Products of CW complexes

Andrew Brooke-Taylor

UNIVERSITY OF LEEDS
CW complexes

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For $n \in \mathbb{N}$, denote by
- $D^n$ the closed ball of radius 1 about the origin in $\mathbb{R}^n$ (the $n$-disc),
- $\overset{\circ}{D^n}$ its interior, and
- $S^{n-1}$ its boundary (the $(n-1)$-sphere).
CW complexes

Definition

A Hausdorff space \( X \) is a **CW complex** if there exists a set of continuous functions \( \varphi_\alpha : D^n \rightarrow X \) (characteristic maps), for \( \alpha \) in an arbitrary index set and \( n \in \mathbb{N} \) a function of \( \alpha \), such that:

1. \( \varphi_\alpha \mid D^n \) is a homeomorphism to its image, and \( X \) is the disjoint union as \( \alpha \) varies of these homeomorphic images \( \varphi_\alpha[D^n] \) (“cells”).

- Closure-finiteness: For each \( \varphi_\alpha \), \( \varphi_\alpha[S^{n-1}] \) is contained in finitely many cells all of dimension less than \( n \).

- Weak topology: A set is closed if and only if its intersection with each closed cell \( \varphi_\alpha[D^n] \) is closed.

We often denote \( \varphi_\alpha[D^n] \) by \( e_n^\alpha \) or just \( e^\alpha \).
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We often denote $\varphi_\alpha[D^n]$ by $e^n_\alpha$ or just $e_\alpha$. 
Let $X$ be the "star" with a central vertex $x_0$ and countably many edges $e_1, \ldots, e_n (n \in \mathbb{N})$ emanating from it (and the countably many "other end" vertices of those edges).

$X$ is not metrizable, as $x_0$ does not have a countable neighbourhood base.

Proof: Identify each edge with the unit interval, with $x_0$ at 0. For every $f: \mathbb{N} \to \mathbb{N}$, consider the open neighbourhood $U(x_0; f)$ of $x_0$ whose intersection with $e_1, \ldots, e_n$ is the interval $[0, 1/(f(n) + 1)]$.

These form a neighbourhood base, but for any countably many $f_i$, there is a $g$ that is not dominated by any of them, so $U(x_0; g)$ does not contain any of the $U(x_0; f_i)$.
Not necessarily metrizable

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The Cartesian product of two CW complexes \(X\) and \(Y\), with the product topology, need not be a CW complex. Since \(D_m \times D_n \sim = D_{m+n}\), there is a natural cell structure on \(X \times Y\), which satisfies closure-finiteness, but the product topology is generally not as fine as the weak topology.

Convention

In this talk, \(X \times Y\) is always taken to have the product topology, so "\(X \times Y\) is a CW complex" means "the product topology on \(X \times Y\) is the same as the weak topology."
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Let $X$ be the “star” with a central vertex $x_0$ and countably many edges $e^{1}_{X,n}$ ($n \in \mathbb{N}$) emanating from it (and the countably many “other end” vertices of those edges).
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Let \( U \times V \) be a member of the open neighbourhood base about \((x_0, y_0)\) in the product topology on \( X \times Y \) — so \( x_0 \in U \) an open subset of \( X \), and \( y_0 \in V \) an open subset of \( Y \).
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Then \( \left( \frac{1}{g(k)+1}, \frac{1}{g(k)+1} \right) \in U \times V \cap H \). So in the product topology, \((x_0, y_0) \in \bar{H} \).
More preliminaries: subcomplexes

A *subcomplex* $A$ of a CW complex $X$ is what you would expect.

E.g. For any CW complex $X$ and $n \in \mathbb{N}$, the $n$-skeleton $X^n$ of $X$ is the subcomplex of $X$ which is the union of all cells of $X$ of dimension at most $n$.

Every subcomplex $A$ of $X$ is closed in $X$. By closure-finiteness, every $x$ in a CW complex $X$ lies in a finite subcomplex.

**Definition**

Let $\kappa$ be a cardinal. We say that a CW complex $X$ is *locally less than* $\kappa$ if for all $x$ in $X$ there is a subcomplex $A$ of $X$ with fewer than $\kappa$ many cells such that $x$ is in the interior of $A$. We write *locally finite* for locally less than $\aleph_0$, and *locally countable* for locally less than $\aleph_1$.
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A subcomplex $A$ of a CW complex $X$ is a subspace which is a union of cells of $X$, such that if $e_{\alpha}^n \subseteq A$ then its closure $\overline{e}_{\alpha}^n = \varphi_{\alpha}[D^n]$ is contained in $A$.
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Proposition

If $\kappa$ is a regular uncountable cardinal, then a CW complex $W$ is locally less than $\kappa$ if and only if every connected component of $W$ has fewer than $\kappa$ many cells.

