Convergence of Borel measures and filters on ω

Damian Sobota

Kurt Gödel Research Center for Mathematical Logic
University of Vienna

Joint work with J. Kąkol, W. Marciszewski and L. Zdomskyy.

Preliminaries

X — an infinite Tychonoff space

$C_p(X)$ — the space of real-valued continuous functions on X with the pointwise topology

K — an infinite compact Hausdorff space

$C(K)$ — the Banach space of real-valued continuous functions on K with the supremum norm

Preliminaries

Measures

A measure μ on a Tychonoff space X is a real-valued set function defined on the Borel σ-field $Bor(X)$ of X, which is regular and finite, i.e.
$\|\mu\| = \sup\{|\mu(A)| + |\mu(B)|: \ A, B \in Bor(X), \ A \cap B = \emptyset\} < \infty$.

Preliminaries

Measures

A measure μ on a Tychonoff space X is a real-valued set function defined on the Borel σ-field $Bor(X)$ of X, which is regular and finite, i.e.
$\|\mu\| = \sup\{|\mu(A)| + |\mu(B)| : \ A, B \in Bor(X), \ A \cap B = \emptyset\} < \infty$.

If $x \in X$, then δ_x is a measure on X (*the Dirac measure at* x).

A measure μ on X is *finitely supported* if $\mu = \sum_{x \in F} \alpha_x \delta_x$ for some finite F and non-zero $\alpha_x \in \mathbb{R}$.

The set F is called *the support* of μ, denoted by $\mathrm{supp}(\mu)$, and $\|\mu\| = \sum_{x \in F} |\alpha_x|$.

The Josefson–Nissenzweig theorem for $C(K)$-spaces

Theorem (Josefson '75, Nissenzweig '75)

For every infinite compact space K there exists a sequence $\langle \mu_n \colon \ n \in \omega \rangle$ of measures on K such that $\|\mu_n\| = 1$ and $\int_K f d\mu_n \to 0$ for every $f \in C(K)$.

The Josefson–Nissenzweig theorem for $C(K)$-spaces

Theorem (Josefson '75, Nissenzweig '75)

For every infinite compact space K there exists a sequence $\langle \mu_n \colon\ n \in \omega \rangle$ of measures on K such that $\|\mu_n\| = 1$ and $\int_K f d\mu_n \to 0$ for every $f \in C(K)$.

An application (one out of many!):

$c_0 = \{x \in \mathbb{R}^\omega \colon\ x(n) \to 0\}$ with the supremum norm

$C(\beta\omega \times \beta\omega)$ may be written as the sum $E \oplus c_0$ where E is a closed subspace, even though $C(\beta\omega)$ may not (Cembranos '84).

The Josefson–Nissenzweig theorem for $C_p(X)$-spaces

Theorem (Banakh–Kąkol–Śliwa '18)

For every infinite Tychonoff space X, TFAE:

1. $C_p(X)$ *may be written as a sum* $E \oplus (c_0)_p$ *where* E *is a closed subspace and projections are continuous;*
2. X *admits a sequence* $\langle \mu_n\colon\ n \in \omega\rangle$ *of finitely supported measures such that* $\|\mu_n\| = 1$ *and* $\int_K f d\mu_n \to 0$ *for every* $f \in C(X)$.

$(c_0)_p = \{x \in \mathbb{R}^\omega\colon\ x(n) \to 0\}$ with the pointwise topology

The Josefson–Nissenzweig theorem for $C_p(X)$-spaces

Theorem (Banakh–Kąkol–Śliwa '18)

For every infinite Tychonoff space X, TFAE:

1. *$C_p(X)$ may be written as a sum $E \oplus (c_0)_p$ where E is a closed subspace and projections are continuous;*
2. *X admits a sequence $\langle \mu_n \colon n \in \omega \rangle$ of finitely supported measures such that $\|\mu_n\| = 1$ and $\int_K f d\mu_n \to 0$ for every $f \in C(X)$.*

$(c_0)_p = \{x \in \mathbb{R}^\omega \colon x(n) \to 0\}$ with the pointwise topology

Definition

For a Tychonoff space X we say that $C_p(X)$ has *the Josefson–Nissenzweig Property* (*JNP*) if X satisfies (2) of the theorem. A sequence $\langle \mu_n \colon n \in \omega \rangle$ from (2) is called *a JN-sequence* on X.

