say that X has *dimension* $\leq \alpha$, $\dim(X) \leq \alpha$ in symbols, if every $x \in X$ has arbitrarily small neighborhoods whose boundaries have dimension $< \alpha$.
We say that a space X has *dimension* α, $\dim(X) = \alpha$ in symbols, if $\dim(X) \leq \alpha$ and $\dim(X) \not\leq \beta$ for all $\beta < \alpha$.
Finally, we say that a space X has *dimension* ∞, $\dim(X) = \infty$ in symbols, if $\dim(X) \not\leq \alpha$ for every $\alpha \in \mathrm{On}$.

$\mathbf{\Delta}^0_\alpha$-Degrees in Uncountable Quasi-Polish Spaces

Theorem

[RSS12] Let X be an uncountable quasi-Polish space and $\mathcal{F}$ be a family of reducibilities.

(1) *If $\dim(X) \neq \infty$ then the $\mathcal{F}$-hierarchy on X is isomorphic to the $\mathcal{F}$-hierarchy on $\mathcal{N}$ whenever $\mathcal{F} \supseteq \mathrm{D}_3^{\mathrm{W}}$. Hence the $(\mathbf{B}, \mathcal{F})$-hierarchy on X is semi-well-ordered, and assuming AD the $\mathcal{F}$-hierarchy on X is is is semi-well-ordered as well.*

(2) *If X is universal for Polish (respectively, quasi-Polish) spaces and $\mathcal{F} \supseteq \mathrm{D}_3^{\mathrm{W}}$, then the $\mathcal{F}$-hierarchy on X is isomorphic to the $\mathcal{F}$-hierarchy on $[0,1]^\omega$ (respectively, on $P\omega$).*

(3) *If $\mathcal{F} \supseteq \mathcal{B}_\omega$ then the $\mathcal{F}$-hierarchy on X is isomorphic to the $\mathcal{F}$-hierarchy on $\mathcal{N}$.*

$\mathbf{\Delta}^0_\alpha$-Degrees in Uncountable QP-Spaces

The next result shows that the previous theorem cannot be improved to the D_2-reducibility.

Theorem

[RSS12]

1. *Suppose X is an uncountable locally connected Polish space. Then the $(\mathbf{B}, \mathrm{D}_2)$-hierarchy on X has a 4-antichain.*
2. *There are uncountable antichains in the D_2-hierarchy on $[0, 1]$.*
3. *The quasi-order $(P(\omega), \subseteq^*)$ of inclusion modulo finite sets on $P(\omega)$ embeds into $(\mathbf{\Sigma}^0_2(\mathbb{R}^2), \leq_{\mathrm{D}_2})$.*

$\mathbf{\Delta}^0_\alpha$-Degrees in Countable Spaces

Theorem

[RSS12]

1. *Let X be a countable Polish space or a finite T_0-space. Then the* D_1*-structure on X is semi-well-ordered.*
2. *There is a scattered ω-algebraic domain $(X, \leq)$ such that $(P(X), \leq_{\mathrm{W}})$ is isomorphic to the poset $(\bar{2} \cdot \omega) + \bar{4}$.*

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Definition

Let X be a space and $\mu, \nu : X \to k$ be k-partitions of X and $\mathcal{C}$ a class of k-partitions of C. We say that μ is *Wadge reducible* to ν (in symbols, $\mu \leq_W \nu$) if $\mu = \nu \circ f$ for some continuous function f on X. For $k = 2$ this definition coincides with the Wadge reducibility of subsets of X. Let $\mathcal{C} \leq_W \nu$ denote that any element of $\mathcal{C}$ is Wadge reducible to ν, and $\nu \equiv \mathcal{C}$ denote that ν is Wadge complete in $\mathcal{C}$, i.e. $\nu \in \mathcal{C}$ and $\mathcal{C} \leq_W \nu$.
Since for many natural spaces (e.g., for the space of reals) the structure of Wadge degrees of $\mathbf{\Delta}^0_2$ is complicated (Hertling 1996) we restrict our attention to the Baire space.

