# Multiverse, Recursive Saturation and Well-Foundedness Mirage

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# Outline

# The Multiverse View on Foundations of Set Theory

Question: how to make sense of the incompleteness phenomenon, independence results and modern set-theoretic practice?

Main idea of set-theoretic pluralism: there is no absolute set-theoretic background for mathematics.

There is a *multiverse* of universes of set theory, mathematical worlds that have equal (to a certain degree – different Multiverse conceptions differ) theoretical status:

- forcing extensions, canonical and non-canonical inner models, universes with and without large cardinals etc.
- each of the universes in the Multiverse instantiates a different concept of set and all these concepts of set are (theoretically – there can be pragmatically better ones) equally valid.

Question: what what kinds of universes can exist and how they are related to one another?

- Should all models in the multiverse have the same height?
- Should all models in the multiverse have the same width?
- Should all models in the multiverse be closed under certain set-theoretic operations (such as the ultrapower construction)?
- Do ill-founded models belong in the multiverse?
- Do models of fragments of $\mathrm{ZFC}$ (e.g. $\mathrm{ZF}$) belong in the multiverse?

## The Radical Pluralism

(Hamkins) Without an absolute set theoretic background, there cannot be an absolute notion of countability or well-foundedness. The relativity of the notion of set must extend to the notions of well-foundedness, height, and even the natural numbers (these are after all first-order concepts):

Every universe will be revealed to be countable and ill-founded from the perspective of another universe.

The natural numbers of a given universe will be revealed to be ill-founded from the perspective of another universe.

# Multiverse – different width

Some basic axioms we might ask a multiverse to satisfy (these express that the multiverse contains universes of different width):

- ***Closure Under Set Forcing***. If $M$ is a world and $\mathbb{P} \in M$ is a poset, then there is a world $M[G]$ where $G \subseteq \mathbb{P}$ is generic over $M$.
- ***Closure Under Class Forcing***. If $M$ is a world and $\mathbb{P} \subseteq M$ is a tame class forcing notion, then there is a world $M[G]$ where $G \subseteq \mathbb{P}$ is generic over $M$.
- ***Closure Under Inner Models***. If $M$ is a world and $W \subseteq M$ is an inner model, then $W$ is a world.
- ***Closure Under Grounds***. If $M$ is a world and $W \subseteq M$ is a ground—meaning that $M = W[G]$ for $G \subseteq \mathbb{P} \in W$ a $W$-generic—then $W$ is a world.

# Multiverse – different width

Remark: already enough to describe a robust multiverse notion. If $M$ is any countable model of ZFC then the *generic multiverse of M*, the closure of $M$ under taking (set) forcing extensions and grounds, will satisfy the Closure Under Set Forcing and Closure Under Grounds axioms.

More broadly, if we fix an ordinal $\alpha$ s.t. there is a transitive $M \models$ ZFC of height $\alpha$, then the collection of transitive models of height $\alpha$ form a multiverse satisfying the above closure axioms.

We can weaken Closure Under Set Forcing or Closure Under Class Forcing by restricting what forcing notions are allowed, e.g. a class forcing notion $\mathbb{P}$ is said to be $\mathrm{Ord}$-*cc* if every antichain of $\mathbb{P}$ is a set (and every $\mathrm{Ord}$-cc class forcing is tame).

- ***Closure Under*** $\mathrm{Ord}$-***cc Class Forcing*** If $M$ is a world and $\mathbb{P} \subseteq M$ is an $\mathrm{Ord}$-cc class forcing, then there is a world $M[G]$ where $G \subseteq \mathbb{P}$ is generic over $M$.

# Multiverse – different height

We might also think that universes can have different heights – the following axioms express that the multiverse contains shorter worlds:

- ***Closure Under Rank-Initial Segments***. If $M$ is a world and $\theta$ is an ordinal in $M$ so that ${V_\theta}^M \models \mathrm{ZFC}$, then ${V_\theta}^M$ is a world.
- ***Closure Under $\in$-Initial Segments***. If $M$ is a world and $N \in M$ is a transitive set so that $N \models \mathrm{ZFC}$, then $N$ is a world.

The following axioms can be seen as strengthenings of the previous two axioms, as well as the Closure Under Inner Models and Closure Under Grounds axioms. The latter two of the following axioms are the first we will see which force the multiverse to contain nonstandard models:

- ***Standard Realizability***. If $M$ is a world and $N \subseteq M$ is a definable transitive class model of ZFC, then $N$ is a world.
- ***Set-Like Realizability***. If $M$ is a world and $N \subseteq M$ is a definable set-like class model of ZFC, then $N$ is a world. Here, by saying $N$ is set-like we mean that $M$ thinks each element of $N$ has set-many $\in^N$-predecessors.
- ***Realizability***. If $M$ is a world and $N \subseteq M$ is a definable class model of ZFC, then $N$ is a world.

# Multiverse – different height

In the other direction, we might think that there is no tallest universe:

- ***Countability***. If $M$ is a world then there is a world $N$ so that $M \in N$ and $N \models M$ is countable.

Observe that if the multiverse satisfies Closure Under Set Forcing and if every world is an element of another world, then Countability is satisfied for free.

A more radical multiverse where every world is seen to be $\omega$-nonstandard by some larger world. This is captured by Well-Foundedness Mirage axiom:

- ***Well-Foundedness Mirage***. If $M$ is a world then there is a world $N$ so that $M \in N$ and $N$ thinks that $M$ is $\omega$-nonstandard.
- ***Weak Well-Foundedness Mirage***. If $M$ is a world then there is a world $N$ so that $M \in N$ and $N$ thinks that $M$ is nonstandard.

# Recursive saturation

Gitman and Hamkins showed that the collection of countable, recursively saturated models of ZFC form a multiverse which satisfies Closure Under Set and Class Forcing, Realizability, Countability, and Well-Foundedness Mirage.

