Definability of maximal families of reals in forcing extensions

Jonathan Schilhan

Institute of Mathematics, University of Vienna

KGRC Seminar, 01.10.2020

generous support through FWF Project Y1012-N35

History and motivation

Many types of special sets of reals are central in field such as set theory, topology, measure theory or algebra:

Well-orders, ultrafilters, mad families, Vitali sets (E_0-transversals), Hamel bases, maximal independent families (mif), maximal sets of orthogonal measures (mof), maximal Turing-independent families, maximal cofinitary groups (mcg), eventually different families (med), towers, scales (in ω^ω), ...

Their existence is guaranteed by the *Axiom of Choice*, which has the controversy of not giving explicit definitions.

Under certain circumstances though, these sets can be *nicely definable* (OD, OD($\mathbb{R}$), projective, $\boldsymbol{\Delta}^1_2$, $\boldsymbol{\Sigma}^1_2$, $\boldsymbol{\Pi}^1_1$, Borel).

History and motivation

One of the earliest results in this direction is due to Gödel:

Theorem (Gödel 1940)

There is a Δ^1_2-definable well-order of the reals in the constructible universe L.

The technique is very general and can be used to construct Δ^1_2 witnesses in L for all examples above. In some cases, this is the best possible:

Fact

A Vitali set is a non-measurable set of reals. In particular, it cannot be $\boldsymbol{\Sigma}^1_1 \cup \boldsymbol{\Pi}^1_1$.

E.g. the same holds true for ultrafilters. Also well-orders.

Theorem (Erdős, Kunen, Mauldin 1981)

There is a Π^1_1 scale in L.

Their technique was streamlined by A. Miller who applied it to many other examples.

Theorem (Miller 1989)

There is a Π^1_1 mad family, maximal independent family, Hamel basis in L.

History and motivation

On the other hand, it was known that $\mathbf{\Sigma}^1_1$ definitions typically do not work.

Theorem (Mathias 1977)

There is no $\mathbf{\Sigma}^1_1$-definable mad family.

Theorem (Miller 1989)

There is no $\mathbf{\Sigma}^1_1$-definable maximal independent family or Hamel basis.

Some mysterious exceptions:

Theorem (Horowitz, Shelah 2016)

There is a Borel mcg and med.

Recent results

In the last decade, research in the area has become very active (with a lot of emphasis on mad families). A few new phenomena have been discovered.

For example:

- If there is a $\Sigma^1_2(r)$ mad family, then there is also a $\Pi^1_1(r)$ one. (Törnquist 2013)
- If there is a $\Sigma^1_2(r)$ mif, then there is also a $\Pi^1_1(r)$ one. (Brendle, Fischer, Khomskii 2019)
- If there is a $\Sigma^1_2(r)$ ultrafilter, then there is a $\Pi^1_1(r)$ ultrafilter base. (S. 2019)

Also:

- If there is a $\Sigma^1_2(r)$ mad family, then $\omega_1 = \omega_1^{L[r]}$.
- If there is a $\Sigma^1_2(r)$ mif, then $\omega_1 = \omega_1^{L[r]}$.
- ...

In particular, we can not be too far away from L for $\mathbf{\Sigma}^1_2$ definability.

Recent results

So far we have only mentioned positive definability results in L. What happens in **forcing extensions** of L?

Fact

(V=L) There is a Π^1_1-definable Cohen-, Sacks-, Random-, Miller-indestructible mad family.

Known techniques for indestructible mad families + making the construction Σ^1_2-definable + evaluates to the same set in the extension + $\Sigma^1_2 \to \Pi^1_1$

What about other forcing notions?

Theorem (Brendle, Khomskii 2013)

There is a Π^1_1 mad family in the Hechler extension of L.

Completely different technique: All mad families in L are destroyed. Really the **definition** is preserved.

Definition

$\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A}$ is a mad family$\}$
$\mathfrak{a}_B = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathbf{\Delta}^1_1, \bigcup \mathcal{B}$ is a mad family$\}$

Obviously $\mathfrak{a}_B \leq \mathfrak{a}$. Note that if there is a $\mathbf{\Sigma}^1_2$ mad family, then $\mathfrak{a}_B = \aleph_1$ ($\mathfrak{a}_B > \aleph_0$ since there is no Borel mad family).

Brendle and Khomskii in fact first showed

Theorem

$\mathfrak{a}_B = \aleph_1$ *in the Hechler model (so* $\mathrm{CON}(\mathfrak{a}_B < \mathfrak{b} = \mathfrak{a})$*).*

They construct a sequence $\langle B_\alpha : \alpha < \omega_1 \rangle$ of Borel sets coded in L, such that $\Vdash \bigcup_{\alpha<\omega_1} B_\alpha$ is a mad family. Then, using the standard techniques, this construction can be turned into a Σ^1_2-definition.

Hypergraphs

Ultimate goal: Understand the definability of various types of families in forcing extensions of L.

Observation: Many of the examples we gave can be framed as maximal independent sets in hypergraphs.

Definition

A *hypergraph* on a set X is a collection E (the edges) of finite non-empty subsets of X, i.e. $E \subseteq [X]^{<\omega} \setminus \{\emptyset\}$. We say that $Y \subseteq X$ is *E-independent* if $[Y]^{<\omega} \cap E = \emptyset$. Y is *maximal E-independent* if Y is maximal under inclusion as an E-independent subset of X.