Proof sketch.

$\Leftarrow$ is trivial. For $\Rightarrow$, given any point $w$, recursively fill out to get an open (hence clopen) subcomplex containing $w$ with fewer than $\kappa$ many cells, using the fact that the cells are compact to control the number of cells along the way if $\kappa < 2^{\aleph_0}$. $\Box$
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**Theorem (J. Milnor, 1956)**

*If $X$ and $Y$ are both (locally) countable, then $X \times Y$ is a CW complex.*
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**Theorem (J. Milnor, 1956)**

*If $X$ and $Y$ are both (locally) countable, then $X \times Y$ is a CW complex.*

**Theorem (Y. Tanaka, 1982)**

*If neither $X$ nor $Y$ is locally countable, then $X \times Y$ is not a CW complex.*
What was known, beyond ZFC

Theorem (Liu Y.-M., 1978)

Assuming the Continuum Hypothesis, \( X \times Y \) is a CW complex if and only if either
- one of them is locally finite, or
- both are locally countable.

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Assuming \( b = \aleph_1 \), \( X \times Y \) is a CW complex if and only if either
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Can we do better?

Question
Can we show, without assuming any extra set-theoretic axioms, that the product $X \times Y$ of CW complexes $X$ and $Y$ is a CW complex if and only if either
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Answer (follows from Tanaka’s work)
No.
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Can we characterise exactly when the product of two CW complexes is a CW complex, without assuming any extra set-theoretic axioms?
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Answer (B.-T.)
Yes!
Pushing Dowker’s example harder

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Recall

- For $f, g \in \mathbb{N}^\mathbb{N}$, we write $f \leq^* g$ if for all but finitely many $n \in \mathbb{N}$, $f(n) \leq g(n)$.
- The bounding number $b$ is the least cardinality of a set of functions that is unbounded with respect to $\leq^*$, i.e. such that no one $g$ is $\geq^*$ them all, i.e.,

$$b = \min \{|\mathcal{F}| : \mathcal{F} \subseteq \mathbb{N}^\mathbb{N} \land \forall g \in \mathbb{N}^\mathbb{N} \exists f \in \mathcal{F} \neg (f \leq^* g)\}.$$
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Let $X$ be the “star” with a central vertex $x_0$ and countably many edges $e_{X,n}^1$ $(n \in \mathbb{N})$ emanating from it (and the countably many “other end” vertices of those edges).
Let $Y$ be the “star” with a central vertex $y_0$ and $b$ many edges $e_{Y,f}^1 (f \in \mathcal{F})$ emanating from it (and the other ends) where $\mathcal{F} \subseteq \mathbb{N}^\mathbb{N}$ is unbounded w.r.t. $\leq^*$.

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$$H = \left\{ \left( \frac{1}{f(n) + 1}, \frac{1}{f(n) + 1} \right) \in e_{X,n}^1 \times e_{Y,f}^1 : n \in \mathbb{N}, f \in \mathcal{F} \right\}$$

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Consider the edge \( e^1_{Y,g} \) of \( Y \):

Let \( k \in \mathbb{N} \) be such that \( \frac{1}{g(k)+1} \in e^1_{Y,g} \cap V \).

Then \( \left( \frac{1}{g(k)+1}, \frac{1}{g(k)+1} \right) \in U \times V \cap H \). So in the product topology, \((x_0, y_0) \in \bar{H} \).
Example (Folklore based on Dowker, 1952)

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Consider the edges \( e_{X,n}^1 \) of \( X \):

Let \( g : \mathbb{N} \to \mathbb{N}^+ \) be an increasing function such that \([0, \frac{1}{g(n)}) \subset e_{X,n}^1 \cap U\) for every \( n \in \mathbb{N} \). Take \( f \in F \) such that \( f \not \leq^* g \).

Consider the edge \( e_{Y,f}^1 \) of \( Y \):

Let \( k \in \mathbb{N} \) be such that \( \frac{1}{f(k)+1} \in e_{Y,f}^1 \cap V \) and \( f(k) > g(k) \).

Then \( \left( \frac{1}{f(k)+1}, \frac{1}{f(k)+1} \right) \in U \times V \cap H \). So in the product topology, \((x_0, y_0) \in \overline{H} \).
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Yes!
A complete characterisation

Theorem (B.-T.)

Let $X$ and $Y$ be CW complexes. Then $X \times Y$ is a CW complex if and only if one of the following holds:

1. $X$ or $Y$ is locally finite.
2. One of $X$ and $Y$ is locally countable, and the other is locally less than $b$. 
Proof


So it remains to show that if $X$ and $Y$ are CW complexes such that $X$ is locally countable and $Y$ is locally less than $b$, then $X \times Y$ is a CW complex.

