Structural reflection and shrewd cardinals

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Introduction

In this talk, I want to present work dealing with the interplay between extensions of the *Downward Löwenheim–Skolem Theorem*, large cardinal axioms and set-theoretic reflection principles.

I will focus on the characterization of large cardinal notions through reflection principles for certain classes of structures.

Throughout this talk, we use the term *large cardinal property* to refer to properties of cardinals that the imply weak inaccessibility of the given cardinal.

Overview

- Structural reflection
- Shrewd cardinals and embedding characterizations
- Weakly shrewd cardinals and structural reflection
    - Reflection below the continuum
    - Local versions of *Vopěnka's Principle*
- Characterizations of small large cardinals
    - Hamkins' *weakly compact embedding property*
    - Structural reflection and cardinal invariants of the continuum

The starting point of the work presented in this talk is the following *principle of structural reflection*:

Definition (Bagaria)

Given an infinite cardinal κ and a class $\mathcal{C}$ of structures of the same type, we let $\mathrm{SR}_{\mathcal{C}}(\kappa)$ denote the statement that for every structure A in $\mathcal{C}$, there exists a structure B in $\mathcal{C}$ of cardinality less than κ and an elementary embedding of B into A.

Principles of this form can be viewed as extensions of the *Downward Löwenheim–Skolem theorem* to second-order properties defined through set-theoretic formulas.

Proposition (Bagaria et al.)

$\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds for every uncountable cardinal κ and every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_1-formula with parameters in $\mathrm{H}(\kappa)$.

In contrast, the work of Bagaria and his collaborators shows that the validity of principles of the form $\mathrm{SR}_{\mathcal{C}}(\kappa)$ for classes $\mathcal{C}$ of structures defined by more complex formulas closely corresponds to the existence of large cardinals.

Moreover, such principles can be used to characterize various important objects in the upper reaches of the large cardinal hierarchy.

Let $PwSet$ denote the Π_1-definable class of all pairs of the form $\langle x, \mathcal{P}(x)\rangle$.

Theorem (Bagaria et al.)

The following statements are equivalent for every infinite cardinal κ:

- *κ is the least supercompact cardinal.*
- *κ is the least cardinal such that $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ that is definable by a $\Sigma_1(PwSet)$-formula without parameters.*
- *κ is the least cardinal such that $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ that is definable by a Σ_2-formula with parameters in $\mathrm{H}(\kappa)$.*

Bagaria and his collaborators extended the above result to Σ_{n+2}-definable classes of structures and so-called $C^{(n)}$*-extendible cardinals*.

Motivated by the aim to characterize cardinals in the lower part of the large cardinal hierarchy through principles of structural reflection, Bagaria and Väänänen introduced the following weakening of the above principle:

Definition (Bagaria–Väänänen)

Given an infinite cardinal κ and a class $\mathcal{C}$ of structures of the same type, we let $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ denote the statement that for every structure A in $\mathcal{C}$ of cardinality κ, there exists a structure B in $\mathcal{C}$ of cardinality less than κ and an elementary embedding of B into A.

In the following, we will isolate a narrow interval in the large cardinal hierarchy that is bounded from below by total indescribability and from above by subtleness, and contains all large cardinals that can be characterized through the principle SR^-.

These results heavily make use of the notion of *shrewd cardinals* introduced by Rathjen in a proof-theoretic context.

Definition (Rathjen)

A cardinal κ is *shrewd* if for every $\mathcal{L}_\in$-formula $\Phi(v_0, v_1)$, every ordinal α and every subset A of V_κ such that $\Phi(A, \kappa)$ holds in $\mathrm{V}_{\kappa+\alpha}$, there exist ordinals $\bar{\alpha}, \bar{\kappa} < \kappa$ such that $\Phi(A \cap \mathrm{V}_{\bar{\kappa}}, \bar{\kappa})$ holds in $\mathrm{V}_{\bar{\kappa}+\bar{\alpha}}$.

It is easy to see that shrewd cardinals are both totally indescribable and stationary limits of totally indescribable cardinals.

Moreover, Rathjen showed that if δ is a subtle cardinal, then the set of cardinals κ that are shrewd cardinals in V_δ is stationary in δ.

Let Cd denote the Π_1-definable class of all cardinals.