If X is a Polish space, then $[X]^{<\omega}$ also has a natural Polish topology and we can study definable hypergraphs E and definable maximal E-independent sets.

Fact

In L, every analytic hypergraph on a Polish space X has a $\mathbf{\Delta}^1_2$ maximal independent set.

Note: $\mathbf{\Delta}^1_2 \leftrightarrow \mathbf{\Sigma}^1_2$

Examples

Example (MIF)

$Y \subseteq \mathcal{P}(\omega)$ is an independent family if for all finite disjoint $A, B \subseteq Y$, $\bigcap_{x \in A} x \cap \bigcap_{x \in B} \omega \setminus x$ is infinite. Letting

$$E_i := \{A \dot\cup B \in [\mathcal{P}(\omega)]^{<\omega} : \bigcap_{x \in A} x \cap \bigcap_{x \in B} \omega \setminus x \text{ is finite}\}$$

an independent family is an E_i-independent set.

The definability of maximal independent families has been recently studied by Brendle, Fischer and Khomskii. One of their main open questions was

Question

Is it consistent that $\mathfrak{i} > \aleph_1$, while there is a Π^1_1 maximal independent family? Is $\mathfrak{i}_B < \mathfrak{i}$ consistent?

Can we destroy all ground model mif's while preserving a Π^1_1 **definition**?

Examples

Example (Ultrafilter)

Let $E_u := \{A \in [\mathcal{P}(\omega)]^{<\omega} : \bigcap A \text{ is finite}\}$. Then an ultrafilter is a maximal E_u-independent set.

In a recent paper, we studied the definability of ultrafilters and asked

Question

Is it consistent that $\mathfrak{u} > \aleph_1$, while there is a Δ^1_2 ultrafilter? Is $\mathfrak{u}_B < \mathfrak{u}$ consistent?

Can we destroy all ground model ultrafilters (ultrafilter bases) while preserving a Δ^1_2 **definition**?

Examples

Example (Hamel basis)

Let $E_h := \{A \in [\mathbb{R}]^{<\omega} : A$ is linearly dependent over $\mathbb{Q}\}$. Then a Hamel basis is a maximal E_h-independent set.

Every Hamel basis has size $2^{\aleph_0}$. This is reflected by the fact that adding a single real destroys every ground model Hamel basis.

Question

Is it consistent that $\neg$ CH, while there is a Δ^1_2 Hamel basis?

Can we destroy all ground model Hamel bases (i.e. add a new real) while preserving a Δ^1_2 **definition**?

For mad families, Vitali sets or mof's, 2-dimensional hypergraphs (i.e. usual graphs) suffice.

Theorem (Schrittesser 2016)

After forcing with a csi of Sacks forcing over L, every analytic (2-dimensional hyper)graph on a Polish space has a $\mathbf{\Delta}^1_2$ *maximal independent set.*

Adding a single real, destroys every ground model maximal orthogonal family of measures.

How to increase $\mathfrak{u}$ and $\mathfrak{i}$? How to destroy ultrafilters and maximal independent families (and, well, Hamel bases)? Add splitting reals!

Definition

A real $x \in [\omega]^\omega$ is splitting over V if for every $y \in [\omega]^\omega \cap V$, $|x \cap y| = \omega$ and $|y \setminus x| = \omega$.

Classical forcing notions adding splitting reals are: Cohen, Random, Silver and forcings adding dominating reals.

Unfortunately they don't work:

Theorem (S.)

In extensions via the posets above there are reals that are splitting over any $\Sigma^1_2(r)$ set with the finite intersection property, for any $r \in V$. (for Cohen, Random, Silver: any OD(V) *set)*

(For a definable independent family, there are countably many (similarly) definable families with the FIP so that any splitting real over all of them witnesses the non-maximality of it.)

Splitting forcing

There is another less known forcing adding splitting reals.

Definition

A set $A \subseteq 2^{<\omega}$ is called *fat* if there is $m = m(A) \in \omega$ so that for every $n \geq m$, $i \in 2$, there is $s \in A$ so that $s(n) = i$.
Let $T \subseteq 2^{<\omega}$ be a perfect tree. Then T is a *splitting tree* if for every $s \in T$, T_s is fat.
(Recall: $T_s = \{t \in T : t \not\perp s\}$)
Splitting forcing $\mathbb{SP}$ consists of all splitting trees ordered by inclusion ($T \leq S$ iff $T \subseteq S$), as usual.

Fact

- *$\mathbb{SP}$ adds a generic splitting real $x_G \in 2^\omega (\cong \mathcal{P}(\omega))$,*
- *$\mathbb{SP}$ is proper (Axiom A),*
- *$\mathbb{SP}$ has continuous reading of names: whenever $\dot{y}$ is a name for an element of a Polish space X (coded in the ground model), $S \in \mathbb{SP}$, there is $T \leq S$ and $f : [T] \to X$ continuous such that $T \Vdash \dot{y} = f(x_G)$*
- *$V^{\mathbb{SP}}$ is a minimal extension of V.*

Recall: Sacks forcing $\mathbb{S}$ consists of all perfect subtrees of $2^{<\omega}$.

How to preserve?

Does splitting forcing work? Can we maybe treat ultrafilters and maximal independent families in the same way? What about Hamel bases?