Borel Partitions

We consider the Wadge reducibility of k-partitions for the Baire and Cantor spaces. To our knowledge, the first result about the Wadge reducibility of k-partitions of the Baire and Cantor spaces is a theorem of van Engelen-Miller-Steel of 1987. The following assertion is a particular case of that theorem.

Theorem

The structure $(\mathbf{\Delta}^1_1(X)_k; \leq_W)$ of Borel-measurable k-partitions is a well preorder.

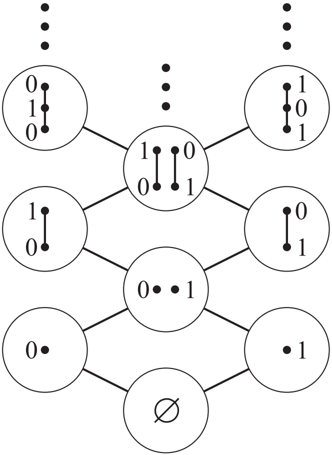
This assertion gives important information about the structure $(\mathbf{\Delta}^1_1(X); \leq_W)$ but it leaves open many questions. In order to understand some of its initial segments we need the following notions.

Homomorphic Preorder

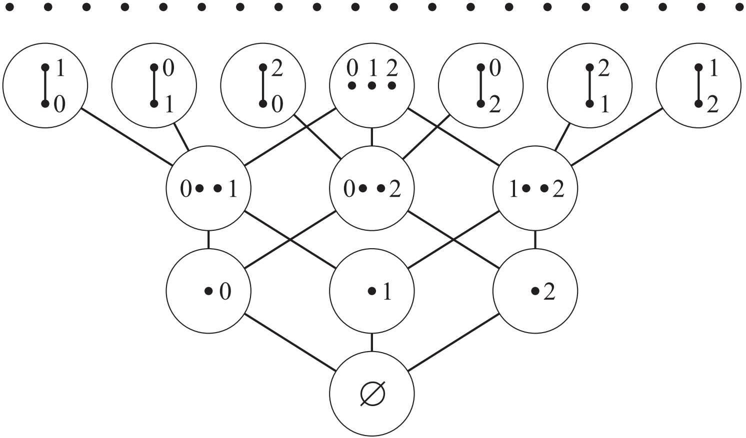
A poset $(P;\leq)$ will be often shorter denoted just by P. Any subset of P may be considered as a poset with the induced partial ordering. In particular, this applies to the "cones" $\check{x}=\{y\in P\mid x\leq y\}$ and $\hat{x}=\{y\in P\mid y\leq x\}$ defined by any $x\in P$. A *well partial order* is a poset P that has neither infinite descending chains nor infinite antichains; for such posets there is a canonical rank function *rk* assigning ordinals to the elements of P. By *a forest* we mean a finite poset in which every upper cone $\check{x}$ is a chain. *A tree* is a forest having the biggest element (called *the root* of the tree). Note that any forest is uniquely representable as a disjoint union of trees, the roots of the trees being the maximal elements of the forest.

Homomorphic Preorder

A *k-labeled forest* (or just a k-forest) is an object $(P; \leq, c)$ consisting of a finite forest $(P; \leq)$ and a labeling $c : P \to k$. A *homomorphism* f between k-forests is a monotone function $f : (P; \leq) \to (P'; \leq')$ respecting the labelings, $c = c' \circ f$.
Let $\mathcal{F}_k$ and $\mathcal{T}_k$ be the sets of all finite k-forests and finite k-trees, respectively. Define a preorder $\leq$ on $\mathcal{F}_k$ as follows:
$(P, c) \leq (P', c')$, if there is a morphism from (P, c) to (P', c').
The quotient structure of $(\mathcal{F}_k; \leq)$ is a well poset intimately related to the Boolean hierarchy of k-partitions.
Sets $\widetilde{\mathcal{F}}_k$ and $\widetilde{\mathcal{T}}_k$ are defined similarly, only for countable forests without infinite chains.