Theorem

The following statements are equiconsistent over the theory $\mathbf{ZFC}$*:*

- *There exists a shrewd cardinal.*
- *There exists a cardinal* κ *such that* $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ *holds for every class* $\mathcal{C}$ *that is definable by a* $\Sigma_1(Cd)$*-formula without parameters.*
- *There exists a cardinal* κ *such that* $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ *holds for every class* $\mathcal{C}$ *that is definable by a* Σ_2*-formula with parameters in* $\mathrm{H}(\kappa)$*.*

This results shows that for all large cardinal properties whose consistency strength is smaller than the existence of a shrewd cardinal, there is no reasonable characterization of these notions through the principle SR^- for classes of structures that are $\Sigma_1(R)$-definable for some Π_1-predicate R, because the consistency strength of this principle for the class Cd of all cardinals is already equal to the existence of a shrewd cardinal.

The proof of the above result is based on the following weakening of shrewdness:

Definition

An infinite cardinal κ is *weakly shrewd* if for every $\mathcal{L}_\in$-formula $\Phi(v_0, v_1)$, every cardinal $\theta > \kappa$ and every subset A of κ with the property that $\Phi(A, \kappa)$ holds in $\mathrm{H}(\theta)$, there exist cardinals $\bar{\kappa} < \bar{\theta}$ with the property that $\bar{\kappa} < \kappa$ and $\Phi(A \cap \bar{\kappa}, \bar{\kappa})$ holds in $\mathrm{H}(\bar{\theta})$.

The notion of weak shrewdness turns out to be closely connected to principles of structural reflection.

The next result shows that this large cardinal property can be characterized through the principle SR^{-} for $\Sigma_1(PwSet)$-definable classes of structures.

Theorem

The following statements are equivalent for every infinite cardinal κ:

- κ *is the least weakly shrewd cardinal.*
- κ *is the least cardinal such that* $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ *holds for every class* $\mathcal{C}$ *that is definable by a* $\Sigma_1(PwSet)$*-formula without parameters.*
- κ *is the least cardinal such that* $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ *holds for every class* $\mathcal{C}$ *that is definable by a* Σ_2*-formula with parameters in* $\mathrm{H}(\kappa)$*.*

In combination, the above theorems directly yield the following equiconsistency:

Corollary

The following statements are equiconsistent over the theory $\mathbf{ZFC}$:

- *There exists a shrewd cardinal.*
- *There exists a weakly shrewd cardinal.* □

Shrewd cardinals

Definition (Rathjen)

A cardinal κ is *shrewd* if for every $\mathcal{L}_{\in}$-formula $\Phi(v_0, v_1)$, every ordinal α and every subset A of V_κ such that $\Phi(A, \kappa)$ holds in $\mathrm{V}_{\kappa+\alpha}$, there exist ordinals $\bar{\kappa}$ and $\bar{\alpha}$ below κ such that $\Phi(A \cap \mathrm{V}_{\bar{\kappa}}, \bar{\kappa})$ holds in $\mathrm{V}_{\bar{\kappa}+\bar{\alpha}}$.

The key technique used in the proofs of the above results is the characterization of shrewdness and weak shrewdness through the existence of certain elementary embeddings.

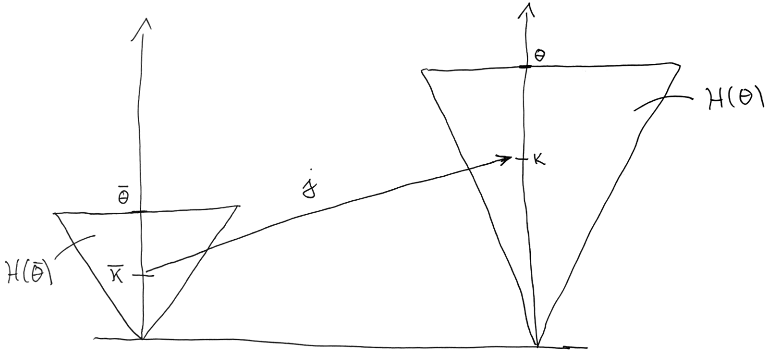
These characterizations are motivated by the following classical result:

Theorem (Magidor)

The following statements are equivalent for every cardinal κ:

- κ *is supercompact.*
- *For every cardinal* $\theta > \kappa$ *and all* $z \in \mathrm{H}(\theta)$, *there exists*
    - *cardinals* $\bar{\kappa} < \bar{\theta} < \kappa$, *and*
    - *an elementary embedding* $j : \mathrm{H}(\bar{\theta}) \longrightarrow \mathrm{H}(\theta)$

  such that $\mathrm{crit}(j) = \bar{\kappa}$, $j(\bar{\kappa}) = \kappa$ *and* $z \in \mathrm{ran}(j)$.