The Josefson–Nissenzweig Property of $C_p(X)$-spaces

Theorem (Banakh–Kąkol–Śliwa '18)

1. *$C_p(\beta\omega)$ does not have the JNP.*

The Josefson–Nissenzweig Property of $C_p(X)$-spaces

Theorem (Banakh–Kąkol–Śliwa '18)

1. *$C_p(\beta\omega)$ does not have the JNP.*
2. *If X contains a non-trivial convergent sequence, then $C_p(X)$ has the JNP.*

The Josefson–Nissenzweig Property of $C_p(X)$-spaces

Theorem (Banakh–Kąkol–Śliwa '18)

1. *$C_p(\beta\omega)$ does not have the JNP.*
2. *If X contains a non-trivial convergent sequence, then $C_p(X)$ has the JNP.*
3. *There exists a compact space K containing many copies of $\beta\omega$ but no non-trivial convergent sequences, yet such that $C_p(K)$ has the JNP.*

Properties of JN-sequences

Assume that a space X admits a JN-sequence $\langle \mu_n \colon \; n \in \omega \rangle$. Then:

Properties of JN-sequences

Assume that a space X admits a JN-sequence $\langle \mu_n \colon \ n \in \omega \rangle$. Then:

1. for $P_n = \{x \in \operatorname{supp}(\mu_n) \colon \ \mu_n(\{x\}) > 0\}$ and $N_n = \operatorname{supp}(\mu_n) \setminus P_n$ we have:
$$\lim_n \|\mu_n \restriction P_n\| = \lim_n \|\mu_n \restriction N_n\| = 1/2;$$

Properties of JN-sequences

Assume that a space X admits a JN-sequence $\langle \mu_n \colon\ n \in \omega \rangle$. Then:

1. for $P_n = \{x \in \mathrm{supp}(\mu_n) \colon\ \mu_n(\{x\}) > 0\}$ and $N_n = \mathrm{supp}(\mu_n) \setminus P_n$ we have:

$$\lim_n \|\mu_n \restriction P_n\| = \lim_n \|\mu_n \restriction N_n\| = 1/2;$$

2. every $f \in C(X)$ is bounded on $\bigcup_n \mathrm{supp}(\mu_n)$;

Properties of JN-sequences

Assume that a space X admits a JN-sequence $\langle \mu_n \colon\ n \in \omega \rangle$. Then:

1. for $P_n = \{x \in \mathrm{supp}(\mu_n) \colon\ \mu_n(\{x\}) > 0\}$ and $N_n = \mathrm{supp}(\mu_n) \setminus P_n$ we have:
$$\lim_n \|\mu_n \restriction P_n\| = \lim_n \|\mu_n \restriction N_n\| = 1/2;$$
2. every $f \in C(X)$ is bounded on $\bigcup_n \mathrm{supp}(\mu_n)$;
3. X admits a JN-sequence with pairwise disjoint supports;

Properties of JN-sequences

Assume that a space X admits a JN-sequence $\langle \mu_n \colon \ n \in \omega \rangle$. Then:

1. for $P_n = \{x \in \operatorname{supp}(\mu_n) \colon \ \mu_n(\{x\}) > 0\}$ and $N_n = \operatorname{supp}(\mu_n) \setminus P_n$ we have:
$$\lim_n \|\mu_n \restriction P_n\| = \lim_n \|\mu_n \restriction N_n\| = 1/2;$$
2. every $f \in C(X)$ is bounded on $\bigcup_n \operatorname{supp}(\mu_n)$;
3. X admits a JN-sequence with pairwise disjoint supports;
4. if X is compact, then either X admits a JN-sequence with supports of size 2, or $\lim_n |\operatorname{supp}(\mu_n)| = \infty$.

Spaces admitting the JNP

Theorem

If K is a compact space satisfying one of the following conditions, then $C_p(K)$ has the JNP:

1. *K is the Alexandrov Duplicate of a compact space;*

Spaces admitting the JNP

Theorem

If K is a compact space satisfying one of the following conditions, then $C_p(K)$ has the JNP:

1. *K is the Alexandrov Duplicate of a compact space;*

If L is compact, then the Alexandrov Duplicate of L is the space $L \times \{0,1\}$ endowed with the topology defined as follows: for every $x \in L$ the point $(x,1)$ is isolated and basic nhbds of $(x,0)$ are given by sets of the form $(U \times \{0\}) \cup ((U \setminus \{x\}) \times \{1\})$, where U is a nhbd of x in L.

$C_p(AD(L))$ is isomorphic to $C_p(L \cup \alpha(|L|))$.

Spaces admitting the JNP

Theorem

If K is a compact space satisfying one of the following conditions, then $C_p(K)$ has the JNP:

1. *K is the Alexandrov Duplicate of a compact space;*

If L is compact, then the Alexandrov Duplicate of L is the space $L \times \{0,1\}$ endowed with the topology defined as follows: for every $x \in L$ the point $(x,1)$ is isolated and basic nhbds of $(x,0)$ are given by sets of the form $(U \times \{0\}) \cup ((U \setminus \{x\}) \times \{1\})$, where U is a nhbd of x in L.

$C_p(AD(L))$ is isomorphic to $C_p(L \cup \alpha(|L|))$.