Maybe we should first ask a more general question: What does it mean for a forcing $\mathbb{P}$ to preserve a union of Borel sets $Y = \bigcup \mathcal{B} \subseteq X$ maximal E-independent?
Let $\dot{y}$ be a $\mathbb{P}$-name for an element of X. Potentially $\dot{y}$ could be a threat to the maximality of (the reinterpretation of) Y. Say $\mathcal{B}$ is closed under finite unions.

Then, necessarily, for a dense set of $q \in \mathbb{P}$, there is $B \in \mathcal{B}$ so that

1. either $q \Vdash \dot{y} \in B$,
2. or $q \Vdash \{\dot{y}\} \cup B$ is not E-independent.

On the other hand the following is sufficient:

For every name $\dot{y}$, every analytic hypergraph H and $p \in \mathbb{P}$, there is $q \leq p$ and an H-independent Borel set B such that

1. either $q \Vdash \dot{y} \in B$,
2. or $q \Vdash \{\dot{y}\} \cup B$ is not H-independent.

How to construct?

Why?

Let $\langle \dot{y}_\alpha, p_\alpha : \alpha < \omega_1 \rangle$ enumerate all pairs of (nice) $\mathbb{P}$-names for elements of X ($|\mathbb{P}| = \aleph_1$ for now, $\mathbb{P}$ proper) and conditions in $\mathbb{P}$. We recursively construct Borel sets $\langle B_\alpha : \alpha < \omega_1 \rangle$.

At stage α : Let H_α be the hypergraph on X where $\{x_0, \dots, x_{n-1}\} \in H_\alpha$ iff $\{x_0, \dots, x_{n-1}\} \cup \bigcup_{i<\alpha} B_i$ is not E-independent. Then there is $q \leq p_\alpha$ and an H_α-independent Borel set B so that

1. either $q \Vdash \dot{y}_\alpha \in B$,
2. or $q \Vdash \{\dot{y}_\alpha\} \cup B$ is not H-independent.

Translated this means that $B_\alpha = B \cup \bigcup_{i<\alpha} B_i$ is E-independent and

1. either $q \Vdash \dot{y}_\alpha \in B_\alpha$,
2. or $q \Vdash \{\dot{y}_\alpha\} \cup B_\alpha$ is not E-independent.

Finally let $Y = \bigcup_{\alpha<\omega_1} B_\alpha$. By genericity, we have taken care of every potential threat $\dot{y}$.

Combinatorial reformulation

Remember the desirable property:

For every name $\dot{y}$, every analytic hypergraph H and $p \in \mathbb{P}$, there is $q \leq p$ and an H-independent Borel set B such that

1. either $q \Vdash \dot{y} \in B$,
2. or $q \Vdash \{\dot{y}\} \cup B$ is not H-independent.

If $\mathbb{P}$ is a tree forcing (say subtrees of $2^{<\omega}$) with continuous reading of names, we can forget about names and pull everything back to conditions $T \in \mathbb{P}$:

For every analytic hypergraph H on 2^ω and $T \in \mathbb{P}$, there is $S \leq T$ such that

1. either $[S]$ is H-independent,
2. or there are continuous functions $\phi_0, \ldots, \phi_{N-1} \colon [S] \to 2^\omega$ so that

$$\bigcup_{i<N} \phi_i''[S] \text{ is } H\text{-independent, but } \forall x \in [S](\{x\} \cup \{\phi_0(x), \ldots, \phi_{n-1}(x)\} \in H).$$

This is a purely combinatorial statement about trees in $\mathbb{P}$.

Mutual Cohen genericity

The key idea is going to be **mutual genericity**.

Definition

Let M be a countable transitive model of set theory (ctm), $X \in M$ a (code for a) Polish space.Then $x \in X$ is called *Cohen generic in X over M* if for every open dense subset $O \in M$ (coded in M) of X, $x \in O$. $x_0, \ldots, x_{n-1} \in X$ are *mutually Cohen generic (mCg) in X over M* if $(y_0, \ldots, y_{n'-1})$ is generic in $X^{n'}$ over M, where $y_0, \ldots, y_{n'-1}$ enumerate $x_0, \ldots, x_{n-1}$.

Lemma

Let M be a ctm, $T \in M$ a perfect subtree of $2^{<\omega}$, i.e. $T \in \mathbb{S}$.

- *There is $S \leq T$, a perfect tree (i.e. $S \in \mathbb{S}$), so that any $x_0, \ldots, x_{n-1} \in [S]$ are mCg in $[T]$ over M.*
- *If $T \in \mathbb{SP}$, there is $S \leq T$, $S \in \mathbb{SP}$, so that any $x_0, \ldots, x_{n-1} \in [S]$ are mCg in $[T]$ over M.*
- *In fact, if $\mathbb{P}$ is any* **weighted tree forcing**, *and $T \in \mathbb{P}$ then there is $S \leq T$, $S \in \mathbb{P}$, so that any $x_0, \ldots, x_{n-1} \in [S]$ are mCg in $[T]$ over M.*

Definition

A tree forcing $\mathbb{P}$ is *weighted*, if ...something technical...

$\mathbb{S}$ and $\mathbb{SP}$ are examples + generalizations.

Key Lemma 1

Key Lemma

Let H be an analytic hypergraph on X. Then there is a ctm M so that

1. *either, for any $x_0, \ldots, x_{n-1} \in X$ that are mCg over M, $\{x_0, \ldots, x_{n-1}\}$ is H-independent,*
2. *or, there are $c_0, \ldots, c_{N-1} \in X$ and a non-empty open set $O \subseteq X$, so that for any $x \in O$ Cohen generic over M, $\{c_0, \ldots, c_{N-1}\}$ is H-independent but $\{c_0, \ldots, c_{N-1}\} \cup \{x\} \in H$.*

Proof.