θ
H(θ)
κ
j
θ̄
H(θ̄)
κ̄

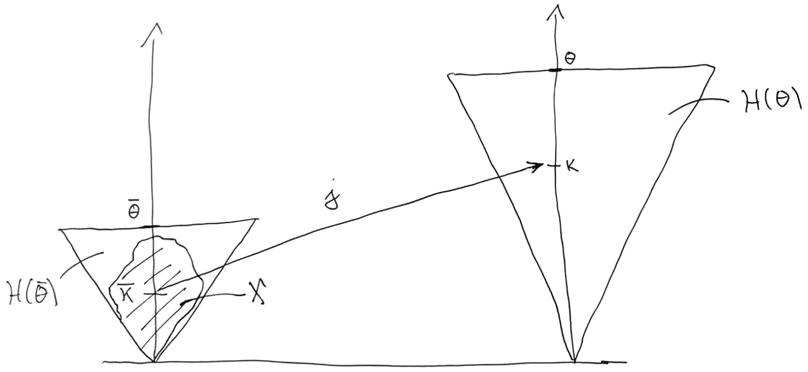
Lemma

The following statements are equivalent for every cardinal κ:

- κ *is a shrewd cardinal.*
- *For all cardinals* $\theta > \kappa$ *and all* $z \in \mathrm{H}(\theta)$, *there exist*
  - *cardinals* $\bar{\kappa} < \bar{\theta} < \kappa$,
  - *an elementary submodel* X *of* $\mathrm{H}(\bar{\theta})$, *and*
  - *an elementary embedding* $j : X \longrightarrow \mathrm{H}(\theta)$

  such that $\bar{\kappa} + 1 \subseteq X$, $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ *and* $z \in \mathrm{ran}(j)$.

Note that, in general, the elementary submodel X will not be transitive.


θ
H(θ)
κ
j
θ̄
H(θ̄)
κ̄
X

We present an easy application of the above embedding characterization.

Remember that, given $n > 0$, a cardinal κ is Σ_n-*reflecting* if it inaccessible and $\mathrm{V}_\kappa \prec_{\Sigma_n} \mathrm{V}$ holds.

Corollary

Shrewd cardinals are Σ_2*-reflecting.*

Proof.

Assume that there is a Σ_2-formula $\varphi(v)$ and $z \in \mathrm{V}_\kappa$ with the property that the statement $\varphi(z)$ holds in V and fails in V_κ.

By Σ_1-absoluteness, there exists a cardinal $\theta > \kappa$ with the property that $\varphi(z)$ holds in $\mathrm{H}(\theta)$.

Pick cardinals $\bar{\kappa} < \bar{\theta} < \kappa$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ such that $\bar{\kappa} + 1 \subseteq X \prec \mathrm{H}(\bar{\theta})$, $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.

Then $\mathrm{V}_{\bar{\kappa}} \subseteq X$ and $j \restriction \mathrm{V}_{\bar{\kappa}} = \mathrm{id}_{\mathrm{V}_{\bar{\kappa}}}$, since shrewd cardinals are inaccessible.

In particular, we know that $z \in \mathrm{V}_{\bar{\kappa}}$ and $j(z) = z$.

But then $\varphi(z)$ holds in $\mathrm{H}(\bar{\theta}) \subseteq \mathrm{V}_\kappa$ and hence Σ_1-absoluteness implies that this statement also holds in V_κ, a contradiction. $\square$

Weakly shrewd cardinals

Definition

An infinite cardinal κ is *weakly shrewd* if for every $\mathcal{L}_\in$-formula $\Phi(v_0, v_1)$, every cardinal $\theta > \kappa$ and every subset A of κ with the property that $\Phi(A, \kappa)$ holds in $\mathrm{H}(\theta)$, there exist cardinals $\bar{\kappa} < \bar{\theta}$ with the property that $\bar{\kappa} < \kappa$ and $\Phi(A \cap \bar{\kappa}, \bar{\kappa})$ holds in $\mathrm{H}(\bar{\theta})$.