### Definition

A model $M$ of set theory is **recursively saturated** if it realizes every finitely consistent, computable type.

That is, if $p(x) = \{\varphi_n(x, a) : n \in \omega\}$ is a type with a parameter $a \in M$ s.t. $\{\ulcorner \varphi_n \urcorner : n \in \omega\}$ is a computable subset of $\omega$ and

$$\forall n \in \omega\, \exists c \in M \;\; M \models \bigwedge_{i<n} \varphi_i(c, a),$$

then

$$\exists d \in M\, \forall n \in \omega \;\; M \models \varphi_n(d, a).$$

# Recursive saturation & Paris models

Observe that if $M$ is recursively saturated then it must be $\omega$-nonstandard—consider the type $\{x > n : n \in \omega\} \cup \{x \in \omega\}$.

But not every $\omega$-nonstandard model will be recursively saturated. One way to see this goes through the observation that if $M$ is recursively saturated then the definable ordinals of $M$ must be bounded, because the type asserting that $x$ has higher rank than every definable ordinal is computable. But there are models all of whose ordinals are definable.

## Definition (Paris)

A model of set theory is a *Paris model* if each of its ordinals is definable without parameters.

## Theorem (Paris (1973))

*If $T$ is a consistent extension of* ZFC *then $T$ has a countable $\omega$-nonstandard Paris model.*

# Recursive saturation & nonstandardness

The following proposition demonstrates the centrality of recursive saturation to the Gitman–Hamkins multiverse.

**Proposition**

*Suppose $M \models \mathrm{ZFC}$ is $\omega$-nonstandard and $N \in M$ is a model of set theory. Then $N$ is recursively saturated.*

**Proof.**

Let $p(x) = \{\varphi_n(x) : n \in \omega\}$ be a computable type with a parameter from $N$ which is finitely consistent over $N$.

W. l. o. g. for all $n > m$ it holds that $\varphi_n(x) \Rightarrow \varphi_m(x)$.

By assumption, $\varphi_n(x)$ is realized for each standard $n$. So by overspill applied in $M$ there must be a nonstandard $e$ so that $M$ thinks $N$ has an element $x$ satisfying $\varphi_e(x)$

The argument uses some absoluteness, namely the fact that $p$ is computable (thus coded in any $\omega$-nonstandard model) and that $N$ is a set in $M$, so $M$ has the truth definition for $N$.

So also for each standard $n$ we have $N \models \varphi_n(x)$ □

# Weakenings of the mirage

As a consequence, if a multiverse satisfies the Well-Foundedness Mirage axiom, then every world in the multiverse must be recursively saturated.

Main question: could this be avoided by weakening the Well-Foundedness Mirage axiom?

The first main theorem of this article answers this question in the positive, giving a natural model of the Weak Well-Foundedness Mirage axiom.

### Main Theorem

*Assume that every real is in a transitive model of* ZFC*. Then the collection of countable, nonstandard models of* ZFC *form a multiverse satisfying the Closure Under Set and Class Forcing, Realizability, Countability, and Weak Well-Foundedness Mirage axioms.*

*Necessarily, this multiverse contains worlds which are not recursively saturated.*

# Weakenings of the mirage

An alternative way to weaken Well-Foundedness Mirage to allow non-recursively saturated worlds:

**Definition**

Given models of set theory $M \subseteq N$ say that $N$ **end-extends** $M$ if $M$ is a transitive sub-class of $N$, i.e. $a \in^N b \in M \Rightarrow a \in M$, and $N$ **top-extends** $M$ if every element of $N \setminus M$ has higher rank then all ordinals of $M$.

Given a top-extension (or elementary end-extension) $N$ of $M$, $M$ is **topped** by $N$ if $M$ is an element of $N$, i.e. $\exists m \in N$ s.t. $M = \{a \in N : a \in^N m\}$. $M$ is **covered** by $N$ if there is $m \in N$ so that $M \subseteq \{a \in N : a \in^N m\}$.

If $N$ is an elementary end-extension of $M$, then it is already a top-extension (also for nonstandard models). Without elementarity, the implication does not hold.

Also, if $N$ is a transitive model (that is a top-extension of $M$), then $M$ is topped in $N$ if and only if $M$ is covered by $N$. For nonstandard models covering is strictly weaker topping.

The discussion above: if $M$ is topped by an $\omega$-nonstandard model $N$, then $M$ is recursively saturated. But this need not be the case if $M$ is merely covered by $N$. This is how we will weaken the Well-Foundedness Mirage axiom (we must weaken the Countability axiom for the same reason).

# Weakenings of the mirage

- ***Covering Countability***. If $M$ is a world then there is a world $N$ with $(m, e) \in N$ which end-extends $M$ so that $N$ thinks $(m, e)$ is countable.
- ***Covering Well-Foundedness Mirage***. If $M$ is a world then there is a world $N$ with $(m, e) \in N$ which end-extends $M$ so that $N$ thinks $(m, e)$ is $\omega$-nonstandard.

The second main theorem gives a multiverse which satisfies these two axioms whose worlds which contains non-recursively saturated worlds:

## Main Theorem

*Assume that* ZFC *is consistent. Then there is a multiverse, some of whose worlds are not recursively saturated, which satisfies Closure Under Set Forcing and* $\mathrm{Ord}$-*cc Class Forcing, Set-Like Realizability, Covering Countability, and Covering Well-Foundedness Mirage.*

# End-extensions

A classical result due to Keisler and Morley is that elementary end-extensions always exist, for countable models.

**Theorem (Keisler–Morley Theorem (1968))**

*Let $M \models \mathrm{ZFC}$ be countable. Then there is $N$ a nontrivial elementary end-extension of $M$.*

This result does not hold in general for uncountable models: define that $M$ is **rather classless** if all its amenable classes are definable. Kaufmann showed that rather classless models exist under the assumption of $\diamondsuit$ and Shelah showed this extra assumption could be removed. Observe that rather classless models must be uncountable, since cofinal $\omega$-sequences over a model of ZFC are always amenable and never definable.