Picture 1: An initial segment of $\widetilde{\mathcal{F}}_2$.



Picture 2: An initial segment of $\widetilde{\mathcal{F}}_3$.

$\mathbf{\Delta}^0_2$-Partitions and beyond

Theorem

1. The quotient structures of $(\mathcal{F}_k \setminus \{\emptyset\}; \leq)$ and of $(B(\mathbf{\Sigma}^0_1(\omega^\omega))_k; \leq_W)$ are isomorphic (Hertling 1993).
2. The quotient structures of $(\widetilde{\mathcal{F}}_k \setminus \{\emptyset\}; \leq$ and of $((\mathbf{\Delta}^0_2(\omega^\omega))_k; \leq_W)$ are isomorphic (Selivanov 2007).

Recently, this result was extended to wider initial segments of Borel-measurable partitions, in particular to the structure of $\mathbf{\Delta}^0_3$-Partitions.

Contents

Wagner hierarchy

In [Wag79] K. Wagner gave in a sense the finest possible topological classification of regular ω-languages (i.e., of the subsets of X^ω for a finite alphabet X recognized by finite automata) known as the Wagner hierarchy. In particular, he completely described the (quotient structure of the) preorder $(\mathcal{R}; \leq_{CA})$ formed by the class $\mathcal{R}$ of regular subsets of X^ω and the reducibility by functions continuous in the Cantor topology on X^ω.
The aim is to generalize this theory from the case of regular ω-regular languages to the case of regular k-partitions of X^ω, i.e. k-tuples $(A_0, \ldots, A_{k-1})$ of pairwise disjoint regular sets satisfying $A_0 \cup \cdots \cup A_{k-1} = X^\omega$. Note that the ω-languages are in a bijective correspondence with 2-partitions of X^ω.

Wagner hierarchy

1) The structure $(\mathcal{R}; \leq_{CA})$ is almost well-ordered with the order type ω^ω, i.e. there are $A_\alpha \in \mathcal{R}$, $\alpha < \omega^\omega$, such that $A_\alpha <_{CA} A_\alpha \oplus \overline{A}_\alpha <_{CA} A_\beta$ for $\alpha < \beta < \omega^\omega$ and any regular set is CA-equivalent to one of the sets $A_\alpha, \overline{A}_\alpha, A_\alpha \oplus \overline{A}_\alpha (\alpha < \omega^\omega)$.
2) The CA-reducibility coincides on $\mathcal{R}$ with the DA-reducibility, i.e. the reducibility by functions computed by deterministic asynchronous finite transducers, and $\mathcal{R}$ is closed under the DA-reducibility.
3) Any level $\mathcal{R}_\alpha = \{C \mid C \leq_{DA} A_\alpha\}$ of the Wagner hierarchy is decidable.

Muller k-Acceptors

A Muller k-acceptor is a pair $(\mathcal{A}, c)$ where $\mathcal{A}$ is an automaton and $c : C_{\mathcal{A}} \to k$ is a k-partition of $C_{\mathcal{A}} = \{f_{\mathcal{A}}(\xi) \mid \xi \in X^{\omega}\}$ where $f_{\mathcal{A}}(\xi)$ is the set of states which occur infinitely often in the sequence $f(i, \xi) \in Q^{\omega}$. Note that in this paper we consider only deterministic finite automata. Such a k-acceptor recognizes the k-partition $L(\mathcal{A}, c) = c \circ f_{\mathcal{A}}$ where $f_{\mathcal{A}} : X^{\omega} \to C_{\mathcal{A}}$ is the map defined above. We have the following characterization of the ω-regular partitions.

Proposition

A partition $L : X^{\omega} \to k$ is regular iff it is recognized by a Muller k-acceptor.