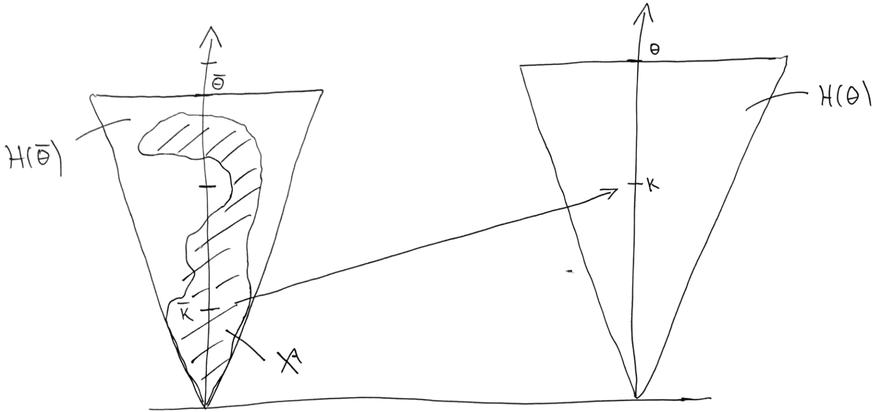
It is possible to show that analogous embedding characterizations exist for weakly shrewd cardinals.

Lemma

The following statements are equivalent for every infinite cardinal κ:

- κ *is a weakly shrewd cardinal.*
- *For all cardinals* $\theta > \kappa$ *and all* $z \in \mathrm{H}(\theta)$*, there exist*
  - *cardinals* $\bar{\kappa} < \bar{\theta}$*,*
  - *an elementary submodel* X *of* $\mathrm{H}(\bar{\theta})$*, and*
  - *an elementary embedding* $j : X \longrightarrow \mathrm{H}(\theta)$

  with $\bar{\kappa} + 1 \subseteq X$*,* $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$*,* $j(\bar{\kappa}) = \kappa > \bar{\kappa}$ *and* $z \in \mathrm{ran}(j)$*.*


$H(\bar{\theta})$
$\bar{\theta}$
$\bar{\kappa}$
X
θ
κ
$H(\theta)$

Corollary

Let κ be a weakly shrewd cardinal.

- *κ is a weakly Mahlo cardinal.*
- *If $\kappa = \kappa^{<\kappa}$ holds, then κ is inaccessible.*

We now connect weak shrewdness with principles of structural reflection:

Lemma

If κ is weakly shrewd and $\mathcal{C}$ is a class of structures of the same type that is definable by a Σ_2-formula with parameters in $\mathrm{H}(\kappa)$, then $\mathrm{SR}^-_\kappa(\mathcal{C})$ holds.

Proof.

Fix a Σ_2-formula $\varphi(v_0, v_1)$ and z in $\mathrm{H}(\kappa)$ such that $\mathcal{C} = \{A \mid \varphi(A, z)\}$ holds. Pick a structure B in $\mathcal{C}$ of cardinality κ.

Then there exists a cardinal $\theta > \kappa$ with the property that $B \in \mathrm{H}(\theta)$ and $\varphi(B, z)$ holds in $\mathrm{H}(\theta)$. Pick cardinals $\bar{\kappa} < \bar{\theta}$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $\bar{\kappa} + 1 \subseteq X \prec \mathrm{H}(\bar{\theta})$, $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa > \bar{\kappa}$ and $B, z \in \mathrm{ran}(j)$.

Then $j \restriction (\mathrm{H}(\bar{\kappa}) \cap X) = \mathrm{id}_{\mathrm{H}(\bar{\kappa}) \cap X}$, and hence $z \in \mathrm{H}(\bar{\kappa})$ and $j(z) = z$.

Pick $A \in X$ with $j(A) = B$. Then elementarity and Σ_1-absoluteness implies that $\varphi(A, z)$ holds and hence A is a structure in $\mathcal{C}$.

