Definability of maximal families of reals in forcing extensions

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KGRC Seminar, 01.10.2020

generous support through FWF Project Y1012-N35
History and motivation

Many types of special sets of reals are central in fields such as set theory, topology, measure theory or algebra:

Well-orders, ultrafilters, mad families, Vitali sets ($E_0$-transversals), Hamel bases, maximal independent families (mif), maximal sets of orthogonal measures (mof), maximal Turing-independent families, maximal cofinitary groups (mcg), eventually different families (med), towers, scales (in $\omega^\omega$), ...

Their existence is guaranteed by the Axiom of Choice, which has the controversy of not giving explicit definitions.

Under certain circumstances though, these sets can be *nicely definable* (OD, OD($\mathbb{R}$), projective, $\Delta^1_2$, $\Sigma^1_2$, $\Pi^1_1$, Borel).
One of the earliest results in this direction is due to Gödel:

**Theorem (Gödel 1940)**

*There is a $\Delta^1_2$-definable well-order of the reals in the constructible universe $L$.***

The technique is very general and can be used to construct $\Delta^1_2$ witnesses in $L$ for all examples above. In some cases, this is the best possible:

**Fact**

*A Vitali set is a non-measurable set of reals. In particular, it cannot be $\Sigma^1_1 \cup \Pi^1_1$. E.g. the same holds true for ultrafilters. Also well-orders.*

**Theorem (Erdős, Kunen, Mauldin 1981)**

*There is a $\Pi^1_1$ scale in $L$.***

Their technique was streamlined by A. Miller who applied it to many other examples.

**Theorem (Miller 1989)**

*There is a $\Pi^1_1$ mad family, maximal independent family, Hamel basis in $L$.***
History and motivation

On the other hand, it was known that $\Sigma_1^1$ definitions typically do not work.

**Theorem (Mathias 1977)**

There is no $\Sigma_1^1$-definable mad family.

**Theorem (Miller 1989)**

There is no $\Sigma_1^1$-definable maximal independent family or Hamel basis.

Some mysterious exceptions:

**Theorem (Horowitz, Shelah 2016)**

There is a Borel mcg and med.
Recent results

In the last decade, research in the area has become very active (with a lot of emphasis on mad families). A few new phenomena have been discovered.

For example:

▶ If there is a $\Sigma^1_2(r)$ mad family, then there is also a $\Pi^1_1(r)$ one. (Törnquist 2013)
▶ If there is a $\Sigma^1_2(r)$ mif, then there is also a $\Pi^1_1(r)$ one. (Brendle, Fischer, Khomskii 2019)
▶ If there is a $\Sigma^1_2(r)$ ultrafilter, then there is a $\Pi^1_1(r)$ ultrafilter base. (S. 2019)

Also:

▶ If there is a $\Sigma^1_2(r)$ mad family, then $\omega_1 = \omega^L_1[r]$.
▶ If there is a $\Sigma^1_2(r)$ mif, then $\omega_1 = \omega^L_1[r]$.
▶ ...

In particular, we can not be too far away from $L$ for $\Sigma^1_2$ definability.
Recent results

So far we have only mentioned positive definability results in $L$. What happens in forcing extensions of $L$?

**Fact**

$(V=L)$ There is a $\Pi_1^1$-definable Cohen-, Sacks-, Random-, Miller-indestructible mad family.

Known techniques for indestructible mad families + making the construction $\Sigma_2^1$-definable + evaluates to the same set in the extension + $\Sigma_2^1 \rightarrow \Pi_1^1$

What about other forcing notions?

**Theorem (Brendle, Khomskii 2013)**

There is a $\Pi_1^1$ mad family in the Hechler extension of $L$.

Completely different technique: All mad families in $L$ are destroyed. Really the **definition** is preserved.
Borelized cardinal invariants

**Definition**

\[ a = \min\{|A| : A \text{ is a mad family}\} \]
\[ a_B = \min\{|B| : B \subseteq \Delta^1_1, \bigcup B \text{ is a mad family}\} \]

Obviously \( a_B \leq a \). Note that if there is a \( \Sigma^1_2 \) mad family, then \( a_B = \aleph_1 \) (since \( a_B > \aleph_0 \) since there is no Borel mad family).

Brendle and Khomskii in fact first showed

**Theorem**

\[ a_B = \aleph_1 \text{ in the Hechler model (so } \text{CON}(a_B < b = a)). \]

They construct a sequence \( \langle B_\alpha : \alpha < \omega_1 \rangle \) of Borel sets coded in \( L \), such that \( \models \bigcup_{\alpha<\omega_1} B_\alpha \) is a mad family. Then, using the standard techniques, this construction can be turned into a \( \Sigma^1_2 \)-definition.
Ultimate goal: Understand the definability of various types of families in forcing extensions of $L$.

Observation: Many of the examples we gave can be framed as maximal independent sets in hypergraphs.

**Definition**
A hypergraph on a set $X$ is a collection $E$ (the edges) of finite non-empty subsets of $X$, i.e. $E \subseteq \mathcal{P}(X)^{<\omega} \setminus \{\emptyset\}$. We say that $Y \subseteq X$ is $E$-independent if $[Y]^{<\omega} \cap E = \emptyset$. $Y$ is maximal $E$-independent if $Y$ is maximal under inclusion as an $E$-independent subset of $X$.