It follows from Kaufmann's theorem that elementary end-extensions of models of ZFC always code undefinable classes and that rather classless models cannot have elementary end-extensions.

# End-extensions

We can often ensure that elementary end-extensions preserve properties of the original model, even if those properties are not first-order. A specific instance: having cofinally many ordinals definable from a single parameter.

**Lemma**

*Suppose $M \models \mathrm{ZFC}$ is countable and there is $a \in M$ so that cofinally many ordinals of $M$ are definable from $a$. Then $M$ has an elementary end-extension $M'$ so that there is $a' \in M'$ from which cofinally many ordinals are definable.*

**Proof.**

Let $M^+$ be an elementary end-extension of $M$. Let $M'$ be a Skolem hull of $M^+$ which contains all of $M$ and some fixed $b \in M^+ \setminus M$. $M^+$ will always have definable maps for sending $x$ to the least rank of a witness to $\varphi(x, y)$. As such, each ordinal in $M'$ will be definable from $b$ and some $c \in M$.
Now let $\theta_\varphi(\beta) \in M'$ be the supremum of the ordinals definable using $\varphi(x, a, b)$ where $b \in V_\beta$.
Then $\theta_\varphi(\beta)$ is definable from $a$ and $\beta$.
So the collection of $\theta_\beta(\varphi)$ for arbitrary $\varphi$ and $\beta \in \mathrm{Ord}^M$ is cofinal in the ordinals of $M'$. Note that this is collection is still cofinal if we consider only cofinally many $\beta \in \mathrm{Ord}^M$.
So if we restrict to those $\beta$ which are definable from $a$ then we get that these $\theta_\beta(\varphi)$ are all definable from $a' = (a, b)$. $\square$

## End-extensions

The Keisler–Morley theorem can be seen as expressing a property of the multiverse of all countable models of ZFC. We can ask whether other multiverses satisfy this property.

- ***Keisler–Morley Extension Property***. If $M$ is a world then there is a world which is a nontrivial elementary end-extension of $M$.

**Theorem (Barwise Extension Theorem (1975))**

*Let $M \models \mathrm{ZFC}$ be countable. Then there is $N \models \mathrm{ZFC} + V = L$ which end-extends $M$.*

Again, this theorem can be seen as expressing a property of the multiverse of all countable models of ZFC, one which can possibly be satisfied by other multiverses.

- ***Barwise Extension Property***. If $M$ is a world then there is a world $N \models V = L$ which nontrivially end-extends $M$.

# Iterated Ultrapowers

Let $M = M_0 \models$ ZFC and $U = U_0 \in M$ an ultrafilter on a set $D = D_0 \in M$. Once you take one ultrapower of $M$ by $U$, we can iterate:
Let $M_1$ be the ultrapower of $M_0$ by $U_0$ and let $j_0 : M_0 \to M_1$ be the corresponding map. Set $D_1 = j_0(D_0)$ and $U_1 = j_0(U_0)$. Then $M_1 \models U_1$ is an ultrafilter over $D_1$ and so we can take the ultrapower of $M_1$ by $U_1$ to obtain $M_2$, $j_1$, $D_2$, and $U_2$.

An important point is that, just like $M_1$ and $j_0$ are definable over $M_0$, so are further iterates $M_n$ and the maps $j_n$. Usual interest: $M$ is a transitive set and so $n$ is a standard natural number.

Indeed, $j_{0,n} = j_n \circ \cdots \circ j_0 : M_0 \to M_n$ can itself be seen as an ultrapower map.

This context: $M_0$ is $\omega$-nonstandard and so $n$ may be nonstandard, but still: if $M_n$ is a definable class over $M$, as is $j_n$, and $j_{0,n} : M_0 \to M_n$ can be seen as an ultrapower map, even when $n$ is nonstandard.

Last, but not least: iterated ultrapowers can be seen as a single ultrapower, even in the case where the ultrapower is iterated a nonstandard number of steps.

# Cofinal Embeddings into Ultrapowers

## Lemma

*Let $M \models \mathrm{ZFC}$ and $U \in M$ be an ultrafilter over $D \in M$. Let $N$ be the ultrapower of $M$ by $U$. Then $M$ embeds cofinally into $N$: given any $[b]_U \in N$ there is $x \in M$ so that $b \in^N [\mathrm{c}_x]$.*

## Proof.

Since $N$ is an elementary extension of $M$ it suffices to show that the ordinals of $M$ embed cofinally into the ordinals of $N$.

Suppose toward a contradiction that $\alpha \in \mathrm{Ord}^N$ is above every ordinal in the image of $M$ via the ultrapower map by $U_e$.

Then $\alpha = [a]_{U_e}$ is the equivalence class modulo $U$ of some function $a : D \to \mathrm{Ord}^M$ which is in $M$.

Let $\xi \in \mathrm{Ord}^M$ be any ordinal above $\operatorname{ran} a$, which exists by the Replacement axiom applied in $M$.

But then $[a]_{U_e} < [\mathrm{c}_\xi]_U$, contradicting that $\alpha$ is above the image of $M$. □

# Ultrapowers

We can ask whether a multiverse is closed under ultrapowers.

- *Closure Under Ultrapower*. If $M$ is a world and $U \in M$ is an ultrafilter on $D \in M$ then the ultrafilter of $M$ by $U$ is a world.

Lemma on cofinal embeddings implies that Closure Under Ultrapower is implied by Set-Like Realizability. As both multiverses we consider here satisfy Set-Like Realizability they will both satisfy Closure Under Ultrapower.