Since the structure B has cardinality κ in $\mathrm{H}(\theta)$, we know that the structure A has cardinality $\bar{\kappa}$ and the fact that $\bar{\kappa}$ is a subset of X allows us to conclude that j induces an elementary embedding of A into B. $\square$

Let $\mathcal{W}$ denote the $\Sigma_1(PwSet)$-definable class of all structures $\langle X, \in, \kappa\rangle$ with the property that there exists a cardinal θ such that

- κ is an infinite cardinal smaller than θ, and
- X is an elementary submodel of $\mathrm{H}(\theta)$ of cardinality κ with $\kappa + 1 \subseteq X$.

Note that, if $\mathrm{V} = \mathrm{L}$, then $\mathcal{W}$ is $\Sigma_1(Cd)$-definable.

Theorem

The following statements are equivalent for every cardinal κ:

- *κ is the least weakly shrewd cardinal.*
- *κ is the least cardinal such that $\mathrm{SR}^-_{\mathcal{W}}(\kappa)$ holds.*
- *κ is the least cardinal such that $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ that is definable by a Σ_2-formula with parameters in $\mathrm{H}(\kappa)$.*

Hyper-shrewdness

The next step in the proofs of the above results is the analysis of weakly shrewd cardinals that are not shrewd.

It turns out that these cardinals are characterized by a failure of Σ_2-reflection.

Lemma

The following statements are equivalent for all weakly shrewd cardinals κ:

- *κ is not a shrewd cardinal.*
- *κ is not a Σ_2-reflecting cardinal.*
- *There exists a cardinal $\delta > \kappa$ with the property that the set $\{\delta\}$ is definable by a Σ_2-formula with parameters in $\mathrm{H}(\kappa)$.*

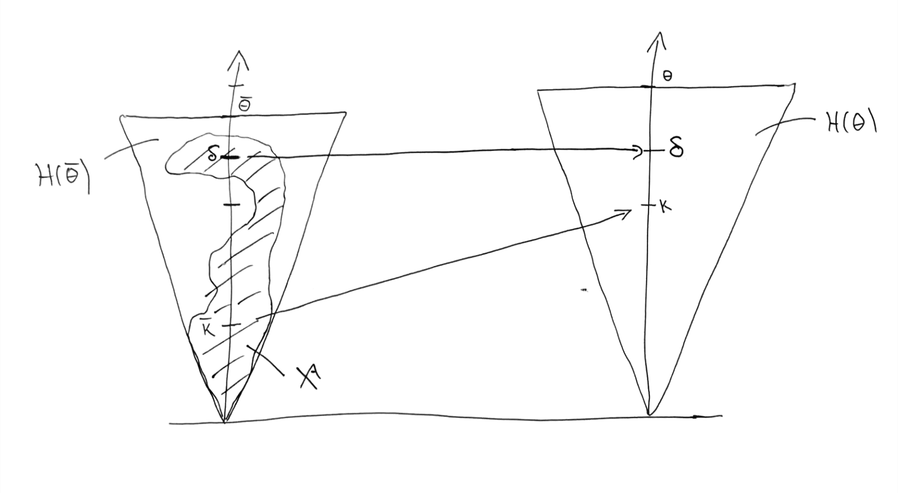
The above equivalence now motivates the following definition:

Definition

Given infinite cardinals $\kappa < \delta$, the cardinal κ is δ-*hyper-shrewd* if for all sufficiently large cardinals $\theta > \delta$ and all $z \in \mathrm{H}(\theta)$, there exist

- cardinals $\bar{\kappa} < \kappa < \delta < \bar{\theta}$,
- an elementary submodel X of $\mathrm{H}(\bar{\theta})$, and
- an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$

with $\bar{\kappa} \cup \{\bar{\kappa}, \delta\} \subseteq X$, $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$, $j(\delta) = \delta$ and $z \in \mathrm{ran}(j)$.



$\bar{\theta}$
H($\bar{\theta}$)
δ
$\bar{\kappa}$
A
θ
H(θ)
δ
κ

The next result shows that weakly shrewd cardinals that are not shrewd are typical examples of hyper-shrewd cardinals:

Lemma

Let κ be a weakly shrewd cardinal that is not a shrewd cardinal.