If $X$ is a Polish space, then $\mathcal{P}(X)^{<\omega}$ also has a natural Polish topology and we can study definable hypergraphs $E$ and definable maximal $E$-independent sets.

**Fact**
In $L$, every analytic hypergraph on a Polish space $X$ has a $\Delta^1_2$ maximal independent set.

Note: $\Delta^1_2 \iff \Sigma^1_2$
Example (MIF)

\( Y \subseteq \mathcal{P}(\omega) \) is an independent family if for all finite disjoint \( A, B \subseteq Y \),
\[ \bigcap_{x \in A} x \cap \bigcap_{x \in B} \omega \setminus x \text{ is infinite}. \]
Letting
\[ E_i := \{ A \cup B \in [\mathcal{P}(\omega)]^{<\omega} : \bigcap_{x \in A} x \cap \bigcap_{x \in B} \omega \setminus x \text{ is finite} \} \]

an independent family is an \( E_i \)-independent set.

The definability of maximal independent families has been recently studied by Brendle, Fischer and Khomskii. One of their main open questions was

**Question**

Is it consistent that \( i > \aleph_1 \), while there is a \( \Pi^1_1 \) maximal independent family? Is \( i_B < i \) consistent?

Can we destroy all ground model mif’s while preserving a \( \Pi^1_1 \) definition?
Example (Ultrafilter)

Let $E_u := \{ A \in [\mathcal{P}(\omega)]^{<\omega} : \cap A \text{ is finite} \}$. Then an ultrafilter is a maximal $E_u$-independent set.

In a recent paper, we studied the definability of ultrafilters and asked

**Question**

Is it consistent that $u > \aleph_1$, while there is a $\Delta^1_2$ ultrafilter? Is $u_B < u$ consistent?

Can we destroy all ground model ultrafilters (ultrafilter bases) while preserving a $\Delta^1_2$ definition?
Examples

Example (Hamel basis)

Let $E_h := \{ A \in [\mathbb{R}]^{<\omega} : A \text{ is linearly dependent over } \mathbb{Q} \}$. Then a Hamel basis is a maximal $E_h$-independent set.

Every Hamel basis has size $2^{\aleph_0}$. This is reflected by the fact that adding a single real destroys every ground model Hamel basis.

Question

Is it consistent that $\neg \text{CH}$, while there is a $\Delta^1_2$ Hamel basis?

Can we destroy all ground model Hamel bases (i.e. add a new real) while preserving a $\Delta^1_2$ definition?

For mad families, Vitali sets or mof’s, 2-dimensional hypergraphs (i.e. usual graphs) suffice.

Theorem (Schrittesser 2016)

After forcing with a csi of Sacks forcing over $L$, every analytic (2-dimensional hyper)graph on a Polish space has a $\Delta^1_2$ maximal independent set.

Adding a single real, destroys every ground model maximal orthogonal family of measures.
How to increase $u$ and $i$? How to destroy ultrafilters and maximal independent families (and, well, Hamel bases)? Add splitting reals!

**Definition**

A real $x \in [\omega]^{\omega}$ is splitting over $V$ if for every $y \in [\omega]^{\omega} \cap V$, $|x \cap y| = \omega$ and $|y \setminus x| = \omega$.

Classical forcing notions adding splitting reals are: Cohen, Random, Silver and forcings adding dominating reals.

Unfortunately they don’t work:

**Theorem (S.)**

*In extensions via the posets above there are reals that are splitting over any $\Sigma_2^1(r)$ set with the finite intersection property, for any $r \in V$. (for Cohen, Random, Silver: any OD($V$) set)*

(For a definable independent family, there are countably many (similarly) definable families with the FIP so that any splitting real over all of them witnesses the non-maximality of it.)
Splitting forcing

There is another less known forcing adding splitting reals.

**Definition**
A set $A \subseteq 2^{<\omega}$ is called *fat* if there is $m = m(A) \in \omega$ so that for every $n \geq m$, $i \in 2$, there is $s \in A$ so that $s(n) = i$.

Let $T \subseteq 2^{<\omega}$ be a perfect tree. Then $T$ is a *splitting tree* if for every $s \in T$, $T_s$ is fat.

(Recall: $T_s = \{ t \in T : t \not\perp s \}$)

**Splitting forcing** $\mathbb{S} \mathbb{P}$ consists of all splitting trees ordered by inclusion ($T \subseteq S$), as usual.

**Fact**
- $\mathbb{S} \mathbb{P}$ adds a generic splitting real $x_G \in 2^\omega (\cong \mathcal{P}(\omega))$,
- $\mathbb{S} \mathbb{P}$ is proper (Axiom A),
- $\mathbb{S} \mathbb{P}$ has continuous reading of names: whenever $\dot{y}$ is a name for an element of a Polish space $X$ (coded in the ground model), $S \in \mathbb{S} \mathbb{P}$, there is $T \subseteq S$ and $f : [T] \to X$ continuous such that $T \models \dot{y} = f(x_G)$,
- $V^{\mathbb{S} \mathbb{P}}$ is a minimal extension of $V$.

Recall: Sacks forcing $\mathbb{S}$ consists of all perfect subtrees of $2^{<\omega}$.  

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How to preserve?

Does splitting forcing work? Can we maybe treat ultrafilters and maximal independent families in the same way? What about Hamel bases?