2. K is the limit of an inverse system based on minimal extensions;

Spaces admitting the JNP

Theorem

If K is a compact space satisfying one of the following conditions, then $C_p(K)$ has the JNP:

1. *K is the Alexandrov Duplicate of a compact space;*

If L is compact, then the Alexandrov Duplicate of L is the space $L \times \{0, 1\}$ endowed with the topology defined as follows: for every $x \in L$ the point $(x, 1)$ is isolated and basic nhbds of $(x, 0)$ are given by sets of the form $(U \times \{0\}) \cup ((U \setminus \{x\}) \times \{1\})$, where U is a nhbd of x in L.

$C_p(AD(L))$ is isomorphic to $C_p(L \cup \alpha(|L|))$.

2. *K is the limit of an inverse system based on minimal extensions;*
3. *K is a product of at least two infinite compact spaces.*

Minimal extensions

Minimal extensions

A compact space K is *obtained from a system of minimal extensions* if K is the inverse limit of a system $\langle K_\alpha, \pi^\beta_\alpha\colon\ \alpha < \beta < \delta\rangle$ such that:

Minimal extensions

Minimal extensions

A compact space K is *obtained from a system of minimal extensions* if K is the inverse limit of a system $\langle K_\alpha, \pi_\alpha^\beta \colon \alpha < \beta < \delta\rangle$ such that:

- K_γ is the inverse limit of $\langle K_\alpha, \pi_\alpha^\beta \colon \alpha < \beta < \gamma\rangle$,

Minimal extensions

Minimal extensions

A compact space K is *obtained from a system of minimal extensions* if K is the inverse limit of a system $\langle K_\alpha, \pi_\alpha^\beta\colon\ \alpha < \beta < \delta\rangle$ such that:

- K_γ is the inverse limit of $\langle K_\alpha, \pi_\alpha^\beta\colon\ \alpha < \beta < \gamma\rangle$,
- $K_{\alpha+1}$ is a minimal extension of K_α, i.e. there is a unique point $x_\alpha \in K_\alpha$ such that $\left|\left(\pi_\alpha^{\alpha+1}\right)^{-1}(x_\alpha)\right| = 2$ and $\left|\left(\pi_\alpha^{\alpha+1}\right)^{-1}(x)\right| = 1$ for every $x \neq x_\alpha$,

Minimal extensions

Minimal extensions

A compact space K is *obtained from a system of minimal extensions* if K is the inverse limit of a system $\langle K_\alpha, \pi_\alpha^\beta\colon\ \alpha < \beta < \delta\rangle$ such that:

- K_γ is the inverse limit of $\langle K_\alpha, \pi_\alpha^\beta\colon\ \alpha < \beta < \gamma\rangle$,
- $K_{\alpha+1}$ is a minimal extension of K_α, i.e. there is a unique point $x_\alpha \in K_\alpha$ such that $\left|\left(\pi_\alpha^{\alpha+1}\right)^{-1}(x_\alpha)\right| = 2$ and $\left|\left(\pi_\alpha^{\alpha+1}\right)^{-1}(x)\right| = 1$ for every $x \neq x_\alpha$,
- $K_0 = 2^\omega$ and every K_α is perfect.

Minimal extensions

Minimal extensions

A compact space K is *obtained from a system of minimal extensions* if K is the inverse limit of a system $\langle K_\alpha, \pi^\beta_\alpha\colon\ \alpha < \beta < \delta \rangle$ such that:

- K_γ is the inverse limit of $\langle K_\alpha, \pi^\beta_\alpha\colon\ \alpha < \beta < \gamma \rangle$,
- $K_{\alpha+1}$ is a minimal extension of K_α, i.e. there is a unique point $x_\alpha \in K_\alpha$ such that $\left|\left(\pi^{\alpha+1}_\alpha\right)^{-1}(x_\alpha)\right| = 2$ and $\left|\left(\pi^{\alpha+1}_\alpha\right)^{-1}(x)\right| = 1$ for every $x \neq x_\alpha$,
- $K_0 = 2^\omega$ and every K_α is perfect.

Remark: Many consistent examples of Efimov spaces are obtained by minimal extensions, e.g.

Fedorchuk ($\Diamond$), Dow and Pichardo-Mendoza (CH), Dow and Shelah (MA+$\neg$CH) etc.

Products of compact spaces and the JNP or c_0

Theorem

For every infinite compact spaces K and L the space $C_p(K \times L)$ has the JNP. In particular, $C_p(K \times L)$ contains a complemented copy of the space $(c_0)_p$.

Products of compact spaces and the JNP or c_0

Theorem

For every infinite compact spaces K and L the space $C_p(K \times L)$ has the JNP. In particular, $C_p(K \times L)$ contains a complemented copy of the space $(c_0)_p$.

Theorem (Cembranos '84, Freniche '84)

For every infinite compact spaces K and L the Banach space $C(K \times L)$ contains a complemented copy of the space c_0.

Idea of the proof

Fix infinite compact K and L.