# Weak Well-Foundedness Multiverse

**Theorem**

*Suppose that every countable set is an element of a transitive model of* ZFC. *Then the collection of countable nonstandard models of* ZFC *form a multiverse which satisfies Closure Under Set and Class Forcing, Realizability, Countability, Weak Well-Foundedness Mirage, the Keisler–Morley Extension Property, and the Barwise Extension Property.*

Remark: this multiverse contains worlds which are $\omega$-standard and thus, *a fortiori*, non-recursively saturated.
And of course there are many countable $\omega$-nonstandard models which are not recursively saturated.

**Proof...**

Closure Under Set and Class Forcing, Realizability, Countability, and the Keisler–Morley and Barwise Extension Properties are immediate.

# Weak Well-Foundedness Multiverse

Proof...

For the Weak Well-Foundedness Mirage axiom: take a world $M$ in this multiverse and let $W$ be the well-founded part of $M$, which we identify with its transitive collapse.

Let $U$ be a ctm of ZFC with $M, W \in U$, w.l.o.g. $U$ thinks $M$ is countable'.

Inside $U$ there is an admissible set $A$ which $U$ thinks is countable and s.t. $M, W \in A$.

Let $\mathcal{L}_A$ be the admissible fragment of $\mathcal{L}_{\omega_1,\omega}$ associated with $A$ and consider $\mathcal{L}_A$ theory $T$: ZFC plus the collection of sentences: $x \in a \Leftrightarrow \bigvee_{b \in a} x = b$ for each $a \in A$.

Then any model of $T$ must end-extend $A$ and $T$ is $\Sigma_1$-definable over $A$ – $T$ has a model, namely $U$.

Apply the Barwise completeness theorem in $V$: $A$ thinks that $T$ is consistent.

Apply the Barwise completeness theorem in $U$: $U$ has a countable model $N$ of $T$.

# Weak Well-Foundedness Multiverse

Proof.

Because the well-founded part of $M$ contains $A \ni W$, $M$ correctly computes $W$ is the well-founded part of $M$ (in $N$ we can build by transfinite recursion an embedding of $W$ onto an initial segment of $M$ which eventually gives full embedding).

Finally, it may not be that $N$ is nonstandard.

But: we can take an elementary end-extension of $N$ which is ill-founded, and this will give us the desired world in this multiverse witnessing Weak Well-Foundedness Mirage for $M$. $\square$

# Weak Well-Foundedness Multiverse

**Corollary**

*With the same consistency assumption as before, the collection of countable nonstandard but $\omega$-standard models gives a multiverse which satisfies Closure Under Set and Class Forcing, Standard Realizability, Countability, Weak Well-Foundedness Mirage, the Keisler–Morley Extension Property, and the Barwise Extension Property.*

**Proof.**

The previous arguments apply.

Realizability cannot be satisfied by the multiverse because every $\omega$-standard model of set theory has realizable $\omega$-nonstandard models.

Standard Realizability is satisfied (if $M$ is nonstandard but $\omega$-standard then any set or class well-founded in $M$ must also be $\omega$-standard. □

Remark: we can get a multiverse satisfying these collections of axioms from a weaker consistency assumption, namely just the existence of a single countable transitive model.

If $U$ is a countable transitive model then the collection of models of ZFC which $U$ thinks are countable and nonstandard will form a multiverse satisfying Weak Well-Foundedness Mirage, as well as the other axioms from Main Theorem

# Covering extensions

Now: a multiverse which satisfies the covering axioms, while having worlds which are not recursively saturated.
To be more precise, we describe a multiverse, call it $\mathcal{C}(M)$, generated from a given fixed countable $\omega$-nonstandard $M \models \mathrm{ZFC}$.
Our construction of $\mathcal{C}(M)$ goes in steps:

- Describe the construction of the *trellis* of $\mathcal{C}(M)$, a $\mathbb{Z} \times \omega$ grid of models on which the rest of $\mathcal{C}(M)$ will be grown. The two main elements to the construction of the trellis are the Keisler–Morley theorem and iterated ultrapowers.
- Start with a fixed $\omega$-nonstandard countable $M \models \mathrm{ZFC}$ and fix $U \in M$ a nonprincipal ultrafilter on $\omega^M$ (we will also ask for $M$ to be a Paris model) and set $M_0^0 = M$.
- For $n \in \omega$ given $M_0^n$, let $M_0^{n+1}$ be an elementary end-extension of $M_0^n$, which exists by the Keisler–Morley theorem. We moreover require that $M_0^n$ has a single parameter from which cofinally many ordinals are definable, which can be done by one the lemmata above.

This establishes how we vertically build up our grid.

# The trellis

Next we have to build it out horizontally:

- Inside $M_0^n$ we can take iterated ultrapowers of $M_0^n$ by $U$. For each $e \in \omega^M$ let $M_e^n$ be the $e$-th iterated ultrapower of $M_0^n$ via $U_0 = U$. And let $U_e$ be the image of $U_0$ under the map $M_0^n \to M_e^n$, so that $M_{e+1}^n$ is the ultrapower of $M_e^n$ by $U_e$.
- This gives us a $\omega^M \times \omega$ grid of models (the columns of this grid form elementary chains of elementary end-extensions).
  For the $e$-th column, let $M_e^\omega$ be the union of the chain $\langle M_e^n : n \in \omega \rangle$.
- The **trellis** will be a section of this grid. Fix a nonstandard $\mathbb{Z}$-block $Z \subseteq \omega^M$. That is, $Z$ is maximal so that all its elements are nonstandard and differ by an integer. Consider the grid of models $M_e^n$ for $n \in \omega$ and $e \in Z$. This grid forms the trellis for $\mathcal{C}(M)$, with each model in the trellis being a world in $\mathcal{C}(M)$.