Maybe we should first ask a more general question: What does it mean for a forcing $\mathbb{P}$ to preserve a union of Borel sets $Y = \bigcup B \subseteq X$ maximal $E$-independent?

Let $\dot{y}$ be a $\mathbb{P}$-name for an element of $X$. Potentially $\dot{y}$ could be a threat to the maximality of (the reinterpretation of) $Y$. Say $B$ is closed under finite unions.

Then, necessarily, for a dense set of $q \in \mathbb{P}$, there is $B \in \mathcal{B}$ so that

1. either $q \Vdash \dot{y} \in B$,
2. or $q \Vdash \{\dot{y}\} \cup B$ is not $E$-independent.

On the other hand the following is sufficient:

For every name $\dot{y}$, every analytic hypergraph $H$ and $p \in \mathbb{P}$, there is $q \leq p$ and an $H$-independent Borel set $B$ such that

1. either $q \Vdash \dot{y} \in B$,
2. or $q \Vdash \{\dot{y}\} \cup B$ is not $H$-independent.
How to construct?

Why?

Let $\langle \dot{y}_\alpha, p_\alpha : \alpha < \omega_1 \rangle$ enumerate all pairs of (nice) $\mathbb{P}$-names for elements of $X$ ($|\mathbb{P}| = \aleph_1$ for now, $\mathbb{P}$ proper) and conditions in $\mathbb{P}$. We recursively construct Borel sets $\langle B_\alpha : \alpha < \omega_1 \rangle$.

At stage $\alpha$ : Let $H_\alpha$ be the hypergraph on $X$ where $\{x_0, \ldots, x_{n-1}\} \in H_\alpha$ iff $\{x_0, \ldots, x_{n-1}\} \cup \bigcup_{i < \alpha} B_i$ is not $E$-independent. Then there is $q \leq p_\alpha$ and an $H_\alpha$-independent Borel set $B$ so that

1. either $q \models \dot{y}_\alpha \in B$,
2. or $q \models \{\dot{y}_\alpha\} \cup B$ is not $H$-independent.

Translated this means that $B_\alpha = B \cup \bigcup_{i < \alpha} B_i$ is $E$-independent and

1. either $q \models \dot{y}_\alpha \in B_\alpha$,
2. or $q \models \{\dot{y}_\alpha\} \cup B_\alpha$ is not $E$-independent.

Finally let $Y = \bigcup_{\alpha < \omega_1} B_\alpha$. By genericity, we have taken care of every potential threat $\dot{y}$. 

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Combinatorial reformulation

Remember the desirable property:

For every name \( \dot{y} \), every analytic hypergraph \( H \) and \( p \in \mathbb{P} \), there is \( q \leq p \) and an \( H \)-independent Borel set \( B \) such that

1. either \( q \Vdash \dot{y} \in B \),
2. or \( q \Vdash \{ \dot{y} \} \cup B \) is not \( H \)-independent.

If \( \mathbb{P} \) is a tree forcing (say subtrees of \( 2^{<\omega} \)) with continuous reading of names, we can forget about names and pull everything back to conditions \( T \in \mathbb{P} \):

For every analytic hypergraph \( H \) on \( 2^\omega \) and \( T \in \mathbb{P} \), there is \( S \leq T \) such that

1. either \( [S] \) is \( H \)-independent,
2. or there are continuous functions \( \phi_0, \ldots, \phi_{N-1} : [S] \to 2^\omega \) so that

\[
\bigcup_{i < N} \phi_i''[S] \text{ is } H\text{-independent, but } \forall x \in [S] (\{x\} \cup \{\phi_0(x), \ldots, \phi_{n-1}(x)\} \in H).
\]

This is a purely combinatorial statement about trees in \( \mathbb{P} \).
Mutual Cohen genericity

The key idea is going to be mutual genericity.

**Definition**
Let $M$ be a countable transitive model of set theory (ctm), $X \in M$ a (code for a) Polish space. Then $x \in X$ is called *Cohen generic in $X$ over $M$* if for every open dense subset $O \in M$ (coded in $M$) of $X$, $x \in O$. $x_0, \ldots, x_{n-1} \in X$ are *mutually Cohen generic (mCg) in $X$ over $M$* if $(y_0, \ldots, y_{n'-1})$ is generic in $X^{n'}$ over $M$, where $y_0, \ldots, y_{n'-1}$ enumerate $x_0, \ldots, x_{n-1}$.

**Lemma**
Let $M$ be a ctm, $T \in M$ a perfect subtree of $2^{<\omega}$, i.e. $T \in S$.

- There is $S \leq T$, a perfect tree (i.e. $S \in S$), so that any $x_0, \ldots, x_{n-1} \in [S]$ are mCg in $[T]$ over $M$.
- If $T \in SP$, there is $S \leq T$, $S \in SP$, so that any $x_0, \ldots, x_{n-1} \in [S]$ are mCg in $[T]$ over $M$.
- In fact, if $\mathbb{P}$ is any weighted tree forcing, and $T \in \mathbb{P}$ then there is $S \leq T$, $S \in \mathbb{P}$, so that any $x_0, \ldots, x_{n-1} \in [S]$ are mCg in $[T]$ over $M$.