Let M_0 be the transitive collapse of a countable elementary model containing H and X. Suppose 1. fails for $M = M_0$. Then there is a counter-example of minimal size: $c_0, \ldots, c_{N-1}, c_N$ mCg over M, $\{c_0, \ldots, c_{N-1}\}$ H-independent, but $\{c_0, \ldots, c_{N-1}, c_N\} \in H$. Consider $M = M_0[c_0, \ldots, c_{N-1}]$. As c_N is generic over M, there is an open set (a condition) $O \in M$ with $c_N \in O$ and

$$O \Vdash \{\dot{c}\} \cup \{c_0, \ldots, c_{N-1}\} \in H.$$

Thus for any generic $x \in O$, $M[x] \models \{x, c_0, \ldots, c_{N-1}\} \in H$.
By $\mathbf{\Sigma}^1_1$-absoluteness indeed, $\{x, c_0, \ldots, c_{N-1}\} \in H$. □

Key Lemma 1

Key Lemma

Let H be an analytic hypergraph on X. Then there is a ctm M so that

1. *either, for any $x_0, \dots, x_{n-1} \in X$ that are mCg over M, $\{x_0, \dots, x_{n-1}\}$ is H-independent,*
2. *or, there are $c_0, \dots, c_{N-1} \in X$ and a non-empty open set $O \subseteq X$, so that for any $x \in O$ Cohen generic over M, $\{c_0, \dots, c_{N-1}\}$ is H-independent but $\{c_0, \dots, c_{N-1}\} \cup \{x\} \in H$.*

Putting things together:

H a hypergraph on $[T]$, T a condition. Applying the key lemma, get the model M:

1. Let $S \leq T$ be as in the first lemma: all $x_0, \dots, x_{n-1} \in [S]$ are mCg in $[T]$ over M $\rightarrow \{x_0, \dots, x_{n-1}\} \notin H \rightarrow [S]$ is H-independent
2. Define $\phi_0, \dots, \phi_{N-1}$ constant, $\phi_i(x) = c_i$. Let $s \in T$, $[s] \subseteq O$ and apply the first lemma to get $S \leq T_s$: every $x \in [S]$ is Cohen generic in $[T] \cap O$ over M $\rightarrow$

$$\bigcup_{i<N} \phi_i''[S] \text{ is } H\text{-independent, but } \forall x \in [S](\{x\} \cup \{\phi_0(x), \dots, \phi_{n-1}(x)\} \in H).$$

Partial answer/result

Theorem

(V=L) For any $\Sigma^1_1(r)$ hypergraph E, there is a (ground model coded) $\Delta^1_2(r)$ maximal E-independent set after adding a single Sacks or a single splitting real (via $\mathbb{SP}$).
We can preserve the **definition** *of an ultrafilter/mif/Hamel basis while destroying all ultrafilters/mifs/Hamel bases.*

More generally: Any proper weighted tree forcing with continuous reading of names.

This is a good first step. But this is far from a model where $\mathfrak{u}$, $\mathfrak{i}$ or $\mathfrak{c}$ is greater than $\aleph_1$.

What about adding more than one real? $\mathbb{SP}^k$, $\mathbb{S}^k$ for $k \in \omega$?

- Conditions are easy to work with: $(T_0, \dots, T_{k-1})$.
- We have a natural analogue of continuous reading of names: $f \colon \prod_{i<k}[T_i] \to X$, $(T_0, \dots, T_{k-1}) \Vdash f(\bar{x}_G) = \dot{y}$.
- Maybe a similar argument works? The combinatorial reformulation is straightforward: For every hypergraph H on $(2^\omega)^k$, $\bar{T} \in \mathbb{P}^k$, there is $\bar{S} \leq \bar{T}$ so that either, $[\bar{S}] = \prod_{i<k}[S_i]$ is H-independent or, there are $\phi_0, \dots, \phi_{N-1}$ continuous, $\bigcup_{i<N} \phi_i''[\bar{S}]$ is H-independent, $\{\bar{x}, \phi_i(\bar{x}) : i < N\} \in H$ for $\bar{x} \in [\bar{S}]$.

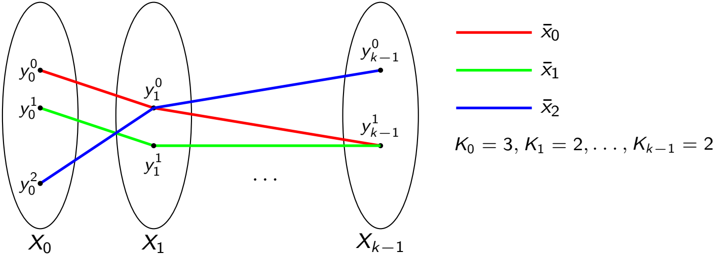
Mutual Cohen genericity revisited

Definition

Let M be a ctm, $\langle X_l : l < k \rangle \in M$ be (codes for) Polish spaces. Then we say that $\bar{x}_0, \ldots, \bar{x}_{n-1} \in \prod_{l<k} X_l$ are *mutually Cohen generic (mCg) with respect to the product* $\prod_{l<k} X_l$ *over* M, if

$$(y_0^0, \ldots, y_0^{K_0-1}, \ldots, y_{k-1}^0, \ldots, y_{k-1}^{K_{k-1}-1}) \text{ is Cohen generic in } \prod_{l<k} X_l^{K_l} \text{ over } M,$$

where $\langle y_l^i : i < K_l \rangle$ is some, equivalently any, enumeration of $\{x_i(l) : i < n\}$ for each $l < k$.