1. $\Omega_n = \{-1,1\}^n$, $\Sigma_n = n \times \{n\}$, $\Omega = \bigcup_n \Omega_n$, $\Sigma = \bigcup_n \Sigma_n$

Idea of the proof

Fix infinite compact K and L.

1. $\Omega_n = \{-1,1\}^n$, $\Sigma_n = n \times \{n\}$, $\Omega = \bigcup_n \Omega_n$, $\Sigma = \bigcup_n \Sigma_n$
2. define a measure μ_n on $\Omega_n \times \Sigma_n$:

$$\mu_n = \sum_{\substack{s \in \Omega_n \\ (i,n) \in \Sigma_n}} \frac{s(i)}{n2^n} \cdot \delta_{(s,(i,n))}$$

Idea of the proof

Fix infinite compact K and L.

1. $\Omega_n = \{-1,1\}^n$, $\Sigma_n = n \times \{n\}$, $\Omega = \bigcup_n \Omega_n$, $\Sigma = \bigcup_n \Sigma_n$
2. define a measure μ_n on $\Omega_n \times \Sigma_n$:

$$\mu_n = \sum_{\substack{s \in \Omega_n \\ (i,n) \in \Sigma_n}} \frac{s(i)}{n2^n} \cdot \delta_{(s,(i,n))}$$

3. $\langle \mu_n \colon\ n \in \omega \rangle$ is a JN-sequence on $\beta\Omega \times \beta\Sigma$ with supports in $\Omega \times \Sigma$

Idea of the proof

Fix infinite compact K and L.

1. $\Omega_n = \{-1,1\}^n$, $\Sigma_n = n \times \{n\}$, $\Omega = \bigcup_n \Omega_n$, $\Sigma = \bigcup_n \Sigma_n$
2. define a measure μ_n on $\Omega_n \times \Sigma_n$:

$$\mu_n = \sum_{\substack{s \in \Omega_n \\ (i,n) \in \Sigma_n}} \frac{s(i)}{n2^n} \cdot \delta_{(s,(i,n))}$$

3. $\langle \mu_n \colon \; n \in \omega \rangle$ is a JN-sequence on $\beta\Omega \times \beta\Sigma$ with supports in $\Omega \times \Sigma$
4. $\beta\Omega \times \beta\Sigma \cong \beta\omega \times \beta\omega$, so $C_p(\beta\omega \times \beta\omega)$ has the JNP

Idea of the proof, cntd.

5. use the Stone Extension Property to prove that $C_p(K \times L)$ has the JNP:

Idea of the proof, cntd.

5. use the Stone Extension Property to prove that $C_p(K \times L)$ has the JNP:

   $D \subseteq K, E \subseteq L$ — countable discrete

Idea of the proof, cntd.

5. use the Stone Extension Property to prove that $C_p(K \times L)$ has the JNP:

   $D \subseteq K, E \subseteq L$ — countable discrete

   $\varphi\colon \omega \to D,\ \psi\colon \omega \to E$ — bijections

Idea of the proof, cntd.

5. use the Stone Extension Property to prove that $C_p(K \times L)$ has the JNP:

   $D \subseteq K, E \subseteq L$ — countable discrete

   $\varphi\colon \omega \to D,\ \psi\colon \omega \to E$ — bijections

   $\Phi\colon \beta\omega \to K,\ \Psi\colon \beta\omega \to L$ — cont. extensions of φ and ψ

Idea of the proof, cntd.

5. use the Stone Extension Property to prove that $C_p(K \times L)$ has the JNP:

   $D \subseteq K, E \subseteq L$ — countable discrete

   $\varphi\colon \omega \to D,\ \psi\colon \omega \to E$ — bijections

   $\Phi\colon \beta\omega \to K,\ \Psi\colon \beta\omega \to L$ — cont. extensions of φ and ψ

   define a measure ν_n on $K \times L$ as follows:

$$\nu_n = \sum_{(x,y)\in \operatorname{supp}(\mu_n)} \mu_n(\{(x,y)\}) \cdot \delta_{(\varphi(x),\psi(y))}$$

Idea of the proof, cntd.

5. use the Stone Extension Property to prove that $C_p(K \times L)$ has the JNP:

   $D \subseteq K, E \subseteq L$ — countable discrete

   $\varphi\colon \omega \to D,\ \psi\colon \omega \to E$ — bijections

   $\Phi\colon \beta\omega \to K,\ \Psi\colon \beta\omega \to L$ — cont. extensions of φ and ψ

   define a measure ν_n on $K \times L$ as follows:

$$\nu_n = \sum_{(x,y)\in\mathrm{supp}(\mu_n)} \mu_n(\{(x,y)\}) \cdot \delta_{(\varphi(x),\psi(y))}$$

6. $\langle \nu_n\colon\ n \in \omega \rangle$ is a JN-sequence on $K \times L$:

$$\int_{K\times L} f(x,y) d\nu_n(x,y) = \int_{\beta\omega\times\beta\omega} f(\Phi(x),\Psi(y)) d\mu_n(x,y)$$

Product of pseudocompact spaces

Definition

A Tychonoff space X is *pseudocompact* if every $f \in C(X)$ is bounded on X.