**Definition**
A tree forcing $\mathbb{P}$ is weighted, if ...something technical...

$S$ and $SP$ are examples + generalizations.
Key Lemma 1

Key Lemma

Let $H$ be an analytic hypergraph on $X$. Then there is a ctm $M$ so that

1. either, for any $x_0, \ldots, x_{n-1} \in X$ that are mCg over $M$, $\{x_0, \ldots, x_{n-1}\}$ is $H$-independent,

2. or, there are $c_0, \ldots, c_{N-1} \in X$ and a non-empty open set $O \subseteq X$, so that for any $x \in O$ Cohen generic over $M$, $\{c_0, \ldots, c_{N-1}\}$ is $H$-independent but $\{c_0, \ldots, c_{N-1}\} \cup \{x\} \in H$.

Proof.

Let $M_0$ be the transitive collapse of a countable elementary model containing $H$ and $X$. Suppose 1. fails for $M = M_0$. Then there is a counter-example of minimal size: $c_0, \ldots, c_{N-1}, c_N$ mCg over $M$, $\{c_0, \ldots, c_{N-1}\}$ $H$-independent, but $\{c_0, \ldots, c_{N-1}, c_N\} \in H$. Consider $M = M_0[c_0, \ldots, c_{N-1}]$. As $c_N$ is generic over $M$, there is an open set (a condition) $O \in M$ with $c_N \in O$ and

$$O \models \{\dot{c}\} \cup \{c_0, \ldots, c_{N-1}\} \in H.$$ 

Thus for any generic $x \in O$, $M[x] \models \{x, c_0, \ldots, c_{N-1}\} \in H$. By $\Sigma^1_1$-absoluteness indeed, $\{x, c_0, \ldots, c_{N-1}\} \in H$. □
Key Lemma 1

Key Lemma

Let $H$ be an analytic hypergraph on $X$. Then there is a ctm $M$ so that

1. either, for any $x_0,\ldots,x_{n-1} \in X$ that are mCg over $M$, $\{x_0,\ldots,x_{n-1}\}$ is $H$-independent,

2. or, there are $c_0,\ldots,c_{N-1} \in X$ and a non-empty open set $O \subseteq X$, so that for any $x \in O$ Cohen generic over $M$, $\{c_0,\ldots,c_{N-1}\}$ is $H$-independent but $\{c_0,\ldots,c_{N-1}\} \cup \{x\} \in H$.

Putting things together:

$H$ a hypergraph on $[T]$, $T$ a condition. Applying the key lemma, get the model $M$:

1. Let $S \leq T$ be as in the first lemma: all $x_0,\ldots,x_{n-1} \in [S]$ are mCg in $[T]$ over $M$ $\rightarrow \{x_0,\ldots,x_{n-1}\} \notin H \rightarrow [S]$ is $H$-independent

2. Define $\phi_0,\ldots,\phi_{N-1}$ constant, $\phi_i(x) = c_i$. Let $s \in T$, $[s] \subseteq O$ and apply the first lemma to get $S \leq T_s$: every $x \in [S]$ is Cohen generic in $[T] \cap O$ over $M \rightarrow$

$$\bigcup_{i < N} \phi_i''[S] \text{ is } H\text{-independent, but } \forall x \in [S](\{x\} \cup \{\phi_0(x),\ldots,\phi_{n-1}(x)\} \in H).$$
Partial answer/result

**Theorem**

(V=L) For any $\Sigma^1_1(r)$ hypergraph $E$, there is a (ground model coded) $\Delta^1_2(r)$ maximal $E$-independent set after adding a single Sacks or a single splitting real (via $\mathbb{SP}$).

We can preserve the definition of an ultrafilter/mif/Hamel basis while destroying all ultrafilters/mifs/Hamel bases.

More generally: Any proper weighted tree forcing with continuous reading of names.

This is a good first step. But this is far from a model where $u$, $i$ or $c$ is greater than $\aleph_1$.

What about adding more than one real? $\mathbb{SP}^k$, $\mathbb{S}^k$ for $k \in \omega$?

- Conditions are easy to work with: $(T_0, \ldots, T_{k-1})$.
- We have a natural analogue of continuous reading of names: $f : \prod_{i<k}[T_i] \to X$, $(T_0, \ldots, T_{k-1}) \models f(\bar{x}_G) = \check{y}$.
- Maybe a similar argument works? The combinatorial reformulation is straightforward: For every hypergraph $H$ on $(2^\omega)^k$, $\bar{T} \in \mathbb{P}^k$, there is $\bar{S} \leq \bar{T}$ so that either, $[\bar{S}] = \prod_{i<k}[S_i]$ is $H$-independent or, there are $\phi_0, \ldots, \phi_{N-1}$ continuous, $\bigcup_{i<N} \phi_i''[\bar{S}]$ is $H$-independent, $\{\bar{x}, \phi_i(\bar{x}) : i < N\} \in H$ for $\bar{x} \in [\bar{S}]$. 
Definition

Let $M$ be a ctm, $\langle X_l : l < k \rangle \in M$ be (codes for) Polish spaces. Then we say that $\bar{x}_0, \ldots, \bar{x}_{n-1} \in \prod_{l<k} X_l$ are *mutually Cohen generic (mCg)* with respect to the product $\prod_{l<k} X_l$ over $M$, if

$$(y^0_0, \ldots, y^{K_0-1}_0, \ldots, y^0_{k-1}, \ldots, y^{K_{k-1}-1}_{k-1})$$ is Cohen generic in $\prod_{l<k} X^K_l$ over $M$, where $\langle y^i_l : i < K_l \rangle$ is some, equivalently any, enumeration of $\{x_i(l) : i < n\}$ for each $l < k$.
Key Lemma 2: finite products