Key Lemma 2: finite products

Lemma

Let M be a ctm, $T_0, \dots, T_{k-1} \in M \cap \mathbb{P}$, where $\mathbb{P}$ is a weighted tree forcing (e.g. $\mathbb{S}$ or $\mathbb{SP}$). Then there are $S_0 \leq T_0, \dots, S_{k-1} \leq T_{k-1}$ so that any $\bar{x}_0, \dots, \bar{x}_{n-1} \in \prod_{i<k}[S_i]$ are mCg wrt $\prod_{i<k}[T_i]$ over M.

Key Lemma

Let H be an analytic hypergraph on $(2^\omega)^k$. Then there is a ctm M so that

1. *either, for any $\bar{x}_0, \dots, \bar{x}_{n-1} \in (2^\omega)^k$ that are mCg over M, $\{\bar{x}_0, \dots, \bar{x}_{n-1}\}$ is H-independent,*
2. *or, there are $\phi_0, \dots, \phi_{N-1} \colon (2^\omega)^k \to (2^\omega)^k$ continuous, $\bar{s} \in (2^{<\omega})^k$, so that for any mCg $\bar{x}_0, \dots, \bar{x}_{n-1} \in [\bar{s}]$ over M, $\{\phi_i(\bar{x}_j) : i < N, j < n\}$ is H-independent but $\{\bar{x}_0, \phi_i(\bar{x}_0) : i < N\} \in H$.*

Proof.

Much more complicated than before. Uses ideas from Harrington's forcing proof of Halpern-Läuchli. □

Key Lemma 2: finite products

Example

Let $k = 2$, $H \subseteq [2^\omega \times 2^\omega]^2$ where $\{\bar{x}_0 \neq \bar{x}_1\} \in H$ iff $x_0(0) = x_1(0)$.

Case 1 is impossible. So we are in case 2: Let $c \in 2^\omega$ be arbitrary, $c \in M$ a ctm and let $\phi(\bar{x}) = (x(0), c)$.Then $\{\bar{x}, \phi(\bar{x})\} \in H$ for every $\bar{x}$ with $x(1) \neq c$ (e.g. $\bar{x}$ generic over M). On the other hand, if $\phi(\bar{x}_0) \neq \phi(\bar{x}_1)$, then $\phi(\bar{x}_0)(0) \neq \phi(\bar{x}_1)(0)$ so $\{\phi(\bar{x}_0), \phi(\bar{x}_1)\} \notin H$.

Example

Let $k = 2$, $H \subseteq [2^\omega \times 2^\omega]^2$ where $\{\bar{x}_0 \neq \bar{x}_1\} \in H$ iff $x_0(0) = x_1(0)$ or $x_0(1) = x_1(1)$.

Again, case 1 is impossible. Instead of a constant $c \in 2^\omega$, let $f : 2^\omega \to 1^\frown 2^\omega$ be a continuous injection, $f \in M$ a ctm and let $\phi(\bar{x}) = (x(0), f(x(0)))$, $\bar{s} = (\emptyset, \langle 0 \rangle)$.Then $\{\bar{x}, \phi(\bar{x})\} \in H$ for every $\bar{x} \in [\bar{s}]$. If $\phi(\bar{x}_0) \neq \phi(\bar{x}_1)$, then $x_0(0) \neq x_1(0)$ and then $\phi(\bar{x}_0)$ and $\phi(\bar{x}_1)$ are different in both coordinates, so $\{\phi(\bar{x}_0), \phi(\bar{x}_1)\} \notin H$.

Partial answer/result 2

Theorem

(V=L) For $\Sigma^1_1(r)$ hypergraph E, there is a $\Delta^1_2(r)$ maximal E-independent set after forcing with $\mathbb{S}^k$ or $\mathbb{SP}^k$, $k \in \omega$.

More generally: any finite product of proper weighted tree forcings with crn, e.g. $\mathbb{S}^{k_0} \times \mathbb{SP}^{k_1}$.

Great! We only need to generalize to infinite products. The csp of $\mathbb{S}$, $\mathbb{SP}$ is proper and has continuous reading of names.

Counterexample

Consider E_1 on $(2^\omega)^\omega$ where $\{\bar{x}_0 \neq \bar{x}_1\} \in E_1$ iff $\forall^\infty n \in \omega (x_0(n) = x_1(n))$.
Let $(S_i)_{i\in\omega}$ be perfect trees (a condition in $\mathbb{S}^\omega$ or $\mathbb{SP}^\omega$). Then $\prod_{i\in\omega}[S_i]$ is never E_1-independent (i.e. a partial transversal for E_1). On the other hand, any continuous $\phi\colon \prod_{i\in\omega}[S_i] \to \prod_{i\in\omega}[S_i]$ so that

$$\{\phi(\bar{x}), \bar{x}\} \in E_1 \text{ for every } \bar{x} \in \prod_{i\in\omega}[S_i] \text{ and } \phi'' \prod_{i\in\omega}[S_i] \text{ is } E_1\text{-independent,}$$

is a continuous selector for $E_1 \restriction \prod_{i\in\omega}[S_i] \cong_B E_1$.