Theorem

Let X and Y be two infinite pseudocompact spaces.

1. *$X \times Y$ need not be pseudocompact.*

Product of pseudocompact spaces

Definition

A Tychonoff space X is *pseudocompact* if every $f \in C(X)$ is bounded on X.

Theorem

Let X and Y be two infinite pseudocompact spaces.

1. *$X \times Y$ need not be pseudocompact.*
2. *If X or Y is compact, then $X \times Y$ is pseudocompact.*

Product of pseudocompact spaces

Definition

A Tychonoff space X is *pseudocompact* if every $f \in C(X)$ is bounded on X.

Theorem

Let X and Y be two infinite pseudocompact spaces.

1. *$X \times Y$ need not be pseudocompact.*
2. *If X or Y is compact, then $X \times Y$ is pseudocompact.*
3. *(Glicksberg '59) $X \times Y$ is pseudocompact if and only if $\beta X \times \beta Y = \beta(X \times Y)$.*

Product of pseudocompact spaces

Definition

A Tychonoff space X is *pseudocompact* if every $f \in C(X)$ is bounded on X.

Theorem

Let X and Y be two infinite pseudocompact spaces.

1. *$X \times Y$ need not be pseudocompact.*
2. *If X or Y is compact, then $X \times Y$ is pseudocompact.*
3. *(Glicksberg '59) $X \times Y$ is pseudocompact if and only if $\beta X \times \beta Y = \beta(X \times Y)$.*

Corollary

If X and Y are infinite pseudocompact spaces such that $X \times Y$ is pseudocompact, then $C_p(X \times Y)$ has the JNP.

A problem of Arkhangel'ski

Corollary

Let X and Y be Tychonoff spaces. If $X \times Y$ is pseudocompact, then $C_p(X \times Y) \cong C_p(X \times Y) \oplus \mathbb{R}$.

$$C_p(X \times Y) \cong E \oplus (c_0)_p \cong E \oplus (c_0)_p \oplus \mathbb{R} \cong C_p(X \times Y) \oplus \mathbb{R}$$

A problem of Arkhangel'ski

Corollary

Let X and Y be Tychonoff spaces. If $X \times Y$ is pseudocompact, then $C_p(X \times Y) \cong C_p(X \times Y) \oplus \mathbb{R}$.

$$C_p(X \times Y) \cong E \oplus (c_0)_p \cong E \oplus (c_0)_p \oplus \mathbb{R} \cong C_p(X \times Y) \oplus \mathbb{R}$$

Question (Arkhangel'ski '82)

Is $C_p(X)$ linearly homoeomorphic to $C_p(X) \oplus \mathbb{R}$ for every infinite space X?

A problem of Arkhangel'ski

Corollary

Let X and Y be Tychonoff spaces. If $X \times Y$ is pseudocompact, then $C_p(X \times Y) \cong C_p(X \times Y) \oplus \mathbb{R}$.

$$C_p(X \times Y) \cong E \oplus (c_0)_p \cong E \oplus (c_0)_p \oplus \mathbb{R} \cong C_p(X \times Y) \oplus \mathbb{R}$$

Question (Arkhangel'ski '82)

Is $C_p(X)$ linearly homoeomorphic to $C_p(X) \oplus \mathbb{R}$ for every infinite space X?

True, e.g., if X contains a non-trivial convergent sequence or is not pseudocompact.
False for compact spaces (Marciszewski '97).

A problem of Arkhangel'ski

Corollary

Let X and Y be Tychonoff spaces. If $X \times Y$ is pseudocompact, then $C_p(X \times Y) \cong C_p(X \times Y) \oplus \mathbb{R}$.

$$C_p(X \times Y) \cong E \oplus (c_0)_p \cong E \oplus (c_0)_p \oplus \mathbb{R} \cong C_p(X \times Y) \oplus \mathbb{R}$$

Question (Arkhangel'ski '82)

Is $C_p(X)$ linearly homoeomorphic to $C_p(X) \oplus \mathbb{R}$ for every infinite space X?

True, e.g., if X contains a non-trivial convergent sequence or is not pseudocompact.
False for compact spaces (Marciszewski '97).

Fact

If X is not pseudocompact, then
$C_p(X) \cong E \oplus \mathbb{R}^\omega \cong E \oplus \mathbb{R}^\omega \oplus \mathbb{R} \cong C_p(X) \oplus \mathbb{R}$.

Haydon spaces

Construction of a Haydon space

For every $A \in [\omega]^\omega$ let $u_A \in \overline{A}^{\beta\omega}$. Put:

$$X = \omega \cup \{u_a \colon \; A \in [\omega]^\omega\}$$

X with the topology inherited from $\beta\omega$ is *a Haydon space*.

