Analytic eventually different families of functions are not maximal

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Mad families and Adrian Mathias, circa 1967

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1. *A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called almost disjoint (a.d.) if for any two distinct $x, y \in \mathcal{F}$ we have*

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Theorem (Mathias, 1967?)

An infinite analytic a.d. family in ω is not maximal.

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Question. Can an analytic e.d. family be maximal?

The answer - The moreover

Theorem (T., 2013)

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- Unlike Mathias' proof of his theorem, the proof of the above uses only classical ideas from descriptive set theory.
- The same ideas yield a new, "classical" proof of Mathias' theorem.
- Moreover, the ideas also make it possible to answer similar questions about eventually different (a.k.a. **cofinitary**) families and groups of *permutations* (which seem to have been posed by Andreas Blass about a decade ago): Analytic such families and groups are never maximal.

Recall that $A \subseteq \omega^\omega$ is analytic iff there is a tree T on $\omega \times \omega$ such that $A = p[T]$, i.e., A is the **projection** of the (closed) set $[T]$ of infinite branches through T. For $t \in T$, we define $t^0, t^1 \in \omega^{<\omega}$ by $t = (t^0, t^1)$.

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We will say that a tree T is **perfect in the first coordinate** if for every $t \in T$ there are $u, v \supseteq t$ such that $u^0 \perp v^0$.

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1. $p[T] = C \cup p[\hat{T}]$.
2. $\hat{T}$ *(which may be empty) is perfect in the first coordinate.*

The perfect set theorem and ordinal analysis revisited, II

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Since T is countable, there is $\lambda < \omega_1$ where $T^{\lambda+1} = T^\lambda$. Let $\hat{T} = T^\lambda$. If $x \in C = p[T] \setminus p[\hat{T}]$ then there is some $\alpha < \lambda$ and $t \in T^\alpha$ such that x is the only branch in T^α extending t. Whence C is countable.

$\square$

Diagonal sequences

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Definition

Call a sequence $(t_i)_{i\in\omega}$ in T a **diagonal sequence** if for all $i \neq j$, and all $y \in p[T_{t_i}]$, $z \in p[T_{t_j}]$ we have

$$y \cap z \subseteq t_i^0 \cap t_j^0.$$

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Claim: If $s, v \in T$ and $s^0 \perp v^0$, then there are $t, w \in T$ with $t \supseteq s$ and $w \supseteq v$ such that for all $y \in p[T_t]$ and all $z \in p[T_w]$ we have $y \cap z \subseteq t^0 \cap w^0$.

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Proof of claim: Otherwise, find $s \subseteq s_1 \subseteq \cdots$ and $v \subseteq v_1 \subseteq \cdots$ such that $|s_i^0 \cap v_i^0| \geq i$, contradicting that $p[T]$ is e.d. $\square$

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Fixing $s_0, v_0 \in T$ with $s^0 \perp v^0$, get $t \supseteq s_0$ and $w \supseteq v_0$ as in the claim and let $t_0 = t$. Since T is perfect in the first coordinate, find $s_1, v_1 \supseteq w$ with $s_1^0 \perp v_1^0$, and repeat... $\square$

Ordinal analysis for e.d. families

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Definition. We call the triple $(\lambda, (T^\alpha)_{\alpha \leq \lambda}, (t_i^\alpha)_{i < \omega, \alpha < \lambda})$ an *ordinal analysis* of T.

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If T is a tree on $\omega \times \omega$ s.t. $\mathcal{A} = p[T]$ is an e.d. family then there is a countable set $C \subseteq \omega^\omega$ and an ordinal analysis $(\lambda, (T^\alpha)_{\alpha \leq \lambda}, (t_i^\alpha)_{i<\omega, \alpha<\lambda})$ of T such that:

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For each $\alpha \leq \lambda$, let $C_\alpha \subseteq \omega^\omega$ be a countable set such that $p[T^\alpha] = C_\alpha \cup p[\hat{T}_\alpha]$, and let

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For $x \in \mathcal{A}$, fix y such that $(x, y) \in [T]$, and assume $x \notin C$. Let α be least such that $x \notin p[T^\alpha]$, and $\beta < \alpha$ least such that $(x, y) \supseteq t_i^\beta$ for some $i < \omega$. Now $(x, y) \in [T^\beta_{t_i^\beta}]$, and so since $x \notin C$, we must have $x \in p[\hat{T}_{t_i^\beta}]$. $\square$

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Enumerate λ as $(\alpha_j)_{j<\omega}$ and enumerate C as $(c_j)_{j<\omega}$.

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We will construct an increasing sequence of natural numbers

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3. For all $k \in [n_i, n_{i+1})$, $j, l \leq i$ and

$$z \in p[\hat{T}^{\alpha_j}_{t_l^{\alpha_j}}] \cup \{c_k : k \leq i\}$$

   we have

$$f_{i+1}(k) \neq z(k)$$

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Either there is some $j \leq i$ such that

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or there is $q' \leq q$ and $j \leq i$ and

$$h \in p[\hat{T}^{\beta_{q'}}_{t_j^{\beta_{q'}}}]$$

such that

$$h(k_0) = g(k_0).$$

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Theorem (T., 2013)

1. *An analytic cofinitary family in S_∞ is never maximal.*

Cofinitary families and groups

Definition

- An e.d. family in S_∞ (the group of permutations of ω) is called a **cofinitary** family of permutations. A cofinitary family is maximal if it is **maximal** under inclusion.
- A **cofinitary group** is a cofinitary family which forms a subgroup of S_∞. A cofinitary group is **maximal** if it is maximal under inclusion among cofinitary groups.

Theorem (T., 2013)

1. *An analytic cofinitary family in S_∞ is never maximal.*
2. *An analytic cofinitary group in S_∞ is never maximal.*

Happy Birthday, Sy!