Lemma
Let $M$ be a ctm, $T_0, \ldots, T_{k-1} \in M \cap \mathbb{P}$, where $\mathbb{P}$ is a weighted tree forcing (e.g. $\mathbb{S}$ or $\mathbb{SP}$). Then there are $S_0 \leq T_0, \ldots, S_{k-1} \leq T_{k-1}$ so that any $\bar{x}_0, \ldots, \bar{x}_{n-1} \in \prod_{i<k} [S_i]$ are $mCg$ wrt $\prod_{i<k} [T_i]$ over $M$.

Key Lemma
Let $H$ be an analytic hypergraph on $(2^\omega)^k$. Then there is a ctm $M$ so that

1. either, for any $\bar{x}_0, \ldots, \bar{x}_{n-1} \in (2^\omega)^k$ that are $mCg$ over $M$, $\{\bar{x}_0, \ldots, \bar{x}_{n-1}\}$ is $H$-independent,

2. or, there are $\phi_0, \ldots, \phi_{N-1} : (2^\omega)^k \to (2^\omega)^k$ continuous, $\bar{s} \in (2^{<\omega})^k$, so that for any $mCg$ $\bar{x}_0, \ldots, \bar{x}_{n-1} \in [\bar{s}]$ over $M$, $\{\phi_i(\bar{x}_j) : i < N, j < n\}$ is $H$-independent but $\{\bar{x}_0, \phi_i(\bar{x}_0) : i < N\} \in H$.

Proof.
Much more complicated than before. Uses ideas from Harrington's forcing proof of Halpern-Läuchli.
Key Lemma 2: finite products

Example
Let $k = 2$, $H \subseteq [2^\omega \times 2^\omega]^2$ where $\{\bar{x}_0 \neq \bar{x}_1\} \in H$ iff $x_0(0) = x_1(0)$.

Case 1 is impossible. So we are in case 2: Let $c \in 2^\omega$ be arbitrary, $c \in M$ a ctm and let $\phi(\bar{x}) = (x(0), c)$. Then $\{\bar{x}, \phi(\bar{x})\} \in H$ for every $\bar{x}$ with $x(1) \neq c$ (e.g. $\bar{x}$ generic over $M$). On the other hand, if $\phi(\bar{x}_0) \neq \phi(\bar{x}_1)$, then $\phi(\bar{x}_0)(0) \neq \phi(\bar{x}_1)(0)$ so $\{\phi(\bar{x}_0), \phi(\bar{x}_1)\} \notin H$.

Example
Let $k = 2$, $H \subseteq [2^\omega \times 2^\omega]^2$ where $\{\bar{x}_0 \neq \bar{x}_1\} \in H$ iff $x_0(0) = x_1(0)$ or $x_0(1) = x_1(1)$.

Again, case 1 is impossible. Instead of a constant $c \in 2^\omega$, let $f : 2^\omega \to 1 \cup 2^\omega$ be a continuous injection, $f \in M$ a ctm and let $\phi(\bar{x}) = (x(0), f(x(0))), \bar{s} = (\emptyset, \langle 0 \rangle)$. Then $\{\bar{x}, \phi(\bar{x})\} \in H$ for every $\bar{x} \in [\bar{s}]$. If $\phi(\bar{x}_0) \neq \phi(\bar{x}_1)$, then $x_0(0) \neq x_1(0)$ and then $\phi(\bar{x}_0)$ and $\phi(\bar{x}_1)$ are different in both coordinates, so $\{\phi(\bar{x}_0), \phi(\bar{x}_1)\} \notin H$. 
Partial answer/result 2

**Theorem**

(V=L) For $\Sigma_1^1(r)$ hypergraph $E$, there is a $\Delta_2^1(r)$ maximal $E$-independent set after forcing with $S^k$ or $\mathbb{S}P^k$, $k \in \omega$.

More generally: any finite product of proper weighted tree forcings with crn, e.g. $S^{k_0} \times \mathbb{S}P^{k_1}$.

Great! We only need to generalize to infinite products. The csp of $S$, $\mathbb{S}P$ is proper and has continuous reading of names.

**Counterexample**

Consider $E_1$ on $(2^\omega)^\omega$ where $\{\bar{x}_0 \neq \bar{x}_1\} \in E_1$ iff $\forall^\infty n \in \omega (x_0(n) = x_1(n))$.

Let $(S_i)_{i \in \omega}$ be perfect trees (a condition in $S^\omega$ or $\mathbb{S}P^\omega$). Then $\prod_{i \in \omega} [S_i]$ is never $E_1$-independent (i.e. a partial transversal for $E_1$). On the other hand, any continuous

$\phi : \prod_{i \in \omega} [S_i] \to \prod_{i \in \omega} [S_i]$ so that

$$\{ \phi(\bar{x}), \bar{x} \} \in E_1 \text{ for every } \bar{x} \in \prod_{i \in \omega} [S_i] \text{ and } \phi'' \prod_{i \in \omega} [S_i] \text{ is } E_1\text{-independent},$$

is a continuous selector for $E_1 \upharpoonright \prod_{i \in \omega} [S_i] \cong_B E_1$. 
The iteration

Corollary

In an extension by $\mathbb{S}^\omega$, there is no $\Delta^1_2$-definable $E_1$-transversal. For $\mathbb{S}_\omega$, this follows by a simpler homogeneity argument and holds for all sets definable over the ground model.