Haydon spaces

Construction of a Haydon space

For every $A \in [\omega]^\omega$ let $u_A \in \overline{A}^{\beta\omega}$. Put:

$$X = \omega \cup \{u_a : \ A \in [\omega]^\omega\}$$

X with the topology inherited from $\beta\omega$ is *a Haydon space*.

Characterization of Haydon spaces

Let X be a subspace of $\beta\omega$ containing ω. TFAE:

1. X is a Haydon space;

Haydon spaces

Construction of a Haydon space

For every $A \in [\omega]^\omega$ let $u_A \in \overline{A}^{\beta\omega}$. Put:

$$X = \omega \cup \{u_a : \ A \in [\omega]^\omega\}$$

X with the topology inherited from $\beta\omega$ is *a Haydon space*.

Characterization of Haydon spaces

Let X be a subspace of $\beta\omega$ containing ω. TFAE:

1. X is a Haydon space;
2. X is pseudocompact and $|X| = 2^\omega$;

Haydon spaces

Construction of a Haydon space

For every $A \in [\omega]^\omega$ let $u_A \in \overline{A}^{\beta\omega}$. Put:

$$X = \omega \cup \{u_a : \ A \in [\omega]^\omega\}$$

X with the topology inherited from $\beta\omega$ is *a Haydon space*.

Characterization of Haydon spaces

Let X be a subspace of $\beta\omega$ containing ω. TFAE:

1. X is a Haydon space;
2. X is pseudocompact and $|X| = 2^\omega$;
3. $|X| = 2^\omega$ and every $A \in [\omega]^\omega$ has a limit point in X.

Haydon spaces

Properties of Haydon spaces

Let X be a Haydon space. Then:

1. every compact subset of X is finite;

Haydon spaces

Properties of Haydon spaces

Let X be a Haydon space. Then:

1. every compact subset of X is finite;
2. $C_p(X)$ does not have the JNP.

Haydon spaces

Properties of Haydon spaces

Let X be a Haydon space. Then:

1. every compact subset of X is finite;
2. $C_p(X)$ does not have the JNP.

Proof

$C(\beta\omega)$ does not have any complemented copy of c_0 (Grothendieck '53, Cembranos '84).

Haydon spaces

Properties of Haydon spaces

Let X be a Haydon space. Then:

1. every compact subset of X is finite;
2. $C_p(X)$ does not have the JNP.

Proof

$C(\beta\omega)$ does not have any complemented copy of c_0 (Grothendieck '53, Cembranos '84). Since $\beta\omega = \beta X$, the same applies to $C(X)$.

Haydon spaces

Properties of Haydon spaces

Let X be a Haydon space. Then:

1. every compact subset of X is finite;
2. $C_p(X)$ does not have the JNP.

Proof

$C(\beta\omega)$ does not have any complemented copy of c_0 (Grothendieck '53, Cembranos '84). Since $\beta\omega = \beta X$, the same applies to $C(X)$. By the Closed Graph Theorem, $C_p(X)$ does not have any complemented copy of $(c_0)_p$, so it does not have the JNP.

Products of Haydon spaces

Theorem

Let $X = \omega \cup \{u_A\colon\ A \in [\omega]^\omega\}$ *be a Haydon space such that for distinct* $A, B \in [\omega]^\omega$ *the ultrafilters* u_A, u_B *are not isomorphic. Then, the square* $X \times X$ *is not pseudocompact.*

Products of Haydon spaces

Theorem

Let $X = \omega \cup \{u_A \colon\ A \in [\omega]^\omega\}$ be a Haydon space such that for distinct $A, B \in [\omega]^\omega$ the ultrafilters u_A, u_B are not isomorphic. Then, the square $X \times X$ is not pseudocompact.

Proof

For every disjoint $A, B \in [\omega]^\omega$ and bijection $f \colon A \to B$ the graph $G = \{(x, f(x)) \colon x \in A\}$ is a discrete clopen subset of $X \times X$.

Products of Haydon spaces

Theorem

There exists a Haydon space Y such that $Y \times Y$ is pseudocompact. Consequently, $C_p(Y \times Y)$ has the JNP.

Products of Haydon spaces

Theorem

There exists a Haydon space Y such that $Y \times Y$ is pseudocompact. Consequently, $C_p(Y \times Y)$ has the JNP.

Proof

$\{A_\alpha\colon\ \alpha < 2^\omega\}$ — an enumeration of $[\omega]^\omega$

$\{B_\alpha\colon\ \alpha < 2^\omega\}$ — an enumeration of $[\omega \times \omega]^\omega$

Products of Haydon spaces

Theorem

There exists a Haydon space Y such that $Y \times Y$ is pseudocompact. Consequently, $C_p(Y \times Y)$ has the JNP.