# The trellis

$$
\begin{tikzcd}
M_0^\omega & & & M_{e-1}^\omega & M_e^\omega & M_{e+1}^\omega & \\
\vdots \arrow[u] & & & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \\
M_0^{n+1} \arrow[u] \arrow[r, "U_0"] & \cdots & \cdots \arrow[r, "U_{e-2}"] & M_{e-1}^{n+1} \arrow[u] \arrow[r, "U_{e-1}"] & M_e^{n+1} \arrow[u] \arrow[r, "U_e"] & M_{e+1}^{n+1} \arrow[u] \arrow[r, "U_{e+1}"] & \cdots \\
M_0^n \arrow[u] \arrow[r, "U_0"] & \cdots & \cdots \arrow[r, "U_{e-2}"] & M_{e-1}^n \arrow[u] \arrow[r, "U_{e-1}"] & M_e^n \arrow[u] \arrow[r, "U_e"] & M_{e+1}^n \arrow[u] \arrow[r, "U_{e+1}"] & \cdots \\
\vdots \arrow[u] & & & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \\
M = M_0^0 \arrow[u] \arrow[r, "U_0"] & \cdots & \cdots \arrow[r, "U_{e-2}"] & M_{e-1}^0 \arrow[u] \arrow[r, "U_{e-1}"] & M_e^0 \arrow[u] \arrow[r, "U_e"] & M_{e+1}^0 \arrow[u] \arrow[r, "U_{e+1}"] & \cdots
\end{tikzcd}
$$

$Z$

**trellis**

Figure: Constructing the trellis for $\mathcal{C}(M)$. Horizontal arrows are ultrapower maps while vertical arrows are elementary end-extensions.

# Covering extensions

The rest of $\mathcal{C}(M)$ grows on the trellis, by closing the trellis under enough forcing extensions and then under set-like realizability.

More precisely: et $e \in Z$ be a column in the trellis.

For each $n \in \omega$ and each $\mathbb{P} \subseteq M_e^\omega$ so that $M_e^\omega$ thinks $\mathbb{P}$ is $\mathrm{Ord}$-cc and each $G \subseteq \mathbb{P}$ which is generic over $M_e^\omega$, set $G_n = G \cap M_e^n$. Then $M_e^n[G_n]$ is a world in $\mathcal{C}(M)$. If $N$ is a definable over $M_e^n[G_n]$ model of ZFC so that $M_e^n[G_n]$ thinks $N$ is set-like, then $N$ is a world in $\mathcal{C}(M)$.

# Why Paris?

First: why we require $M$ to be a Paris model?

Recall: every completion of ZFC admits a countable $\omega$-nonstandard Paris model.

## Lemma

*Suppose $M = M_0^0$ is a countable $\omega$-nonstandard Paris model. Then each $M_e^n$ is $\omega$-nonstandard but not recursively saturated.*

## Proof.

$M_e^n$ is an ultrapower by a nonprincipal ultrafilter on $\omega^M$, so $M_e^n$ has to be $\omega$-nonstandard.

$M_0^n$ embeds cofinally into $M_e^n$. Since $M_0^n$ has cofinally many ordinals definable from a parameter, the same holds $M_e^n$ and hence it cannot be recursively saturated. $\square$

# Definable inner models

It suffices for Main Theorem to have that worlds in the trellis are non-recursively saturated. But we actually have more:

### Lemma

*Suppose $M$ is a definable inner model of $N$ and that $N$ is recursively saturated. Then $M$ must also be recursively saturated.*

### Proof.

Consider a computable type $p(x) = \{\varphi_n(x, a) : n \in \omega\}$ with a parameter from $M$, and suppose $p(x)$ is finitely realizable in $M$. Then the type $p'(x) = \{\varphi_n^M(x, a) : n \in \omega\}$ is finitely realizable in $N$. Hence it must be realized in $N$, but then $p(x)$ is also realized in $M$. $\square$

By Woodin-Laver-Hamkins this means that set forcing extensions of models in the trellis are also non-recursively saturated.

# The trellis

For the vertical arrows in the Figure:

**Lemma**

*Fix any $e \in \omega^M$ and any $n \in \omega$. Then $M_e^{n+1}$ is an elementary end-extension of $M_e^n$. That is, the following diagram commutes, where the vertical arrows represent elementary end-extensions.*

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r, "U_{e-1}"] & M_e^{n+1} \arrow[u] \arrow[r, "U_e"] & M_{e+1}^{n+1} \arrow[u] \arrow[r, "U_{e+1}"] & \cdots \\
\cdots \arrow[r, "U_{e-1}"] & M_e^{n} \arrow[u] \arrow[r, "U_e"] & M_{e+1}^{n} \arrow[u] \arrow[r, "U_{e+1}"] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

# Elementary end-extensions

Proof...

Each $M_e^n$ is an internally iterated ultrapower of $M_0^n$ starting from a fixed ultrapower $U_0 \in M_0^n$ on $\omega^{M_0^n} = \omega^{M_0^0}$, and hence can be represented as a single ultrapower of $M_0^n$.

It suffices to show the following - let:

- $N^+$ be an elementary end-extension of $N \models \mathrm{ZFC}$,
- $U \in N$ be an ultrapower on a set $D \in N$
- $K$ be the ultrapower of $N$ by $U$
- $K^+$ be the ultrapower of $N^+$ by $U$.

Then $K^+$ is an elementary end-extension of $K$.

# Elementary end-extensions

Proof.

First: $K^+$ is an end-extension of $K$.
Indeed: fix $[a]_U \in K$ and suppose $[b]_U \in K^+$ is such that $K^+ \models [b]_U \in [a]_U$. Then

$$\{d \in D : N^+ \models b(d) \in a(d)\} \in U.$$

Since $N^+$ end-extends $N$ and $a(d) \in N$ for all $d$, we get that $b(d) \in N$ for a $U$-many $d$, thus: $[b]_U \in K$.

Next: $K^+$ is an elementary extension of $K$.

Indeed: fix $[a]_U \in K$ and consider a formula $\varphi(x)$.