Labeled Trees and Forests

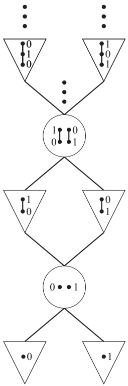
Let $(Q;\leq)$ be a poset. A *Q-poset* is a triple $(P,\leq,c)$ consisting of a finite nonempty poset $(P;\leq)$, $P\subseteq\omega$, and a labeling $c:P\to Q$. A *morphism* $f:(P,\leq,c)\to(P',\leq',c')$ of Q-posets is a monotone function $f:(P;\leq)\to(P';\leq')$ satisfying $\forall x\in P(c(x)\leq c'(f(x)))$. Let $\mathcal{P}_Q$, $\mathcal{F}_Q$ and $\mathcal{T}_Q$ denote the sets of all finite Q-posets, Q-forests and Q-trees, respectively.
The *h-preorder* $\leq_h$ on $\mathcal{P}_Q$ is defined as follows: $P\leq_h P'$, if there is a morphism from P to P'. Note that for the particular case $Q=\bar{k}$ of the antichain with k elements we obtain the preorders $\mathcal{P}_k$, $\mathcal{F}_k$ and $\mathcal{T}_k$.

Labeled Trees and Forests

It is well known that if Q is a wqo then $(\mathcal{F}_Q;\leq_h)$ and $(\mathcal{T}_Q;\leq_h)$ are wqo's. Obviously, $P\subseteq Q$ implies $\mathbb{F}_P\subseteq\mathbb{F}_Q$, and $P\sqsubseteq Q$ (i.e., P is an initial segment of Q) implies $\mathbb{F}_P\sqsubseteq\mathbb{F}_Q$.
Define the sequence $\{\mathcal{F}_k(n)\}_{n<\omega}$ of preorders by induction on n as follows: $\mathcal{F}_k(0)=\overline{k}$ and $\mathcal{F}_k(n+1)=\mathcal{F}_{\mathcal{F}_k(n)}$. Identifying the elements $i<k$ of $\overline{k}$ with the corresponding minimal elements $s(i)$ of $\mathcal{F}_k(1)$, we may think that $\mathcal{F}_k(0)\sqsubseteq\mathcal{F}_k(1)$, hence $\mathcal{F}_k(n)\sqsubseteq\mathcal{F}_k(n+1)$ for each $n<\omega$ and $\mathcal{F}_k(\omega)=\bigcup_{n<\omega}\mathcal{F}_k(n)$ is a wqo.

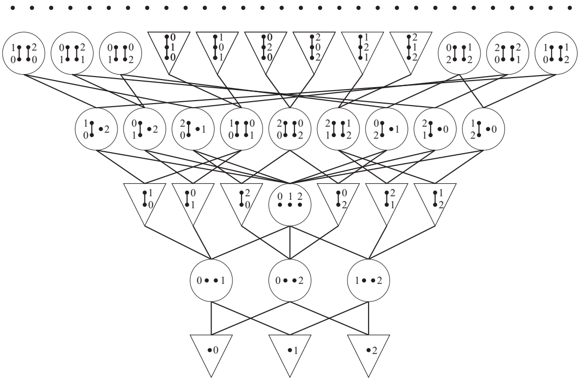
Labeled Trees and Forests

The preorders $\mathcal{F}_k(\omega)$, $\mathcal{T}_k(\omega)$ and the set $\mathcal{T}_k^{\sqcup}(\omega)$ of finite joins of elements in $\mathcal{T}_k(\omega)$, play an important role in the study of the FH of k-partitions because they provide convenient naming systems for the levels of this hierarchy (similar to the previous work where $\mathcal{F}_k$ and $\mathcal{T}_k$ where used to name the levels of the DH of k-partitions). Note that $\mathcal{F}_k(1) = \mathcal{F}_k$ and $\mathcal{T}_k(1) = \mathcal{T}_k$.
For the FH of ω-regular k-partitions, the structure $\mathcal{T}_k^{\sqcup}(2)$ is especially relevant. For $k = 2$ it is isomorphic to the structure of levels of the Wagner hierarchy.