Proof

$\{A_\alpha\colon\ \alpha < 2^\omega\}$ — an enumeration of $[\omega]^\omega$
$\{B_\alpha\colon\ \alpha < 2^\omega\}$ — an enumeration of $[\omega \times \omega]^\omega$

for each $\alpha < 2^\omega$ choose limit points $u_\alpha \in \omega^*$ of A_α and $(p_\alpha, q_\alpha) \in \beta\omega \times \beta\omega$ of B_α

Products of Haydon spaces

Theorem

There exists a Haydon space Y such that $Y \times Y$ is pseudocompact. Consequently, $C_p(Y \times Y)$ has the JNP.

Proof

$\{A_\alpha : \alpha < 2^\omega\}$ — an enumeration of $[\omega]^\omega$
$\{B_\alpha : \alpha < 2^\omega\}$ — an enumeration of $[\omega \times \omega]^\omega$

for each $\alpha < 2^\omega$ choose limit points $u_\alpha \in \omega^*$ of A_α and $(p_\alpha, q_\alpha) \in \beta\omega \times \beta\omega$ of B_α

$Y = \omega \cup \{u_\alpha, p_\alpha, q_\alpha : \alpha < 2^\omega\}$

Products of Haydon spaces

Theorem

There exists a Haydon space Y such that $Y \times Y$ is pseudocompact. Consequently, $C_p(Y \times Y)$ has the JNP.

Proof

$\{A_\alpha : \alpha < 2^\omega\}$ — an enumeration of $[\omega]^\omega$
$\{B_\alpha : \alpha < 2^\omega\}$ — an enumeration of $[\omega \times \omega]^\omega$

for each $\alpha < 2^\omega$ choose limit points $u_\alpha \in \omega^*$ of A_α and $(p_\alpha, q_\alpha) \in \beta\omega \times \beta\omega$ of B_α

$Y = \omega \cup \{u_\alpha, p_\alpha, q_\alpha : \alpha < 2^\omega\}$

Corollary

Let $Z = X \sqcup Y$. Then $Z \times Z$ is not pseudocompact, but $C_p(Z \times Z)$ has the JNP.

Non-pseudocompact squares of Haydon spaces

Theorem

If any of the axioms from the below list holds, then there exists a Haydon space X such that $C_p(X \times X)$ does not have the JNP.

Non-pseudocompact squares of Haydon spaces

Theorem

If any of the axioms from the below list holds, then there exists a Haydon space X such that $C_p(X \times X)$ does not have the JNP.

1. *the Continuum Hypothesis*

Non-pseudocompact squares of Haydon spaces

Theorem

If any of the axioms from the below list holds, then there exists a Haydon space X such that $C_p(X \times X)$ does not have the JNP.

1. *the Continuum Hypothesis*
2. *Martin's axiom*

Non-pseudocompact squares of Haydon spaces

Theorem

If any of the axioms from the below list holds, then there exists a Haydon space X such that $C_p(X \times X)$ does not have the JNP.

1. *the Continuum Hypothesis*
2. *Martin's axiom*
3. *there exists an antichain in the Rudin-Keisler ordering consisting of weak P-points in ω^**

Non-pseudocompact squares of Haydon spaces

Theorem

If any of the axioms from the below list holds, then there exists a Haydon space X such that $C_p(X \times X)$ does not have the JNP.

1. *the Continuum Hypothesis*
2. *Martin's axiom*
3. *there exists an antichain in the Rudin-Keisler ordering consisting of weak P-points in ω^**
4. $\mathfrak{d} = 2^\omega \leqslant \mathfrak{u}^+$

Non-pseudocompact squares of Haydon spaces

Theorem

If any of the axioms from the below list holds, then there exists a Haydon space X such that $C_p(X \times X)$ does not have the JNP.

1. *the Continuum Hypothesis*
2. *Martin's axiom*
3. *there exists an antichain in the Rudin-Keisler ordering consisting of weak P-points in ω^**
4. $\mathfrak{d} = 2^\omega \leqslant \mathfrak{u}^+$
5. *axiom (†)*

Non-pseudocompact squares of Haydon spaces

Theorem

If any of the axioms from the below list holds, then there exists a Haydon space X such that $C_p(X \times X)$ does not have the JNP.

1. *the Continuum Hypothesis*
2. *Martin's axiom*
3. *there exists an antichain in the Rudin-Keisler ordering consisting of weak P-points in ω^**
4. $\mathfrak{d} = 2^\omega \leqslant \mathfrak{u}^+$
5. *axiom (†)*

Theorem

Assume that there exist two RK-incompatible weak P-points in ω^*. Then, there exist Haydon spaces X and Y such that $C_p(X \times Y)$ does not have the JNP.