Since $N^+$ is an elementary extension of $N$ we have that

$$\{d \in D : N \models \varphi(a(d))\} = \{d \in D : N^+ \models \varphi(a(d))\}.$$

So $K \models \varphi([a]_U)$ iff $K^+ \models \varphi([a]_U)$.. $\square$

# Interactions in $\mathcal{C}(M)$

By the lemma on cofinal embedding into the ultrapower again:

**Lemma**

*Let $M_e^n$ be a world in the trellis. Then $M_{e+1}^n$ is set-like realizable from $M_e^n$.* $\square$

The remaining three lemmata show how arbitrary models of $\mathcal{C}(M)$ interact with models in the trellis.

First, let us see that models in the trellis have generics for every $\mathrm{Ord}$-cc forcing.

**Lemma**

*Let $M_e^n$ be a model in the trellis of $\mathcal{C}(M)$ and let $\mathbb{P} \subseteq M_e^n$ be a forcing which $M_e^n$ thinks is $\mathrm{Ord}$-cc. Then there is $G \subseteq \mathbb{P}$ generic over $M_e^n$ so that $M_e^n[G]$ is in $\mathcal{C}(M)$.*

# Interactions in $\mathcal{C}(M)$

### Lemma

*Let $M_e^n$ be a model in the trellis of $\mathcal{C}(M)$ and let $\mathbb{P} \subseteq M_e^n$ be a forcing which $M_e^n$ thinks is $\mathrm{Ord}$-cc. Then there is $G \subseteq \mathbb{P}$ generic over $M_e^n$ so that $M_e^n[G]$ is in $\mathcal{C}(M)$.*

### Proof.

Let $\mathbb{P}^+ \subseteq M_e^\omega$ be $\mathbb{P}$ as defined in $M_e^\omega$ and take $G^+ \subseteq \mathbb{P}^+$ generic over $M_e^\omega$.

Then if $G = G^+ \cap M_e^n$, it follows $M_e^n[G]$ is in $\mathcal{C}(M)$.

Wish: $G \subseteq \mathbb{P}$ is generic over $M_e^n$ – in particular: $M_e^n[G] \models \mathrm{ZFC}$.

Take $A \subseteq \mathbb{P}$ a maximal antichain: since $\mathbb{P}$ is $\mathrm{Ord}$-cc, $A \in M_e^n$.

By elementarity $A$ remains a maximal antichain of $\mathbb{P}^+$ in $M_e^\omega$ so $G^+ \cap A \neq \emptyset$, and then $G \cap A \neq \emptyset$, i.e. $G$ is generic. □

# Interactions in $\mathcal{C}(M)$

How do these forcing extensions relate to elementary end-extensions?

**Lemma**

*Let $M^n_e$ be a model in the trellis of $\mathcal{C}(M)$ and let $M^n_e[G]$ be a world in $\mathcal{C}(M)$ where $G \subseteq \mathbb{P}$ is generic for an $\mathrm{Ord}$-cc forcing over $M^n_e$. Then there is is $G'$ generic over $M^{n+\mathbf{1}}_e$ so that $M^{n+\mathbf{1}}_e[G']$ is in $\mathcal{C}(M)$ and is an elementary end-extension of $M^n_e[G]$.*

**Proof...**

By the construction of $\mathcal{C}(M)$, we have that $G = G^+ \cap M^n_e$ where $G^+ \subseteq \mathbb{P}^+$ is generic over $M^\omega_e$.

ut $G' = G^+ \cap M^{n+\mathbf{1}}_e$. By previous Lemma $G' \subseteq \mathbb{P}'$ is generic over $M^{n+\mathbf{1}}_e$ and $G \subseteq G'$.

By elementarity $M^n_e[G] \subseteq M^{n+\mathbf{1}}_e[G']$.

Take $a \in M^n_e[G]$ and let $M^{n+\mathbf{1}}_e[G'] \models b \in a$ – w.l.o.g. $\dot{b} \in M^n_e$.

For contradiction: $\exists\, p \in G$ s.t. $p \Vdash \dot{b} \notin \dot{a}$. There is $p' \in G'$ s.t. $p' \Vdash \dot{b} \in \dot{a}$, and then $q \leq p, p'$, which is impossible.

Thus: some $p \in G$ forces $\dot{b} \in \dot{a}$. So $b \in M^n_e[G]$, i.e. this extension is an end-extension.

# Interactions in $\mathcal{C}(M)$

Lemma

*Let $M_e^n$ be a model in the trellis of $\mathcal{C}(M)$ and suppose $M_e^n[G]$ is a world in $\mathcal{C}(M)$ where $G \subseteq \mathbb{P}$ is generic for an $\mathrm{Ord}$-cc forcing over $M_e^n$. Then there is is $G'$ generic over $M_e^{n+1}$ so that $M_e^{n+1}[G']$ is in $\mathcal{C}(M)$ and is an elementary end-extension of $M_e^n[G]$.*

Proof.

Finally: elementarity.

For contradiction: $M^+[G^+] \models \varphi(a)$ but $M[G] \models \neg\varphi(a)$, where $a \in M[G]$.

Then $\exists\, p^+ \in G^+$ s.t. $M^+ \models p^+ \Vdash \varphi(\dot{a})$ and $\exists\, p \in G$ s.t. $M \models p \Vdash \neg\varphi(\dot{a})$.

By elementarity, $M^+$ agrees about what $p$ forces, and further $\exists q \leq p, p^+$, contradiction. □

# Interactions in $\mathcal{C}(M)$

Finally: models in $\mathcal{C}(M)$ are covered by moving up the trellis.

**Lemma**

*Suppose $N$ is a proper class set-like realizable from $M_e^n[G]$, where $M_e^n$ is in the trellis of $\mathcal{C}(M)$ and $G$ is generic over $M$ for an $\mathrm{Ord}$-cc forcing. Then there is $G'$ generic over $M_e^{n+1}$ so that $M_e^{n+1}[G'] \ni (v, E)$ which end-extends $N$. Moreover, for any standard $k$ we may arrange so that $(v, E)$ satisfies the first $k$ axioms of* ZFC.