We could ask:

Question

Can we characterize hypergraphs for which countable support products of, say $\mathbb{S}$, work? For which hypergraphs does the combinatorial reformulation hold true?

Iterations on the other hand seem promising, since conditions are “smaller” than in products. For instance, the argument for $E_1$ fails:

Fact

For any $\vec{p} \in \mathbb{S}^*\omega$, there is $\vec{q} \leq \vec{p}$ so that for any $\mathbb{S}^*\omega$-generics $\vec{x}_0 \neq \vec{x}_1$ with $\vec{q}$ in the corresponding generic filter, $x_0(n) \neq x_1(n)$ for all $n \geq \min\{m : x_0(m) \neq x_1(m)\}$.

Conditions in iterations are harder to work with though. Also what does continuous reading of names mean now?
Good master conditions

Let $\langle P_\beta, Q_\beta : \beta \leq \lambda \rangle$ be a countable support iteration, where for each $\beta < \lambda$, $Q_\beta$ is a tree forcing, $Q_\beta$ is an analytic subset of a Polish space and there is a sequence $\langle \leq_\beta, n : n \in \omega \rangle$ of analytic partial orders on $Q_\beta$ witnessing the Axiom A with continuous reading of names.

Assume each $Q_\beta$ consists of subtrees of $2^{<\omega}$.

Lemma

For any $\bar{p} \in P_\lambda$, $M$ a countable elementary model with $P_\lambda, \bar{p} \in M$, there is $\bar{q} \leq \bar{p}$ a master condition over $M$ together with a unique closed set $[\bar{q}] \subseteq (2^\omega)^\lambda$ so that

1. $\bar{q} \Vdash \bar{x}_G \in [\bar{q}]$,
   for every $\beta < \lambda$,

2. $\bar{q} \Vdash \bar{q}(\beta) = \{ s \in 2^{<\omega} : \exists \bar{z} \in [\bar{q}] (\bar{z} \upharpoonright \beta = \bar{x}_G \upharpoonright \beta \land s \subseteq z(\beta)) \}$,

3. the map sending $\bar{x} \in [\bar{q}] \upharpoonright \beta$ to $\{ s \in 2^{<\omega} : \exists \bar{z} \in [\bar{q}] (\bar{z} \upharpoonright \beta = \bar{x} \land s \subseteq z(\beta)) \}$ is continuous and maps to $Q_\beta$,

and for every name $\dot{y} \in M$ for an element of a Polish space $X$,

4. there is a continuous function $f : [\bar{q}] \to X$ so that $\bar{q} \Vdash \dot{y} = f(\bar{x}_G)$.

Moreover, there is a countable set $A \subseteq \lambda$ so that $[\bar{q}] = (2^\omega)^{\lambda \setminus A} \times [\bar{q}] \upharpoonright A$ and all continuous functions above are supported on $A$.

$\bar{q}$ is called a good master condition over $M$. 

Definability of maximal families of reals in forcing extensions 

Institute of Mathematics, University of Vienna
Good master conditions

On the other hand: whenever $A$ is countable, $C \subseteq (2^\omega)^A$ is a closed set where for each $\beta \in A$ and $\bar{x} \in C \upharpoonright \beta$:

$$\{ s \in 2^{<\omega} : \exists \bar{z} \in C(\bar{z} \upharpoonright \beta = \bar{x} \land s \subseteq z(\beta)) \} \in \mathbb{Q}_\beta,$$

then there is a good master condition $\bar{q} \in \mathbb{P}_\lambda$ such that $[\bar{q}] \upharpoonright A \subseteq C$.

Remember that for any perfect tree $T \subseteq 2^{<\omega}$, there is a canonical homeomorphism $\eta_T : [T] \to 2^\omega$. If $\bar{q}$ is a good master condition and $A \subseteq \lambda$ as before, we can use this to define a canonical homeomorphism

$$\Phi_{\bar{q}} : [\bar{q}] \upharpoonright A \to (2^\omega)^\alpha,$$

where $\alpha = \text{otp}(A)$, witnessed by $\iota : A \to \alpha$, and for each $\beta \in A$, $\bar{x} \in [\bar{q}] \upharpoonright A$,

$$\Phi_{\bar{q}}(\bar{x})(\iota(\beta)) = \eta_T(x(\beta)),$$

with $T = \{ s \in 2^{<\omega} : \exists \bar{z} \in [\bar{q}] \upharpoonright A(\bar{z} \upharpoonright \beta = \bar{x} \upharpoonright \beta \land s \subseteq z(\beta)) \}$. 
Mutual Cohen genericity revisited again

This time we have an infinite product \((2^\omega)^\alpha\).