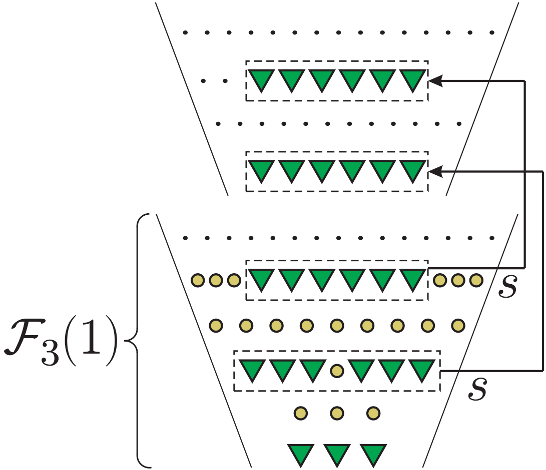


Picture 1: An initial segment of $\mathcal{F}_2$.

Labeled Trees and Forests



Picture 2: An initial segment of $\mathcal{F}_3$.



Picture 3: A fragment of $\mathcal{T}_3^{\sqcup}(2)$.

The structure of ω-regular k-partitions

Theorem

1. The quotient-posets of $(\mathcal{R}_k; \leq_{CA})$ and of $(\mathcal{R}_k; \leq_{DA})$ are isomorphic to the quotient-poset of $\mathcal{T}_k^{\sqcup}(2)$.
2. The relations $\leq_{CA}, \leq_{DA}$ coincide on $\mathcal{R}_k$, the same holds for the relations $\leq_{CS}, \leq_{DS}$.
3. The relations $L(\mathcal{A}, c) \leq_{CA} L(\mathcal{A}, c)$ and $L(\mathcal{A}, c) \leq_{DA} L(\mathcal{A}, c)$ are decidable.

Proof Sketch

1) Extending and modifying some operations of W. Wadge and A. Andretta on subsets of the Cantor space, we embed $\mathcal{T}_k^{\sqcup}(2)$ into $(\mathcal{R}_k; \leq_{CA})$ and $(\mathcal{R}_k; \leq_{DA})$ (an embedding is induced by $F \mapsto r(F)$).
2) We extend the author FH of sets [Se98] to the FH of k-partitions over $(\mathbf{\Sigma}^0_1 \cap \mathcal{R}, \mathbf{\Sigma}^0_2 \cap \mathcal{R})$ in such a way that $r(F)$ is CA-complete in $\mathbf{\Sigma}(F)$ and DA-complete in $\mathbf{\Sigma}\mathcal{R}(F)$.
3) Relate to any Muller k-acceptor $\mathcal{A} = (\mathcal{A}, c)$ the structure $(C_{\mathcal{A}}; \leq_0, \leq_1, c)$ where $C_{\mathcal{A}}$ is the set of cycles of $\mathcal{A}$, $D \leq_0 E$ iff some state in D is reachable in the graph of the automaton $\mathcal{A}$ from some state in E, and $D \leq_1 E$ iff $D \subseteq E$.

Proof Sketch

4) The structure $(C_{\mathcal{A}}; \leq_0, \leq_1, c)$ may be identified with some $P_{\mathcal{A}} \in \mathcal{P}_k(2)$.
5) Using the known facts [Se98] that $(\mathbf{\Sigma}^0_1 \cap \mathcal{R}, \mathbf{\Sigma}^0_2 \cap \mathcal{R})$ have the reduction property conclude that $\mathbf{\Sigma}\mathcal{R}(P_{\mathcal{A}}) = \mathbf{\Sigma}\mathcal{R}(F_{\mathcal{A}})$ where $F_{\mathcal{A}} \in \mathcal{T}_k^{\sqcup}(2)$ is the natural unfolding of $P_{\mathcal{A}}$.
6) Check that $L\mathcal{A})$ is CA-complete in $\mathbf{\Sigma}(F_{\mathcal{A}})$ and DA-complete in $\mathbf{\Sigma}\mathcal{R}(F_{\mathcal{A}})$ and conclude that $L\mathcal{A}) \equiv_{DA} r\mathcal{A})$.

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