Corollary

In an extension by $\mathbb{SP}^\omega$, there is no Δ^1_2-definable E_1-transversal. For $\mathbb{S}^\omega$, this follows by a simpler homogeneity argument and holds for all sets definable over the ground model.

We could ask:

Question

Can we characterize hypergraphs for which countable support products of, say $\mathbb{S}$, work? For which hypergraphs does the combinatorial reformulation hold true?

Iterations on the other hand seem promising, since conditions are "smaller" than in products. For instance, the argument for E_1 fails:

Fact

For any $\bar{p} \in \mathbb{S}^{*\omega}$, there is $\bar{q} \leq \bar{p}$ so that for any $\mathbb{S}^{*\omega}$-generics $\bar{x}_0 \neq \bar{x}_1$ with $\bar{q}$ in the corresponding generic filter, $x_0(n) \neq x_1(n)$ for all $n \geq \min\{m : x_0(m) \neq x_1(m)\}$.

Conditions in iterations are harder to work with though. Also what does continuous reading of names mean now?

Good master conditions

Let $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta \leq \lambda \rangle$ be a countable support iteration, where for each $\beta < \lambda$, $\mathbb{Q}_\beta$ is a tree forcing, $\mathbb{Q}_\beta$ is an analytic subset of a Polish space and there is a sequence $\langle \leq_{\beta,n} : n \in \omega \rangle$ of analytic partial orders on $\mathbb{Q}_\beta$ witnessing the *Axiom A with continuous reading of names*.

Assume each $\mathbb{Q}_\beta$ consists of subtrees of $2^{<\omega}$.

Lemma

For any $\bar{p} \in \mathbb{P}_\lambda$, M a countable elementary model with $\mathbb{P}_\lambda, \bar{p} \in M$, there is $\bar{q} \leq \bar{p}$ a master condition over M together with a unique closed set $[\bar{q}] \subseteq (2^\omega)^\lambda$ so that

1. $\bar{q} \Vdash \bar{x}_G \in [\bar{q}]$,

for every $\beta < \lambda$,

2. $\bar{q} \Vdash \dot{q}(\beta) = \{s \in 2^{<\omega} : \exists \bar{z} \in [\bar{q}](\bar{z} \restriction \beta = \bar{x}_G \restriction \beta \wedge s \subseteq z(\beta))\}$,
3. *the map sending $\bar{x} \in [\bar{q}] \restriction \beta$ to $\{s \in 2^{<\omega} : \exists \bar{z} \in [\bar{q}](\bar{z} \restriction \beta = \bar{x} \wedge s \subseteq z(\beta))\}$ is continuous and maps to $\mathbb{Q}_\beta$,*

and for every name $\dot{y} \in M$ for an element of a Polish space X,

4. *there is a continuous function $f : [\bar{q}] \to X$ so that $\bar{q} \Vdash \dot{y} = f(\bar{x}_G)$.*

Moreover, there is a countable set $A \subseteq \lambda$ so that $[\bar{q}] = (2^\omega)^{\lambda \setminus A} \times [\bar{q}] \restriction A$ and all continuous functions above are supported on A.

$\bar{q}$ is called a *good master condition over M*.

Good master conditions

On the other hand: whenever A is countable, $C \subseteq (2^\omega)^A$ is a closed set where for each $\beta \in A$ and $\bar{x} \in C \restriction \beta$:

$$\{s \in 2^{<\omega} : \exists \bar{z} \in C(\bar{z} \restriction \beta = \bar{x} \wedge s \subseteq z(\beta))\} \in \mathbb{Q}_\beta,$$

then there is a good master condition $\bar{q} \in \mathbb{P}_\lambda$ such that $[\bar{q}] \restriction A \subseteq C$.

Remember that for any perfect tree $T \subseteq 2^{<\omega}$, there is a canonical homeomorphism $\eta_T \colon [T] \to 2^\omega$. If $\bar{q}$ is a good master condition and $A \subseteq \lambda$ as before, we can use this to define a canonical homeomorphism

$$\Phi_{\bar{q}} \colon [\bar{q}] \restriction A \to (2^\omega)^\alpha,$$

where $\alpha = \mathrm{otp}(A)$, witnessed by $\iota \colon A \to \alpha$, and for each $\beta \in A$, $\bar{x} \in [\bar{q}] \restriction A$,

$$\Phi_{\bar{q}}(\bar{x})(\iota(\beta)) = \eta_T(x(\beta)),$$

with $T = \{s \in 2^{<\omega} : \exists \bar{z} \in [\bar{q}] \restriction A(\bar{z} \restriction \beta = \bar{x} \restriction \beta \wedge s \subseteq z(\beta))\}$.

Mutual Cohen genericity revisited again

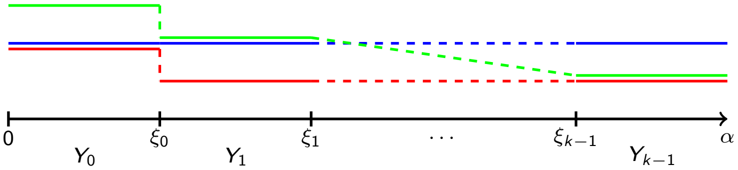
This time we have an infinite product $(2^\omega)^\alpha$.