- *There exists $\delta > \kappa$ with the property that the set $\{\delta\}$ is definable by a Σ_2-formula with parameters in $\mathrm{H}(\kappa)$.*
- *If $\delta > \kappa$ is a cardinal with the property that the set $\{\delta\}$ is definable by a Σ_2-formula with parameters in $\mathrm{H}(\kappa)$, then κ is δ-hyper-shrewd.*

Lemma

If κ is an inaccessible cardinal that is δ-hyper-shrewd for some cardinal $\delta > \kappa$, then the interval (κ, δ) contains an inaccessible cardinal and, if ε is the least inaccessible cardinal above κ, then

$$\mathrm{V}_\varepsilon \models \text{“}\kappa \text{ is a shrewd cardinal”}.$$

Lemma

Weak shrewdness and δ-hyper-shrewdness are downwards-absolute to L.

The above results directly motivate two follow-up questions:

- First, these results suggest to study the interactions between structural reflection and the behavior of the continuum function.

  In particular, it is interesting to ask whether any large cardinal property that entails strong inaccessibility can be characterized through the principle SR^-.

- Second, it is natural to ask which large cardinal properties stronger than weak shrewdness can be characterized through the principle SR^- for classes of structures defined by more complex formulas.

The answers to both questions turn out to be closely related to the existence of weakly shrewd cardinals that are not shrewd.

The size of the continuum

The following results position the consistency strength of weakly shrewd cardinals that are not shrewd in the large cardinal hierarchy:

Theorem

- *If κ is a weakly shrewd cardinal that is not shrewd, then there exists an ordinal $\varepsilon > \kappa$ with the property that ε is inaccessible in L and κ is a shrewd cardinal in L_ε.*
- *The least subtle cardinal is a stationary limit of inaccessible weakly shrewd cardinals that are not shrewd.*

The following result shows that the existence of weakly shrewd cardinals below the size of the continuum is consistent and has consistency strength strictly larger than the existence of a shrewd cardinal.

Theorem

The following statements are equiconsistent over $\mathbf{ZFC}$*:*

- *There exists an inaccessible weakly shrewd cardinal that is not shrewd.*
- *There exists a weakly shrewd cardinal that is not inaccessible.*
- *There exists a weakly shrewd cardinal smaller than* $2^{\aleph_0}$*.*

Lemma

If κ is a cardinal that is δ-hyper-shrewd for some cardinal $\delta > \kappa$ and G is $\mathrm{Add}(\omega, \delta)$-generic over V, then κ is δ-hyper-shrewd in $\mathrm{V}[G]$.

Lemma

If $\mathrm{V} = \mathrm{L}$ holds, κ is a cardinal that is δ-hyper-shrewd for some cardinal $\delta > \kappa$ and G is $\mathrm{Add}(\delta^+, 1)$-generic over V, then, in $\mathrm{V}[G]$, the cardinal κ is a weakly shrewd and not shrewd.

Finally, hyper-shrewdness also allows us to show that subtle cardinals imply the existence of weakly shrewd cardinals that are not shrewd.

Remember that a cardinal δ is *subtle* if for every sequence $\langle d_\alpha \mid \alpha < \delta \rangle$ with $d_\alpha \subseteq \alpha$ for all $\alpha < \delta$ and every closed unbounded subset C of δ, there exist $\alpha, \beta \in C$ with $\alpha < \beta$ and $d_\alpha = d_\beta \cap \alpha$.

Lemma

If δ is a subtle cardinal, then the set of all inaccessible δ-hyper-shrewd cardinals is stationary in δ.

More complex classes

The techniques developed in the proofs of the above results also allow us to show that the existence of a weakly shrewd cardinal does not imply the existence of a cardinal κ with the property that $\mathrm{SR}^-_\kappa(\mathcal{C})$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_3-formula without parameters.

In contrast, the following result shows that the existence of a weakly shrewd cardinal that is not shrewd implies the existence of reflection points for classes of structures of arbitrary complexities.

Theorem

Let κ be weakly shrewd cardinal that is not shrewd.