## Proof...

Fix standard $k$ and formulas $\nu(x)$ and $\varepsilon(x, y)$ which define over $M_e^n[G]$ the set-like, proper class model $(N, \in^N)$.

$\mathrm{Ord}^N$ is a proper class in $M_e^n[G]$, $\forall \alpha \in Ord^N\ \alpha < |V_\xi{}^N|$ for some $V_\xi{}^N$ which is a set in $M_e^n[G]$.

# Interactions in $\mathcal{C}(M)$

**Lemma**

*Suppose $N$ is a proper class and set-like realizable from $M_e^n[G]$, where $M_e^n$ is in the trellis of $\mathcal{C}(M)$ and $G$ is generic over $M$ for an $\mathrm{Ord}$-cc forcing. Then there is $G'$ generic over $M_e^{n+1}$ so that $M_e^{n+1}[G']$ contains $(v, E)$ which end-extends $N$. Moreover, for any standard $k$ we may arrange so that $(v, E)$ satisfies the first $k$ axioms of* ZFC.

**Proof...**

In $M_e^n[G]$ fix an ordinal $\alpha$. Consider $f : V_\alpha \to \mathrm{Ord}^N$ defined as $f(x) = rk^N(x)$ for $x \in N$. Then $rg(f)$ must be bounded in $\mathrm{Ord}^N$, so let $\xi \in \mathrm{Ord}^N$ be a bound.

Then ${V_\xi}^N$ contains as $\in^N$-elements each $x \in N$ with rank $< \alpha$ (as computed in $M_e^n[G]$).
By the Lévy–Montague reflection principle ${V_\xi}^N$ satisfies the first $k$ axioms of ZFC.
Thus $(*)$: For each ordinal $\alpha \in M_e^n[G]$ there is $v \in N$ so that $v$ satisfies ZFC $\restriction k$ and

$$\forall x \in N \; rk^{M[G]}(x) < \alpha \Rightarrow x \in v.$$

# Interactions in $\mathcal{C}(M)$

**Lemma**

*Suppose $N$ is a proper class and set-like realizable from $M_e^n[G]$, where $M_e^n$ is in the trellis of $\mathcal{C}(M)$ and $G$ is generic over $M$ for an $\mathrm{Ord}$-cc forcing. Then there is $G'$ generic over $M_e^{n+1}$ so that $M_e^{n+1}[G']$ contains $(v, E)$ which end-extends $N$. Moreover, for any standard $k$ we may arrange so that $(v, E)$ satisfies the first $k$ axioms of* ZFC.

**Proof.**

Let $\mathbb{P}'$ be $\mathbb{P}$ as defined in $M_e^{n+1}$. By the previous Lemma let $G' \subseteq \mathbb{P}'$ be generic over $M_e^{n+1}$ s.t. $M_e^{n+1}[G']$ is an e.e.e. of $M_e^n[G]$ and let $N'$ be the proper class model defined over $M_e^{n+1}[G']$ by $\nu(x)$ and $\varepsilon(x, y)$.

By elementarity $(*)$ holds $M_e^{n+1}[G']$. Pick $\alpha \in M_e^{n+1}[G'] \setminus M_e^n[G]$ and let $v$ be the witness to the instance of $(*)$ for $\alpha$. Put

$$E = \in^{N'} \restriction y = \{(x, y) \in N' \times N' : x \in^{N'} y \text{ and } x \in^{N'} v \text{ and } y \in^{N'} v\}.$$

Then $(v, E)$ extends $(N, \in^N)$ and $(v, E)$ ZFC $\restriction k$.

Wish: this extension is an end-extension, and thus $(v, E)$ covers $N$.

Indeed: pick $x \in N$. $N$ is set-like in $M_e^n[G]$, thus there is a set $x^* \in M_e^n[G]$ of its predecessors - but this is definable, so it holds in $M_e^{n+1}[G']$, and then:

# Final proof

Now, proceed to checking which multiverse axioms it satisfies.

**Theorem (An elaboration of main theorem)**

*Suppose $M$ is a countable $\omega$-nonstandard Paris model. Then its covering multiverse $\mathcal{C}(M)$ is a multiverse which contains $\omega$-nonstandard but non-recursively saturated worlds which satisfies the following multiverse axioms: Set-Like Realizability, Closure Under Set Forcing and $\mathrm{Ord}$-cc Class Forcing, Covering Countability, and Covering Well-Foundedness Mirage. Moreover, $\mathcal{C}(M)$ satisfies the Keisler–Morley Extension Property.*

**Proof...**

Set-Like Realizability:

Every world $N$ in $\mathcal{C}(M)$ is set-like realizable from a forcing extension of an $M^n_e$ from the trellis. Set-like realizability is transitive, so if $K \subseteq N$ is set-like realizable from $N$ then $K$ must be in $\mathcal{C}(M)$.

# Final proof

Proof...

Closure Under $\mathrm{Ord}$-cc Class Forcing:

Let $N$ be any world in $\mathcal{C}(M)$ and let $\mathbb{P}$ be either a set forcing notion from $N$ or else an $\mathrm{Ord}$-cc class forcing notion from $N$.

$N$ is realizable from some forcing extension $M^n_e[H]$ of some $M^n_e$ in the trellis of $\mathcal{C}(M)$ via a poset $\mathbb{Q}$ from $M^n_e$.

Then $M^n_e[H]$ sees an isomorphic copy of $\mathbb{P}$, call it $\mathbb{P}'$.

Because the product of two $\mathrm{Ord}$-cc forcings is $\mathrm{Ord}$-cc we get that $M^n_e[H][G']$ is a world in $\mathcal{C}(M)$, for some $G' \subseteq \mathbb{P}'$ which is generic over $M^n_e[H]$.