Definition

Let $\alpha < \omega_1$, M a ctm with $\alpha \in M$. Then we say that $\bar{x}_0, \dots, \bar{x}_{n-1}$ are mCg with respect to the product $\prod_{\beta<\alpha} 2^\omega$ over M, if there is a partition $\xi_0 = 0 < \dots < \xi_k = \alpha$, $k \in \omega$, so that

$$\bar{x}_0, \dots, \bar{x}_{n-1} \text{ are mCg with respect to } \prod_{l<k} Y_l \text{ over M,}$$

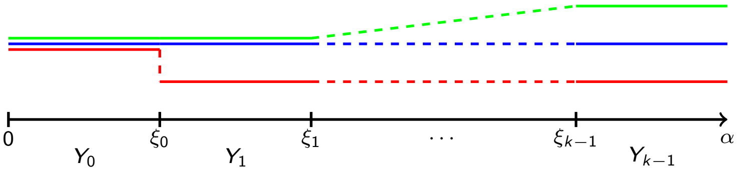
where $Y_l = (2^\omega)^{[\xi_l, \xi_{l+1})}$, $l < k$.



Mutual Cohen genericity revisited again

Definition

Let $\alpha < \omega_1$, M a ctm with $\alpha \in M$. Then we say that $\bar{x}_0, \ldots, \bar{x}_{n-1}$ are **strongly** mCg with respect to the product $\prod_{\beta<\alpha} 2^\omega$ over M, if they are mCg (as before) and for any $i, j < n$ if $\xi = \min\{\beta : x_i(\beta) \neq x_j(\beta)\}$, then for all $\beta \geq \xi$, $x_i(\beta) \neq x_j(\beta)$.



Key Lemma 3: infinite products

Key Lemma

Let $\alpha < \omega_1$ and H an analytic hypergraph on $(2^\omega)^\alpha$. Then there is a ctm M, $\alpha \in M$, so that

1. *either, for any $\bar{x}_0, \ldots, \bar{x}_{n-1} \in (2^\omega)^\alpha$ that are strongly mCg over M (wrt $\prod_{\beta<\alpha} 2^\omega$), $\{\bar{x}_0, \ldots, \bar{x}_{n-1}\}$ is H-independent,*
2. *or, there are $\phi_0, \ldots, \phi_{N-1} \colon (2^\omega)^\alpha \to (2^\omega)^\alpha$ continuous, $\bar{s} \in \bigotimes_{\beta<\alpha} 2^{<\omega}$, so that for any strongly mCg $\bar{x}_0, \ldots, \bar{x}_{n-1} \in [\bar{s}]$ over M (wrt $\prod_{\beta<\alpha} 2^\omega$), $\{\phi_i(\bar{x}_j) : i < N, j < n\}$ is H-independent but $\{\bar{x}_0, \phi_i(\bar{x}_0) : i < N\} \in H$.*

$\bigotimes_{\beta<\alpha} 2^{<\omega}$ is the set of finite partial functions $\alpha \to 2^{<\omega}$. $\bar{s} \in \bigotimes_{\beta<\alpha} 2^{<\omega}$ defines a basic open set $[\bar{s}]$ of $(2^\omega)^\alpha$.

Sketch of the limit case.

Assume the statement is true for all $\xi < \alpha$. We define a hypergraph H_ξ on $(2^\omega)^\xi$ for every $\xi < \alpha$, where $\{\bar{x}_0, \ldots, \bar{x}_{n-1}\} \in H_\xi \cap [(2^\omega)^\xi]^n$ iff $\exists p \in (\bigotimes_{\beta\in[\xi,\alpha)} 2^{<\omega})^n$ so that

$$p \Vdash \{\bar{x}_0{}^\frown \dot{c}_0, \ldots, \bar{x}_{n-1}{}^\frown \dot{c}_{n-1}\} \in H.$$

If 1. holds true for every H_ξ, as witnessed by M_ξ, then we find $M \ni M_\xi$ for every $\xi < \alpha$ and 1. holds true for H and M.

Key Lemma 3: infinite products

...

If 2. holds for some H_ξ, witnessed by M' and $\phi'_0, \dots, \phi'_{N-1}$, $\bar{s}'$, then we can assume wlog that there is a fixed p so that

$$p \Vdash \{\bar{x}^\frown \dot{c}_0, \phi'_0(\bar{x})^\frown \dot{c}_1, \dots \phi'_{N-1}(\bar{x})^\frown \dot{c}_N\} \in H.$$

Now we force continuous functions $\chi_i \colon (2^\omega)^\xi \to (2^\omega)^{[\xi,\alpha)} \cap [p(i+1)]$ for $i < N$ over M' and let $M = M'[\langle \chi_i : i < N \rangle]$. Finally:

$$\phi_i(\bar{x}) = \phi'(\bar{x})^\frown \chi_i(\phi'(\bar{x})), i < N$$

and

$$\bar{s} = \bar{s}'^\frown p(0).$$

□

Together with the lemma for finite products this lets us induct up to ω.

MCG for conditions

Now assume that the $\mathbb{Q}_\beta$ in the iteration $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta \leq \lambda \rangle$ are either $\mathbb{S}$ or $\mathbb{SP}$ (or any "Borel-"weighted tree forcing).