- *There is a cardinal $\delta > \kappa$ with the property that the set $\{\delta\}$ is definable by a Σ_2-formula with parameters in $\mathrm{H}(\kappa)$.*
- *Given $0 < n < \omega$, if $\delta > \kappa$ is a cardinal with the property that the set $\{\delta\}$ is definable by a Σ_2-formula with parameters in $\mathrm{H}(\kappa)$, then there exists a cardinal $\rho < \delta$ such that $\mathrm{SR}^{-}_{\mathcal{C}}(\rho)$ holds for every class $\mathcal{C}$ that is definable by a Σ_n-formula with parameters in $\mathrm{H}(\rho)$.*

A combination of the compactness theorem with the above result now allows us to show that $\mathbf{ZFC}$ is consistent with the existence of cardinals with maximal local structural reflection properties.

The existence of such cardinals can be seen as a localized version of *Vopěnka's Principle*.

Moreover, such cardinals can consistently exist below the cardinality of the continuum.

In particular, this shows that no large cardinal property that implies strong inaccessibility can be characterized through the principle SR^{-}.

Let $\mathcal{L}_c$ denote the first-order language extending $\mathcal{L}_\in$ by a constant symbol $\dot{\kappa}$.

Given $n > 0$, we let SR^-_n denote the $\mathcal{L}_c$-sentence stating that $\mathrm{SR}^-_{\mathcal{C}}(\dot{\kappa})$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_n-formula in $\mathcal{L}_\in$ with parameters in $\mathrm{H}(\dot{\kappa})$.

Corollary

- *The consistency of the $\mathcal{L}_\in$-theory*

  $$\mathbf{ZFC} + \text{“There exists a weakly shrewd cardinal that is not shrewd”}$$

  implies the consistency of the $\mathcal{L}_c$-theory $\mathbf{ZFC} + \{\mathsf{SR}^-_n \mid 0 < n < \omega\}$.

- *The following theories are equiconsistent:*
  - $\mathbf{ZFC}$+*“There exists a weakly shrewd cardinal that is not shrewd”.*
  - $\mathbf{ZFC} + \{\mathsf{SR}^-_n \mid 0 < n < \omega\} + \text{“}\dot{\kappa} < 2^{\aleph_0}\text{”}$.

Less complex classes

We now turn to the characterizations of large cardinal notions below weak shrewdness through principles of structural reflection.

Since the above results show that it is not possible to characterize such notions through canonical Π_1-predicates R and the principle SR^- for $\Sigma_1(R)$-definable classes of structures, we introduce new complexity classes in-between Σ_1- and Σ_2-definability.

Our definition is motivated by the fact that Σ_1-absoluteness implies that the following statements are equivalent for every class Q:

- Q is definable by a Σ_1-formula with parameters z.
- There is a Σ_1-formula $\varphi(v_0, v_1)$ with

$$\mathrm{H}(\delta^+) \cap Q \;=\; \{x \in \mathrm{H}(\delta^+) \mid \mathrm{H}(\delta^+) \models \varphi(x,z)\}$$

  for every infinite cardinal δ with $z \in \mathrm{H}(\delta^+)$.

Definition

Let R be a class, let $n > 0$ be a natural number and let z be a set.

A class S is *uniformly locally $\Sigma_n(R)$-definable in the parameter z* if there is a $\Sigma_n(R)$-formula $\varphi(v_0, v_1)$ with the property that

$$\mathrm{H}(\kappa^+) \cap S \; = \; \{x \in \mathrm{H}(\kappa^+) \mid \langle \mathrm{H}(\kappa^+), \in, R\rangle \models \varphi(x, z)\}$$

holds for every infinite cardinal κ with $z \in \mathrm{H}(\kappa)$.

It is easy to see that for every $n > 0$ and every Π_1-predicate R, all uniformly locally $\Sigma_n(R)$-definable classes are Σ_2-definable in the same parameter.

In contrast, a truth predicate for all $\mathrm{H}(\kappa^+)$ is an example a Σ_2-definable class that is not locally definable.

Definition

Let R and Z be classes and let $n > 0$ be a natural number.

A class $\mathcal{C}$ of structures of the same type is a *local* $\Sigma_n(R)$*-class over* Z if the following statements hold:

- $\mathcal{C}$ is closed under isomorphic copies.
- $\mathcal{C}$ is uniformly locally $\Sigma_n(R)$-definable in parameters in Z.

Theorem

The following statements are equivalent for every infinite cardinal κ:

- κ *is the least weakly inaccessible cardinal.*
- κ *is the least cardinal such that* $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ *holds for every local* $\Sigma_1(Cd)$*-class* $\mathcal{C}$ *over* $\emptyset$.
- κ *is the least cardinal such that* $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ *holds for every local* $\Sigma_1(Cd)$*-class* $\mathcal{C}$ *over* $\mathrm{H}(\kappa)$.