**Definition**

Let \(\alpha < \omega_1\), \(M\) a ctm with \(\alpha \in M\). Then we say that \(\bar{x}_0, \ldots, \bar{x}_{n-1}\) are mCg with respect to the product \(\prod_{\beta < \alpha} 2^\omega\) over \(M\), if there is a partition \(\xi_0 = 0 < \cdots < \xi_k = \alpha, \ k \in \omega\), so that

\[
\bar{x}_0, \ldots, \bar{x}_{n-1} \text{ are mCg with respect to } \prod_{l < k} Y_l \text{ over } M,
\]

where \(Y_l = (2^\omega)[\xi_l, \xi_{l+1}),\ l < k\).

![Diagram showing the partition and products](image-url)
Mutual Cohen genericity revisited again

**Definition**

Let $\alpha < \omega_1$, $M$ a ctm with $\alpha \in M$. Then we say that $\bar{x}_0, \ldots, \bar{x}_{n-1}$ are **strongly** mCg with respect to the product $\prod_{\beta < \alpha} 2^\omega$ over $M$, if they are mCg (as before) and for any $i, j < n$ if $\xi = \min \{ \beta : x_i(\beta) \neq x_j(\beta) \}$, then for all $\beta \geq \xi$, $x_i(\beta) \neq x_j(\beta)$. 

![Graph](image.png)
Key Lemma 3: infinite products

Key Lemma

Let $\alpha < \omega_1$ and $H$ an analytic hypergraph on $(2^\omega)^\alpha$. Then there is a ctm $M$, $\alpha \in M$, so that

1. either, for any $\bar{x}_0, \ldots, \bar{x}_{n-1} \in (2^\omega)^\alpha$ that are strongly mCg over $M$ (wrt $\prod_{\beta < \alpha} 2^\omega$), $\{\bar{x}_0, \ldots, \bar{x}_{n-1}\}$ is $H$-independent,

2. or, there are $\phi_0, \ldots, \phi_{N-1} : (2^\omega)^\alpha \to (2^\omega)^\alpha$ continuous, $\bar{s} \in \bigotimes_{\beta < \alpha} 2^{<\omega}$, so that for any strongly mCg $\bar{x}_0, \ldots, \bar{x}_{n-1} \in [\bar{s}]$ over $M$ (wrt $\prod_{\beta < \alpha} 2^\omega$), $\{\phi_i(\bar{x}_j) : i < N, j < n\}$ is $H$-independent but $\{\bar{x}_0, \phi_i(\bar{x}_0) : i < N\} \in H$.

$\bigotimes_{\beta < \alpha} 2^{<\omega}$ is the set of finite partial functions $\alpha \to 2^{<\omega}$. $\bar{s} \in \bigotimes_{\beta < \alpha} 2^{<\omega}$ defines a basic open set $[\bar{s}]$ of $(2^\omega)^\alpha$.

Sketch of the limit case.

Assume the statement is true for all $\xi < \alpha$. We define a hypergraph $H_\xi$ on $(2^\omega)^\xi$ for every $\xi < \alpha$, where $\{\bar{x}_0, \ldots, \bar{x}_{n-1}\} \in H_\xi \cap [(2^\omega)^\xi]^n$ iff $\exists p \in (\bigotimes_{\beta \in [\xi, \alpha]} 2^{<\omega})^n$ so that

$$p \models \{\bar{x}_0 \bowtie \dot{c}_0, \ldots, \bar{x}_{n-1} \bowtie \dot{c}_{n-1}\} \in H.$$

If 1. holds true for every $H_\xi$, as witnessed by $M_\xi$, then we find $M \supseteq M_\xi$ for every $\xi < \alpha$ and 1. holds true for $H$ and $M$.
Key Lemma 3: infinite products

... 

If 2. holds for some $H_\xi$, witnessed by $M'$ and $\phi'_0, \ldots, \phi'_{N-1}, \bar{s}'$, then we can assume wlog that there is a fixed $p$ so that

\[ p \vdash \{ \bar{x} \sim \dot{c}_0, \phi'_0(\bar{x}) \sim \dot{c}_1, \ldots \phi'_{N-1}(\bar{x}) \sim \dot{c}_N \} \in H. \]

Now we force continuous functions $\chi_i : (2^\omega)^\xi \to (2^\omega)^{[\xi, \alpha)} \cap [p(i + 1)]$ for $i < N$ over $M'$ and let $M = M'[\langle \chi_i : i < N \rangle]$. Finally:

\[ \phi_i(\bar{x}) = \phi'(\bar{x}) \sim \chi_i(\phi'(\bar{x})), i < N \]

and

\[ \bar{s} = \bar{s}' \sim p(0). \]

Together with the lemma for finite products this lets us induct up to $\omega$. 

\[ \square \]
MCG for conditions

Now assume that the $Q_\beta$ in the iteration $\langle P_\beta, \dot{Q}_\beta : \beta \leq \lambda \rangle$ are either $S$ or $\mathcal{SP}$ (or any “Borel-” weighted tree forcing).

Lemma

Let $\alpha < \omega_1$, $M$ be a ctm with $\alpha \in M$ and $\bar{q} \in P_\lambda$ a good master condition, $\Phi_{\bar{q}} : [\bar{q}] \upharpoonright A \rightarrow (2^\omega)^\alpha$ as before. Let $\bar{s} \in \bigotimes_{\beta < \alpha} 2^{<\omega}$. Then there is $\bar{r} \leq \bar{q}$ a good master condition so that any $\bar{x}_0, \ldots, \bar{x}_{n-1} \in [\bar{r}] \upharpoonright A$,

$$\Phi_{\bar{q}}(\bar{x}_0), \ldots, \Phi_{\bar{q}}(\bar{x}_{n-1}) \in (2^\omega)^\alpha \cap [\bar{s}]$$

are strongly mCg wrt $\prod_{\beta < \alpha} 2^\omega$ over $M$.