Since $M^n_e[H]$ knows the isomorphism between $\mathbb{P}'$ and $\mathbb{P}$, $N[G]$ is realizable from $M^n_e[H][G']$.

# Final proof

Proof...

Covering Countability:

It's enough to check that every world in $\mathcal{C}(M)$ is covered by a set-sized structure in another world.

Take an arbitrary $N$ in $\mathcal{C}(M)$. Then $N$ is realizable from some $M_e^n[G]$, where $M_e^n$ thinks $\mathbb{P}$ is $\mathrm{Ord}$-cc.

$M_e^n[G]$ thinks that $N$ is set-like, and thus $N$ is topped by $M_e^n[G]$ (if $N$ is a proper class, one of the lemmata just said that there is $(v, E) \in M_e^{n+1}[G']$ which covers $N$).

# Final proof

Proof...

Covering Well-Foundedness Mirage:

Take an arbitrary world $N$ in $\mathcal{C}(M)$, i.e. $N$ is set-like realizable from $M^n_e[G]$ where $G \subseteq \mathbb{P}$ is generic over $M^n_e$ from the trellis and $M^n_e$ thinks $\mathbb{P}$ is $\mathrm{Ord}$-cc.

Wish: there is a world in $\mathcal{C}(M)$ which covers $N$ with $(v, E)$ so that this world thinks $(v, E)$ is ill-founded.

First: $N$ is realizable from a forcing extension of $M^n_{e-\mathbf{1}}$. Indeed: $M^n_e$ is an ultrapower of $M^n_{e-\mathbf{1}}$ by an ultrapower $U_{e-\mathbf{1}}$ on $\omega^{M^n_{e-\mathbf{1}}}$.

In particular, $M^n_e$ is realizable from $M^n_{e-\mathbf{1}}$, thus, $M^n_{e-\mathbf{1}}$ has an isomorphic copy of $\mathbb{P}$, call it $\mathbb{P}'$, and let $G'$ is the image of $G$ under this isomorphism: then $G'$ is generic over $M^n_{e-\mathbf{1}}$.

By the transitivity of set-like realizability: $N$ is set-like realizable from $M^n_{e-\mathbf{1}}[G']$.

# Final proof

Proof...

Next: $M^n_{e-1}[G']$ thinks that $N$ is ill-founded, because $M^n_{e-1}[G']$ thinks that $M^n_e[G]$ is ill-founded.

Moreover: it is witnessed by a set in $M^n_{e-1}[G']$, i.e. a descending $\omega^{M^n_{e-1}}$-sequence in $\in^N$.

$M^{n+1}_{e-1}[G^+]$ is an elementary end-extension of $M^n_{e-1}[G']$ and it has an element $(v, E)$ which covers $(N, \in^N)$.

More: $M^{n+1}_{e-1}[G^+]$ has that same set witnessing that $(v, E)$ is ill-founded.

# Final proof

Proof.

Keisler–Morley Extension Property:

Consider an arbitrary world $N \in \mathcal{C}(M)$. That is, $N$ is set-like realizable in $M_e^n[G]$, a forcing extension of some world in the trellis by $\mathrm{Ord}$-cc forcing.

If $N$ is a set in $M_e^n[G]$, then collapsing $N$ to be countable there is an elementary end-extension of $N$.

If $N$ is a proper class defined by $\nu(x)$ and $\varepsilon(x,y)$ apply the same definition in $M_e^{n+1}[G'](M_e^{n+1}[G']$ is an e.e.e. of $M_e^n[G])$: $N'$ is set-like realizable from $M_e^{n+1}[G']$ and $N'$ end-extends$N$.

It is an elementary, since $M_e^n[G]$ and $M_e^{n+1}[G']$ agree on $\nu(x)$ and $\varepsilon(x,y)$. □

# Concluding remarks

- There is some flexibility in the choice of $M$ for the construction of $\mathcal{C}(M)$:by lemma on Paris models we may pick $M$ to satisfy any consistent extension of ZFC. In particular, we can ensure that $\mathcal{C}(M)$ contains models with large cardinals, satisfy forcing axioms, or whatever other first-order properties we wish.
- Letting $\{M_i : i \in I\}$ be collection of countable $\omega$-nonstandard Paris models of ZFC then $\bigcup_{i\in I} \mathcal{C}(M_i)$ is a multiverse which satisfies Set-Like Realizability, Closure Under $\mathrm{Ord}$-cc Class Forcing, Covering Countability, and Covering Well-Foundedness Mirage. So we can ensure the multiverse contains any number of models with whatever first-order property we want.
- There is no chance for Barwise Extension Property.
- In general, satisfying Standard Realizability forces a multiverse with worlds satisfying Con(ZFC) to contain recursively saturated models. If we want a multiverse satisfying Covering Well-Foundedness Mirage which has no recursively saturated worlds, then we must give up Standard Realizability - but it's possible.

# Open questions

$\mathcal{C}(M)$ might be unsatisfying: it only has Closure Under $\mathrm{Ord}$-cc Class Forcing and Set-Like Realizability. We would like a covering multiverse which satisfies Closure Under Class Forcing and Realizability, but that did not work out for our construction. We used $\mathrm{Ord}$-cc-ness in the proof and we used the set-like part of Set-Like Realizability essentially, thus:

**Question**

*Is there a multiverse which satisfies Realizability, Closure Under Class Forcing, Covering Countability, and Covering Well-Foundedness Mirage?*

We also want to know whether there is a natural collection of models which satisfies Covering Well-Foundedness Mirage, similar to the Gitman and Hamkins model of the Well-Foundedness Mirage axiom and our model of the Weak Well-Foundedness Mirage axiom, so:

**Question**

*Is there a natural model of the Covering Well-Foundedness Mirage axiom?*

Thank you