Let Rg denote the Π_1-definable class of all regular cardinals.

Theorem

The following statements are equivalent for every infinite cardinal κ:

- *κ is the least weakly Mahlo cardinal.*
- *κ is the least cardinal such that $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ holds for every local $\Sigma_1(Rg)$-class $\mathcal{C}$ over $\emptyset$.*
- *κ is the least cardinal such that $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ holds for every local $\Sigma_1(Rg)$-class $\mathcal{C}$ over $\mathrm{H}(\kappa)$.*

Recall that, given natural numbers m and n, Lévy defined a cardinal κ to be *weakly* Π^m_n*-indescribable* if for all predicates $A_0, \ldots, A_{m-1}$ on κ and all Π^m_n-sentences Φ that hold in $\langle \kappa, A_0, \ldots, A_{m-1} \rangle$, there exists an ordinal $\lambda < \kappa$ such that Φ holds in $\langle \lambda, A_0 \cap \lambda^{\# A_0}, \ldots, A_{m-1} \cap \lambda^{\# A_{m-1}} \rangle$.

Theorem

The following statements are equivalent for every infinite cardinal κ and every $n > 0$:

- *κ is the least weakly Π^1_n-indescribable cardinal.*
- *κ is the least cardinal such that $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ holds for every local Σ_{n+1}-class over $\emptyset$.*
- *κ is the least cardinal such that $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ holds for every local Σ_{n+1}-class over $\mathrm{H}(\kappa)$.*

The proofs of the above results again rely on characterizations of restrictions of weak shrewdness through elementary embeddings.

The following lemma provides the relevant characterization for weak Π^1_1-indescribability.

Lemma

The following statements are equivalent for every infinite cardinal κ:

- κ *is weakly* Π^1_1*-indescribable.*
- *For every cardinal* $\theta > \kappa$ *and all* $z \in \mathrm{H}(\theta)$*, there exists*
  - *a transitive set* N*, and*
  - *a non-trivial elementary embedding* $j : N \longrightarrow \mathrm{H}(\theta)$

  with the property that $\mathrm{crit}(j)$ *is a cardinal,* $j(\mathrm{crit}(j)) = \kappa$*,* $z \in \mathrm{ran}(j)$ *and* $\mathrm{H}(\mathrm{crit}(j)^+)^N \prec_{\Sigma_1} \mathrm{H}(\mathrm{crit}(j)^+)$*.*

The above result allows us to show that a large cardinal property isolated by Hamkins is in fact equal to Lévy's notion of weak Π^1_1-indescribability.

Definition (Hamkins)

A cardinal κ has the *weakly compact embedding property* if for every transitive set M of cardinality κ with $\kappa \in M$, there is a transitive set N and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}(j) = \kappa$

Hamkins proved that, if κ is weakly compact and G is $\mathrm{Add}(\omega, \kappa^+)$- generic over V, then κ has the weakly compact embedding property in $\mathrm{V}[G]$.

Corollary

A cardinal κ has the weakly compact embedding property if and only if it is weakly Π^1_1-indescribable.

Cardinal invariants of the continuum

In unpublished work, Cody, Cox, Hamkins and Johnstone showed that various cardinal invariants of the continuum do not possess the weakly compact embedding property.

Using the above results, we put this implication into a more general context.

Proposition

Given a class I of infinite cardinals, there exists a class $\mathcal{C}$ of structures such that $\mathrm{SR}^-_{\mathcal{C}}(\mathrm{min(I)})$ fails and the following statements hold:

- *If I is $\Sigma_n(R)$-definable in parameter z, then $\mathcal{C}$ is definable in the same way.*
- *If I is uniformly locally $\Sigma_n(R)$-definable in parameter z, then $\mathcal{C}$ is a local $\Sigma_n(R)$-class over $\{z\}$.*

Proposition

The sets $\{2^{\aleph_0}\}$, $[\mathfrak{b}, 2^{\aleph_0}]$ and $[\mathfrak{d}, 2^{\aleph_0}]$ are all uniformly locally Σ_2-definable without parameters.

Thank you for listening!