Lemma

Let $\alpha < \omega_1$, M be a ctm with $\alpha \in M$ and $\bar{q} \in \mathbb{P}_\lambda$ a good master condition, $\Phi_{\bar{q}} \colon [\bar{q}] \restriction A \to (2^\omega)^\alpha$ as before. Let $\bar{s} \in \bigotimes_{\beta<\alpha} 2^{<\omega}$. Then there is $\bar{r} \leq \bar{q}$ a good master condition so that any $\bar{x}_0, \dots, \bar{x}_{n-1} \in [\bar{r}] \restriction A$,

$$\Phi_{\bar{q}}(x_0), \dots, \Phi_{\bar{q}}(x_{n-1}) \in (2^\omega)^\alpha \cap [\bar{s}] \text{ are strongly mCg wrt } \prod_{\beta<\alpha} 2^\omega \text{ over } M.$$

Proof Idea.

We can assume without loss of generality that $[\bar{q}] \restriction A = (2^\omega)^\alpha$, via the map $\Phi_{\bar{q}}$, and imagine $\bar{q}$ to be the trivial condition in an iteration of length α of (slightly different) weighted tree forcings, let's call it $\langle \mathbb{R}_\beta, \dot{\mathbb{S}}_\beta : \beta \leq \alpha \rangle$.

We construct a closed set $C \subseteq (2^\omega)^\alpha \cap [\bar{s}]$ in a way that there is $\bar{r} \in \mathbb{R}_\alpha$ with $[\bar{r}] \subseteq C$. We recursively construct $C_\beta = C \restriction \beta \subseteq (2^\omega)^\beta \cap [\bar{s} \restriction \beta]$ for $\beta \leq \alpha$ "generically" over M in a finite support iteration.

MCG for conditions

...

Each C_β is a set of mCgs over M wrt $\prod_{\xi<\beta} 2^\omega$.

At each step β the iteration adds a continuous function $F\colon C_\beta \to \mathcal{T}$ (perfect subtrees of $2^{<\omega}$) over $M[C_\beta]$ so that $[F(\bar{x}_0)] \cap [F(\bar{x}_1)] = \emptyset$ and $\bigcup_{i<n}[F(\bar{x}_i)]$ consists of mCgs in 2^ω over $M[\bar{x}_0, \ldots, \bar{x}_{n-1}]$ for $\bar{x}_0, \ldots, \bar{x}_{n-1} \in C_\beta$ pairwise distinct.

Also, we ensure that $F(\bar{x}) \in \mathbb{S}_\beta$ for every $\bar{x} \in C_\beta$. Then

$$C_{\beta+1} := \{\bar{x}^\frown z : z \in [F(\bar{x})]\}.$$

$\square$

Main result

Whenever H is an analytic hypergraph on a Polish space X, $f\colon [\bar{q}] \restriction A \to X$ continuous, we can pull back H to $(2^\omega)^\alpha$ via f and $\Phi_{\bar{q}}$ and apply the lemmas to get the desirable property of $\mathbb{P}_\lambda$.

Altogether:

Theorem

After forcing with a csi of Sacks or splitting forcing over L, every analytic hypergraph in a Polish space has a $\boldsymbol{\Delta}^1_2$ maximal independent set.

Remark

- There is a universal analytic hypergraph on $2^\omega \times 2^\omega$, which is coded in the ground model. A maximal independent set then induces one for every analytic hypergraph.
- $|\mathbb{P}_\lambda| > \aleph_1$ and there are more than $\aleph_1$ many names for reals. But we can treat good master conditions and names as reals themselves (of which there are $\aleph_1$ many) through their representation as spaces $[\bar{q}] \restriction A$ and continuous functions $f\colon [\bar{q}] \restriction A \to X$.
- This is a key ingredient to make the construction Σ^1_2-definable.

Answering the questions

Corollary

It is consistent that there is a Π^1_1 mif, a Δ^1_2 ultrafilter and a Δ^1_2 Hamel basis while $\aleph_1 < \mathfrak{i}, \mathfrak{u}, \mathfrak{c}$. In particular, it is consistent that $\mathfrak{i}_B, \mathfrak{u}_B < \mathfrak{i}, \mathfrak{u}$.

Proof.

Force with $\mathbb{SP}$ in a ω_2-length countable support iteration. □

Corollary

The reaping number $\mathfrak{r}$ is never a (ZFC provable) lower bound of "Borelized cardinal invariants" (if they fit in the framework of analytic hypergraphs).

Corollary of the construction

There is a (Δ^1_2) P-point after iterating $\mathbb{SP}$ or $\mathbb{S}$ over L.

The key point is that the Borel sets $\langle B_\alpha : \alpha < \omega_1 \rangle$ that we construct can be chosen to be compact (due to $[\bar{q}]$ being compact). For an F_σ filter B there is a single compact set K so that $B \cup K$ generates a filter and K has a pseudointersection for every countable subset of B.

Concluding remarks

What about other tree forcings?

Theorem (Schrittesser, Törnquist 2018)

After adding a single Miller real over L every Σ^1_1 (2-dimensional hyper)graph on a Polish space has a Δ^1_2 maximal independent set.

A strengthening to the csi should not be too hard. Consider:

Theorem (Spinas 2001)

For every Miller tree T there is a master condition $S \leq T$ so that any $x_0 \neq x_1 \in [S]$ are $\mathbb{M}^2$ generic (over some countable model M).

On the other hand, Miller genericity behaves very different from Cohen genericity. Also, $\mathbb{M}^3$ adds a Cohen real, so finite products of $\mathbb{M}$ do not work.

Question

Does the main result (for hypergraphs) hold true for csi of Miller forcing?

Laver forcing and G_δ hypergraphs?

Thank you!