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Topologies

Any compact subset of a CW complex $X$ is contained in finitely many cells, and each closed cell $\bar{e}^n_\alpha$ is compact. So

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We can also restrict to those compact sets which are continuous images of the compact space $\omega + 1$ (with the order topology).

Definition

A topological space $Z$ is sequential if for every subset $C$ of $Z$, $C$ is closed if and only if $C$ contains the limit of every convergent countable sequence from $C$ ($C$ is sequentially closed).
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Any sequential space is compactly generated. Since $D^n$ is sequential for every $n$, we have that CW complexes are sequential.
Need to show: $X \times Y$ is sequential.
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So suppose

- $H \subseteq X \times Y$ is sequentially closed, and
- $(x_0, y_0) \in X \times Y \setminus H$.

We want to construct open neighbourhoods $U$ of $x_0$ in $X$ and $V$ of $y_0$ in $Y$ such that $(U \times V) \cap H = \emptyset$. 
Constructing neighbourhoods

We can build an open neighbourhood $U$ of a point $x$ in a CW complex $X$ by induction on dimension:
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- If $x \in e^n_\alpha \subset X$, start with the image under $\varphi_\alpha$ of an open ball in $\dot{D}^n$. This defines $U \cap X^n$.
- Once $U \cap X^k$ is defined, for each $(k + 1)$-cell $e^{k+1}_\beta$ whose boundary intersects $U \cap X^k$, take a collar neighbourhood of $\varphi^{-1}_\beta(U \cap X^k)$ in $D^{k+1}$: for any positive integer $m$, we can take a collar of the form

$$\left(\frac{m - 1}{m}, 1\right] \cdot \varphi^{-1}_\beta(U \cap X^k) \subset D^{k+1} \subset \mathbb{R}^{k+1}.$$
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For any function $f$ from the set of indices of cells in $X$ to $\mathbb{N}$ we thus get an open neighbourhood $U(x; f)$, taking radius/collar width $\frac{1}{f(\beta)+1}$ for the cell $\beta$ step.
Lemma

*Such open neighbourhoods form a base for the topology on $X$.**
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Proof.

Follow your nose, recursively constructing a neighbourhood of this form whose closure is a subset of any given open neighbourhood. Since each $S^k$ is compact, there will be a collar width $m$ sufficiently large to do this for each subsequent cell.

$\square$
Constructing neighbourhoods avoiding $H$

Lemma 1 (Adding one cell to finite subcomplexes)

Suppose $W$ and $Z$ are CW complexes, $W'$ is a finite subcomplex of $W$, $Z'$ is a finite subcomplex of $Z$, $U \subseteq W'$ is open in $W'$, $V \subseteq Z'$ is open in $Z'$, and $H$ is a sequentially closed subset of $W \times Z$ such that the closure of $U \times V$ is disjoint from $H$.

Let $e$ be a cell of $Z$ whose boundary is contained in $Z'$. Then there is a $p \in \mathbb{N}$ such that, if $V_{e,p}$ is $V$ extended by the width $1/(p+1)$ collar in $e$, then $U \times V_{e,p}$ has closure disjoint from $H$.

Proof sketch. Use the fact that $W' \times (Z' \cup e)$ is sequential, normal, and compact.
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We shall construct functions $f : \mathbb{N} \to \mathbb{N}$ and $g : J \to \mathbb{N}$, where $J$ is the index set for cells of $Y$, such that $U(x_0; f) \times U(y_0; g)$ has closure disjoint from $H$. 
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First idea
Simultaneous induction on dimension on each side.

For each new cell $e^k_\alpha$ that you consider on the $Y$ side, you get a function $f_\alpha$ defining an open subset of $X^k$ avoiding $H$. Since there are fewer than $b$ many $\alpha$, they can be eventually dominated by a single function $f$, which is taken to define the open set on $X^k$, and with respect to which the $e^k_\alpha$ collar can be chosen.
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This doesn’t work ($f_\alpha \leq^* f$ isn’t good enough).
$\leq^*$ isn’t good enough

If $f_\alpha(n) \leq f(n)$ for all $n$, then $U(x; f_\alpha) \supseteq U(x; f)$.