Proof Idea.

We can assume without loss of generality that $[\bar{q}] \upharpoonright A = (2^\omega)^\alpha$, via the map $\Phi_{\bar{q}}$, and imagine $\bar{q}$ to be the trivial condition in an iteration of length $\alpha$ of (slightly different) weighted tree forcings, let’s call it $\langle R_\beta, \dot{S}_\beta : \beta \leq \alpha \rangle$.

We construct a closed set $C \subseteq (2^\omega)^\alpha \cap [\bar{s}]$ in a way that there is $\bar{r} \in R_\alpha$ with $[\bar{r}] \subseteq C$. We recursively construct $C_\beta = C \upharpoonright \beta \subseteq (2^\omega)^\beta \cap [\bar{s} \upharpoonright \beta]$ for $\beta \leq \alpha$ “generically” over $M$ in a finite support iteration.
Each $C_\beta$ is a set of mCgs over $M$ wrt $\prod_{\xi<\beta} 2^\omega$.

At each step $\beta$ the iteration adds a continuous function $F : C_\beta \to T$ (perfect subtrees of $2^{<\omega}$) over $M[C_\beta]$ so that $[F(\bar{x}_0)] \cap [F(\bar{x}_1)] = \emptyset$ and $\bigcup_{i<n} [F(\bar{x}_i)]$ consists of mCgs in $2^\omega$ over $M[\bar{x}_0, \ldots, \bar{x}_{n-1}]$ for $\bar{x}_0, \ldots, \bar{x}_{n-1} \in C_\beta$ pairwise distinct.

Also, we ensure that $F(\bar{x}) \in S_\beta$ for every $\bar{x} \in C_\beta$. Then
\[
C_{\beta+1} := \{ \bar{x} \mapsto z : z \in [F(\bar{x})] \}.
\]
Main result

Whenever $H$ is an analytic hypergraph on a Polish space $X$, $f : [\bar{q}] \upharpoonright A \to X$ continuous, we can pull back $H$ to $(2^\omega)^\alpha$ via $f$ and $\Phi_{\bar{q}}$ and apply the lemmas to get the desirable property of $\mathcal{P}_\lambda$.

Altogether:

**Theorem**

*After forcing with a csi of Sacks or splitting forcing over $L$, every analytic hypergraph in a Polish space has a $\Delta^1_2$ maximal independent set.*

**Remark**

▶ There is a universal analytic hypergraph on $2^\omega \times 2^\omega$, which is coded in the ground model. A maximal independent set then induces one for every analytic hypergraph.

▶ $|\mathbb{P}_\lambda| > \aleph_1$ and there are more than $\aleph_1$ many names for reals. But we can treat good master conditions and names as reals themselves (of which there are $\aleph_1$ many) through their representation as spaces $[\bar{q}] \upharpoonright A$ and continuous functions $f : [\bar{q}] \upharpoonright A \to X$.

▶ This is a key ingredient to make the construction $\Sigma^1_2$-definable.
Answering the questions

Corollary

It is consistent that there is a \( \Pi^1_1 \) mif, a \( \Delta^1_2 \) ultrafilter and a \( \Delta^1_2 \) Hamel basis while \( \aleph_1 < i, u, c \). In particular, it is consistent that \( i_B, u_B < i, u \).

Proof.

Force with \( \mathcal{SP} \) in a \( \omega_2 \)-length countable support iteration. \( \square \)

Corollary

The reaping number \( r \) is never a (ZFC provable) lower bound of "Borelized cardinal invariants" (if they fit in the framework of analytic hypergraphs).

Corollary of the construction

There is a \( (\Delta^1_2) \) P-point after iterating \( \mathcal{SP} \) or \( \mathbb{S} \) over \( L \).

The key point is that the Borel sets \( \langle B_\alpha : \alpha < \omega_1 \rangle \) that we construct can be chosen to be compact (due to \([\bar{q}]\) being compact). For an \( F_\sigma \) filter \( B \) there is a single compact set \( K \) so that \( B \cup K \) generates a filter and \( K \) has a pseudointersection for every countable subset of \( B \).
Concluding remarks

What about other tree forcings?

**Theorem (Schrittesser, Törnquist 2018)**

*After adding a single Miller real over $L$ every $\Sigma^1_1$ (2-dimensional hyper)graph on a Polish space has a $\Delta^1_2$ maximal independent set.*

A strengthening to the csi should not be too hard. Consider:

**Theorem (Spinas 2001)**

*For every Miller tree $T$ there is a master condition $S \leq T$ so that any $x_0 \neq x_1 \in [S]$ are $\mathbb{M}^2$ generic (over some countable model $M$).*

On the other hand, Miller genericity behaves very different from Cohen genericity. Also, $\mathbb{M}^3$ adds a Cohen real, so finite products of $\mathbb{M}$ do not work.

**Question**

Does the main result (for hypergraphs) hold true for csi of Miller forcing? Laver forcing and $G_\delta$ hypergraphs?
Thank you!