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This is important as large cardinals are needed to show that many interesting statements in set theory are consistent.

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Example: Gitik starts with a model with a totally measurable cardinal (i.e. a cardinal κ such that $o(\kappa) = \kappa^{++}$) and then forces a failure of the singular cardinal hypothesis SCH.
If you want a "definable" failure of SCH you use an L-like ground model with a totally measurable (SDF-Honzik).

Mighty Mouse

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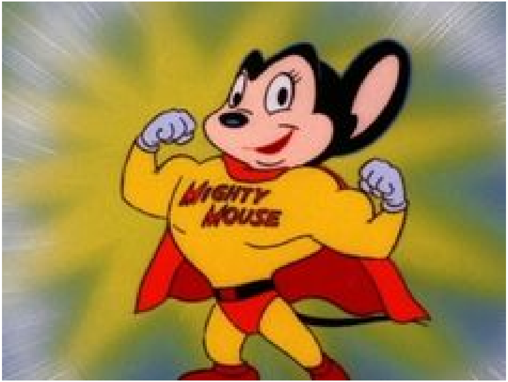

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Suppose that Mighty Mouse exists. Then $V = M[G]$ where M is a definable L-like model (obtained by "iterating" Mighty Mouse) and G is generic over M (for a definable, Ord-cc class forcing).

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The Theorem is false without the hypothesis that Mighty Mouse exists.

Without Mighty Mouse

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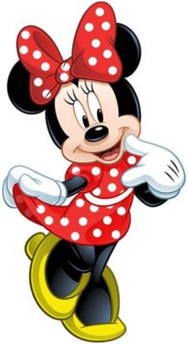

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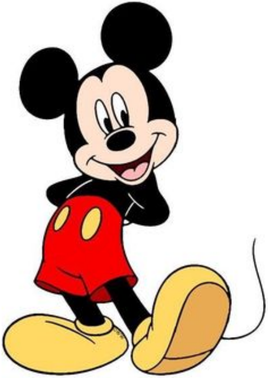

Mickey Mouse

Minnie Mouse

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Now iterate the measurable cardinals of Minnie Mouse below the top-measurable onto SuccLimCard, the $\aleph_{\lambda+\omega}$, λ limit or 0. The top-measurable is used to guarantee that the iteration doesn't "die out" at some stage as it can be used to generate fresh measurables.

Minnie Mouse

Key fact: Suppose that κ with measure U is iterated λ times, generating the sequence $(\kappa_i \mid i \leq \lambda)$ and sending the measure U on κ to the measure U_λ on κ_λ. Suppose that $\text{cof}(\lambda) = \omega$ and $i_0 < i_1 < \cdots$ is cofinal in λ. Then $(\kappa_{i_n} \mid n < \omega)$ is generic (over the λ-th iterate) for the Prikry forcing defined using the measure U_λ.

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Thus the $\aleph_{\lambda+n}$'s form a Prikry sequence for the measure of the iterate on $\aleph_{\lambda+\omega}$. Using a result of Gunter Fuchs, Welch observes that in fact the entire collection of these Prikry sequences is generic for a Prikry product.

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So $L[\mathrm{Card}]$ is a Prikry-Product generic extension of an iterate of Minnie Mouse.

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We want to show that $L[\text{Reg}]$ is a generic extension of an iterate of Mickey Mouse. The idea is to iterate each measurable through ω-many regular cardinals, which will form a Prikry sequence over the iterate and then the regular cardinals will be the union of these Prikry sequences.

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But first note that if there are no weakly inaccessibles then $L[\text{Reg}]$ is just $L[\text{Card}]$ and therefore our iteration should produce an iterate not of Mickey but only of Minnie.

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What happens is that when we iterate Mickey for Ord steps we might not use all of the measures below the etop measurable, but obly a proper initial segment of them.

Mickey Mouse

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The iteration begins by iterating the measurables below the least measurable limit of measurables onto SuccLimCard, the $\aleph_{\lambda+\omega}$, λ limit or 0. At some point this will be achieved for all measurables below the least measurable limit of measurables and it is time to hit that.

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Doing so will create new measurables below the new measurable limit of measurables, which have to be iterated further onto elements of SuccLimCard. Then we hit the new least measurable limit of measurables and repeat this again and again. There are 2 cases.

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If the top measurable gets iterated through ω-many weakly inaccessibles then we stop iterating it and move on to the least measurable above it. Otherwise we move the least measurable limit of measurables all the way to Ord.

Mickey Mouse

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But if the least measurable limit of measurables gets moved all the way to Ord then we don't want the iterate Mickey* of Mickey but the iterate Minnie* of Minnie. Ideally, Minnie* is just Mickey$^*|$Ord, the truncation of Mickey* to Ord.

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The advantage of Modern mice is that they enjoy Gödel-like condensation.

Mickey Mouse

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Classical Mickey Mouse

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Question: What is $L[\mathrm{Cof}]$?

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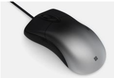

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