If $f_\alpha(n) \leq^* f(n)$, then there may be finitely many $n$ for which $f_\alpha(n) > f(n)$. 
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Solution

Hechler conditions!
The construction is actually by recursion on dimension on the $Y$ side, and simultaneously, constructing $f$ as the limit of a sequence of *Hechler conditions*, that is:

- finite initial segments of $f$, and
- promises to dominate some function $F$ thereafter.
Lemma 2 (Adding a $Y$-side cell, fitting $X$-side promises)

Let $Y'$ be a finite subcomplex of $Y$ containing $y_0$, $F : \mathbb{N} \to \mathbb{N}$ be a function, $i \in \mathbb{N}$, $s$ be a function from the indices of $Y'$ to $\mathbb{N}$ such that $U(x_0; F) \times U(y_0; s) \subseteq X \times Y'$ has closure disjoint from $H$, and $Y'' = Y' \cup e_\alpha$ for some cell $e_\alpha$ of $Y$ not in $Y'$.

Then there is a function $f : \mathbb{N} \to \mathbb{N}$ such that

1. $f(n) \geq F(n)$ for all $n$ in $\mathbb{N}$,
2. for every $f' : \mathbb{N} \to \mathbb{N}$ such that $f' \geq \ast f$ and $f' \geq F$, there is a $q \in \mathbb{N}$ such that $U(x_0; f') \times U(y_0; s \cup \{(\alpha, q)\})$ has closure disjoint from $H$. 

Andrew Brooke-Taylor (Leeds)
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Making it work

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Then there is a function $f : \mathbb{N} \to \mathbb{N}$ such that

1. $f(n) \geq F(n)$ for all $n$ in $\mathbb{N}$, and $f(n) = F(N)$ for all $n < i$,
2. for every $f' : \mathbb{N} \to \mathbb{N}$ such that $f' \succeq^* f$ and $f' \geq F$, there is a $q \in \mathbb{N}$ such that $U(x_0; f') \times U(y_0; s \cup \{(\alpha, q)\})$ has closure disjoint from $H$. 
Proof of Lemma 2

For every finite tuple $r$ of length $n$ such that $r \geq F \upharpoonright n$, $U(x_0; r) \subset U(x_0; F)$, so $U(x_0; r) \times U(y_0; s)$ certainly has closure disjoint from $H$. 
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By Lemma 1, we can then take $q_r \in \mathbb{N}$ such that $U(x_0; r) \times U(y_0; s \cup \{(\alpha, q_r)\})$ has closure disjoint from $H$. 
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By Lemma 1, we can then take $q_r \in \mathbb{N}$ such that $U(x_0; r) \times U(y_0; s \cup \{(\alpha, q_r)\})$ has closure disjoint from $H$.

Then by Lemma 1 again, there is $p \in \mathbb{N}$ such that $U(x_0; r \cup \{(n, p)\}) \times U(y_0; s \cup \{(\alpha, q_r)\})$ has closure disjoint from $H$. 
Now, assuming by induction we have defined $f \upharpoonright n$ for some $n \geq i$, there are only finitely many $r$ with $F \upharpoonright n \leq r \leq f \upharpoonright n$; follow this procedure for all of them, and take the maximum of the resulting values $p$ to be $f(n)$. Recursively do this for all $n \geq i$.

Then for any $f' \geq F$ with $f' \geq^* f$, $f' \geq r \cup (f \upharpoonright [n, \infty))$ for some $n \geq i$ and some $r$ of length $n$ as above, so

$$U(x_0; f' \upharpoonright n + 1) \times U(y_0; s \cup \{(\alpha, q_r)\})$$

has closure disjoint from $H$, and in fact

$$U(x_0; f') \times U(y_0; s \cup \{(\alpha, q_r)\})$$

has closure disjoint from $H$.

Lemma 2
Finishing the proof of the Theorem

With Lemma 2 in hand, the argument is now basically as outlined in the “First idea”:

Proceed by induction on dimension on the $Y$ side. Assume we have defined $f_k : \mathbb{N} \to \mathbb{N}$ and $g \upharpoonright Y^k$. For each $(k + 1)$-dimensional cell $e_\alpha$ on the $Y$ side, use Lemma 2 with

- $f_k$ as $F$,
- $k$ as $i$,
- the minimal (finite) subcomplex of $Y$ containing $e_\alpha$ and $y_0$ as $Y''$, and
- $g \upharpoonright (Y'' \setminus e_\alpha)$ as $s$

to get $f_{\alpha,k+1}$. There are fewer than $b$ many such $f_{\alpha,k+1}$, so take $f_{k+1} \geq f_k$ with $f_{k+1} \upharpoonright k = f_k \upharpoonright k$ eventually dominating all of them. Then take $q$ as given by Lemma 2 (with $f_{k+1}$ as $f'$) as $g(\alpha)$.

Finally, take $f$ to be the (componentwise) limit of the $f_{k+1}$; these $f$ and $g$ are such that $U(x_0; f) \times U(y_0; g)$ has closure disjoint from $H$. \qed