Ramsey-like Operators

Peter Holy

University of Udine

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Large Cardinals

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- U is *κ-amenable* if whenever X is a set of size κ in M, then $X \cap U \in M$.

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Why, for example, we should care about large cardinal ideals:

Two results of Baumgartner

Subtlety

First, I need to introduce even more notions.

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κ is ineffable if the subtle and the Π^1_2-indescribable ideal on κ generate a nontrivial ideal. This then is the ineffable ideal on κ.

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Large Cardinal Operators

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Some basic properties of large cardinal operators:

- $\forall I\ \mathfrak{O}(I) \supseteq I$,
- $\forall I, J\ [I \subseteq J\ \rightarrow\ \mathfrak{O}(I) \subseteq \mathfrak{O}(J)]$.

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We will often define certain ideals I by actually defining the collection of I-positive sets in the following.

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Given an ideal I on κ, let $\mathcal{R}(I)^+$ be the set of all $A \subseteq \kappa$ such that any regressive function $f\colon [\kappa]^{<\omega} \to \kappa$ has a homogeneous set $H \subseteq A$ in I^+.

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Let $\mathcal{R}_M(I)^+$ be the set of all $A \subseteq \kappa$ such that for any $y \subseteq \kappa$, there is a weak κ-model $M \ni y$, and a κ-amenable M-normal M-ultrafilter U on κ with $A \in U$, such that any countable intersection of elements of U is in I^+.

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Theorem (Sharpe, Welch (2011))

For any ideal I on κ,

$$\mathcal{R}_M(I) = \mathcal{R}(I).$$

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Proposition

For any ideal $I \supseteq \mathrm{NS}_\kappa$ on κ,

$$\mathcal{I}_M(I) = \mathcal{I}(I).$$

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We have seen that pre-Ramseyness relates to Ramseyness as does subtlety to ineffability. Hence, subtlety could perhaps be called *pre-ineffability*. This concept of *pre-versions* of large cardinals, and also their associated ideals and operators, can be generalized, in particular when we have suitable characterizations of these objects in terms of the existence of certain models and ultrafilters. For this, we need the (easy) concept of *local instances* of our operators.

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A little more notation: Sequences of Ideals

We refer to a sequence $\vec{I} = \langle I_\alpha \mid \alpha \leq \kappa \rangle$ such that each I_α is an ideal on α, and α ranges over inaccessible cardinals, as a sequence of ideals.

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If $\vec{I}$ is uniformly defined (say for example $I_\alpha = NS_\alpha$ for every α), we sometimes identify $\vec{I}$ and I_κ.

Pre-Operators

Examples

The subtle operator is the operator $\mathcal{I}_0$, where

$$\mathcal{I}_0(\vec{I})^+ = \{A \subseteq \kappa \mid \forall \vec{a}\, \forall C \subseteq \kappa \text{ club } \exists \alpha \in A\ A \cap C \cap \alpha \in \mathcal{I}^{\vec{a}\restriction\alpha}(I_\alpha)^+\}.$$

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General definition

Given an operator $\mathfrak{O}$ with local instances $\mathfrak{O}^p$, given by parameters p with restrictions $p \restriction \alpha$, and a sequence $\vec{I}$ of ideals, let $\mathfrak{O}_0(\vec{I})^+$ be defined as

$$\{A \subseteq \kappa \mid \forall p \forall C \subseteq \kappa \text{ club } \exists \alpha \in A\ A \cap C \cap \alpha \in \mathfrak{O}^{p \restriction \alpha}(I_\alpha)^+\}.$$

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As for the operators $\mathcal{I}$ and $\mathcal{R}$, the above also defines pre-operators $(\mathcal{I}_M)_0$ and $(\mathcal{R}_M)_0$ that correspond to the operators $\mathcal{I}_M$ and $\mathcal{R}_M$.

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In particular, this gives us a way to characterize the subtle and the pre-Ramsey ideal using small models and ultrafilters.

A general framework for large cardinal operators

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So, the difference to the Ramsey operator is that we only ask that $U \subseteq I^+$, rather than that all countable intersections from U be in I^+.

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We can't hope to obtain properness as above with respect to any ideal I. For example, if κ is measurable and I is the complement of any normal ultrafilter on κ, then $I \subseteq \mathcal{I}(I) \subseteq \mathsf{T}(I) \subseteq \mathcal{R}(I) = I$.

A test application for large cardinal operators: Baumgartner's result

By a uniform argument, we obtain the following. For $\mathcal{I}$ and $\mathcal{R}$, our argument proceeds using the model versions $\mathcal{I}_M$ and $\mathcal{R}_M$.

Theorem (for $\mathcal{I}$ and $\mathcal{R}$: Brent Cody (2020))

For many operators $\mathfrak{O}$, in particular also for $\mathfrak{O} \in \{\mathcal{I}, \mathsf{T}, \mathcal{R}\}$, and all $\beta < \kappa$, we have

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By a uniform argument, we obtain the following. For $\mathcal{I}$ and $\mathcal{R}$, our argument proceeds using the model versions $\mathcal{I}_M$ and $\mathcal{R}_M$.

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In most, but not all cases, letting $\Pi^1_{-1}(\kappa) = [\kappa]^{<\kappa}$, the above also holds for $\beta = -1$. In fact, many further results on the ineffability operator and the Ramsey operator can be shown to carry over to a larger class of large cardinal operators, that includes the operators $\mathcal{I}$, T, and $\mathcal{R}$, and potentially many other operators defined by the existence of ultrafilters for weak κ-models, by uniform arguments.