Axiom (†)

Axiom (†)

There exists a function $A \mapsto u_A$ assigning to each $A \in [\omega]^\omega$ a weak P-point $u_A \in \overline{A}^{\beta\omega}$ such that

Axiom (†)

Axiom (†)

There exists a function $A \mapsto u_A$ assigning to each $A \in [\omega]^\omega$ a weak P-point $u_A \in \overline{A}^{\beta\omega}$ such that

for every pair $(f_1, f_2) \in \omega^\omega \times \omega^\omega$ there exists a family $\mathcal{A} \subseteq [\omega]^\omega$ of size $< 2^\omega$ such that

Axiom (†)

Axiom (†)

There exists a function $A \mapsto u_A$ assigning to each $A \in [\omega]^\omega$ a weak P-point $u_A \in \overline{A}^{\beta\omega}$ such that

for every pair $(f_1, f_2) \in \omega^\omega \times \omega^\omega$ there exists a family $\mathcal{A} \subseteq [\omega]^\omega$ of size $< 2^\omega$ such that

for every $A_1 \notin \mathcal{A}$ and $A_2 \in [\omega]^\omega \setminus \{A_1\}$, if $f_1(u_{A_1}), f_2(u_{A_2}) \in \omega^*$, then $f_1(u_{A_1}) \neq f_2(u_{A_2})$.

Axiom (†)

Axiom (†)

There exists a function $A \mapsto u_A$ assigning to each $A \in [\omega]^\omega$ a weak P-point $u_A \in \overline{A}^{\beta\omega}$ such that

for every pair $(f_1, f_2) \in \omega^\omega \times \omega^\omega$ there exists a family $\mathcal{A} \subseteq [\omega]^\omega$ of size $< 2^\omega$ such that

for every $A_1 \notin \mathcal{A}$ and $A_2 \in [\omega]^\omega \setminus \{A_1\}$, if $f_1(u_{A_1}), f_2(u_{A_2}) \in \omega^*$, then $f_1(u_{A_1}) \neq f_2(u_{A_2})$.

Question

Does (†) hold in ZFC?

Proof of the theorem

Let X be constructed using (†). Assume X^2 admits a JN-sequence $\langle \mu_n \colon \ n \in \omega \rangle$.

Proof of the theorem

Let X be constructed using (†). Assume X^2 admits a JN-sequence $\langle \mu_n \colon\ n \in \omega \rangle$.

1. We construct a new disjointly supported JN-sequence $\langle \nu_n \colon\ n \in \omega \rangle$ such that $\bigcup_n \operatorname{supp}(\nu_n)$ is discrete.

   (HERE WE USE WEAK P-POINTS.)

Proof of the theorem

Let X be constructed using (†). Assume X^2 admits a JN-sequence $\langle \mu_n \colon\ n \in \omega \rangle$.

1. We construct a new disjointly supported JN-sequence $\langle \nu_n \colon\ n \in \omega \rangle$ such that $\bigcup_n \operatorname{supp}(\nu_n)$ is discrete.

   (HERE WE USE WEAK P-POINTS.)

2. We prove that $\lim_n |\nu_n|(\Delta_X^c) = 0$ by constructing a disjoint sequence of clopens $\langle V_k^1 \times V_k^2 \colon\ n \in \omega \rangle$ outside of Δ_X such that $\limsup_k |\nu_k|(V_k^1 \times V_k^2) > 0$ and $\bigcup_k \overline{V_k^1 \times V_k^2}^{X \times X}$ is clopen.

   (HERE WE USE THE REST OF (†).)

Proof of the theorem

Let X be constructed using (†). Assume X^2 admits a JN-sequence $\langle \mu_n \colon\ n \in \omega \rangle$.

1. We construct a new disjointly supported JN-sequence $\langle \nu_n \colon\ n \in \omega \rangle$ such that $\bigcup_n \operatorname{supp}(\nu_n)$ is discrete.

   (HERE WE USE WEAK P-POINTS.)

2. We prove that $\lim_n |\nu_n|(\Delta_X^c) = 0$ by constructing a disjoint sequence of clopens $\langle V_k^1 \times V_k^2 \colon\ n \in \omega \rangle$ outside of Δ_X such that $\limsup_k |\nu_k|(V_k^1 \times V_k^2) > 0$ and $\bigcup_k \overline{V_k^1 \times V_k^2}^{X \times X}$ is clopen.

   (HERE WE USE THE REST OF (†).)

3. $\lim_n |\nu_n|(\Delta_X) = 1$, but $\Delta_X \cong X$ and $C_p(X)$ does not have the JNP, a contradiction.

Open questions

Question

Is it consistent that for any infinite pseudocompact space X the space $C_p(X \times X)$ has the JNP?

Open questions

Question

Is it consistent that for any infinite pseudocompact space X the space $C_p(X \times X)$ has the JNP?

Question

Is it consistent that there exists an infinite countably compact space X such that the space $C_p(X \times X)$ does not have the JNP?

The end

Thank you for the attention!