Construction with opposition: cardinal invariants and games

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(joint work with J. Brendle and M. Hrusak)
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Corollary

$\mathfrak{c} = \omega_1 \not\to \diamondsuit$.

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For example, $2^{\omega_1} = \omega_2$ implies $\diamondsuit_{\omega_2}(E^{\omega_2}_{\omega})$.

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If (A, B, R) is an invariant, then its *evaluation* $\langle A, B, R\rangle$ is given by

$$\langle A, B, R\rangle = \min\{|X| : X \subseteq B \text{ and } \forall a \in A \exists b \in X(aRb)\}.$$

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1. $\diamondsuit(2, \neq) \to$ *the Builder has a winning strategy in the tower game* $G_{\mathfrak{t}} \to \mathfrak{t} = \omega_1$.
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Let $\{A_\alpha : \alpha \in \mathrm{odd}(\omega_1)\}$ be an enumeration of $[\omega]^\omega$. Suppose $\langle Y_\alpha : \alpha \leq \beta \rangle$ is a partial match,

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Let $\{A_\alpha : \alpha \in \mathrm{odd}(\omega_1)\}$ be an enumeration of $[\omega]^\omega$. Suppose $\langle Y_\alpha : \alpha \leq \beta \rangle$ is a partial match, where the Spoiler played Y_β. Let

$$Y_{\beta+1} = \begin{cases} Y_\beta \setminus A_\beta & \text{if } Y_\beta \setminus A_\beta \text{ is infinite,} \\ Y_\beta \cap A_\beta & \text{otherwise.} \end{cases}$$

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Given an infinite $\subseteq^*$-decreasing sequence $s = \{Y^s_\xi : \xi < \delta(s)\}$ with $\delta(s)$ limit,

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Given an infinite $\subseteq^*$-decreasing sequence $s = \{Y^s_\xi : \xi < \delta(s)\}$ with $\delta(s)$ limit, we will define a strictly increasing sequence $\{l^s_i : i \in \omega\}$ of natural numbers.

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Fix an increasing sequence $\{\delta_i : i \in \omega\} \subseteq \delta(s)$ converging to $\delta(s)$. Let

$$l^s_0 = \min\left(Y^s_{\delta_0}\right),$$

and

$$l^s_{i+1} = \min\left(\bigcap_{j \leq i+1} Y^s_{\delta_j} \setminus (l^s_i + 1)\right).$$

$\diamondsuit(2, \neq)$ implies the Builder has a winning strategy in $G_{\mathfrak{t}}$

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$$F(s,C) = \begin{cases} 0 & \text{if } C \subseteq^* \{I^s_{2i} : i \in \omega\}, \\ 1 & \text{otherwise.} \end{cases}$$

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$$Y_{\delta(s)} = \begin{cases} \{I^s_{2i} : i \in \omega\} & \text{if } g(\delta(s)) = 0, \\ \{I^s_{2i+1} : i \in \omega\} & \text{otherwise.} \end{cases}$$

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Let $C \subseteq \omega$. Then if δ is an infinite limit ordinal such that $F(s{\restriction}_\delta, C) \neq g(\delta)$, it is straightforward to see that $C \not\subseteq^* Y_\delta$.

□

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Corollary

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Corollary

$\diamondsuit(2, =) \not\leftarrow$ *the Builder has a winning strategy in the tower game* $G_{\mathfrak{t}}$.

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Assume CH. Let $\mathcal{Y} = (Y_\alpha : \alpha < \omega_1)$ be a tower. Let $(f_\alpha : \alpha < \omega_1)$ list all partial functions from $\omega \to \omega$ with infinite range.

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- $A_\alpha \subseteq^* B_\alpha$, $B_\alpha \subseteq^* A_\beta$ for $\beta < \alpha$,
- B_α is chosen according to a given rule, and
- if $\mathrm{ran}(f_\alpha \restriction_{B_\alpha})$ is infinite, then $\mathrm{ran}(f_\alpha \restriction_{A_\alpha})$ is almost disjoint from some Y_{β_α}.

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To choose A_α note that there is $\beta < \omega_1$ such that $\mathrm{ran}(f_\alpha{\restriction}_{B_\alpha}) \setminus Y_{\beta_\alpha}$ is infinite because $\mathcal{Y}$ is a tower.

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Let $\mathcal{F}$ be the filter generated by the A_α. Consider Laver forcing $\mathbb{L}_{\mathcal{F}}$ with $\mathcal{F}$.
Assume the following:

Claim

$\mathbb{L}_{\mathcal{F}}$ preserves $\mathcal{Y}$.

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On the other hand, for each strategy Σ of the Builder in $V^{\mathbb{P}_{\omega_2}}$, there is $\gamma < \omega_2$ such that $\Sigma\restriction_{V^{\mathbb{P}_\gamma}}$ is a strategy in $V^{\mathbb{P}_\gamma}$ and was used to construct the B_α and $\mathcal{F}$.

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$\square$

The ultrafilter number game

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Builder	U_0		$\cdots$	U_α		$\cdots$
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Builder	U_0		$\cdots$	U_α		$\cdots$
Spoiler		U_1	$\cdots$		$U_{\alpha+1}$	$\cdots$

The Builder wins the match if the filter generated by $\{U_\alpha : \alpha < \omega_1\}$ is an ultrafilter; otherwise, the Spoiler wins.

The ultrafilter number game

[1] Here $B \subseteq^* A$ means that $B \setminus A$ is finite, and $A \cap B =^* \emptyset$ says that $A \cap B$ is finite.

The ultrafilter number game

Remember the *reaping number* $\mathfrak{r} = \langle [\omega]^\omega, [\omega]^\omega, \mathbf{R} \rangle$, where $A\mathbf{R}B$ if $B \subseteq^* A$ or $A \cap B =^* \emptyset$[1].

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Case 2: $|g(\delta) \setminus F(s\restriction_\delta, C)| < \aleph_0$. Let $j \in \omega$ such that $g(\delta) \setminus j \subseteq F(s\restriction_\delta, C)$. Then $U_\delta \setminus k_j^{s\restriction_\delta} \subseteq C$ if $\{i \in \omega : k_i^{s\restriction_\delta} \in C\}$ is infinite, and $U_\delta \setminus k_j^{s\restriction_\delta} \subseteq \omega \setminus C$ otherwise.

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Start with $V \models \mathrm{CH} + 2^{\omega_1} = \omega_2$, and force with $\mathbb{P}^{\omega_2}$, the countable support iteration used by Shelah to construct a model with a unique P-point.

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Builder	A_0		$\cdots$	A_α		$\cdots$
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Builder	A_0		$\cdots$	A_α		$\cdots$
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The Builder wins the match if the family $\{A_\alpha : \alpha < \omega_1\}$ is a maximal almost disjoint family; otherwise, the Spoiler wins.

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2. the collection $s \cup \{B\}$ is an almost disjoint family of infinite subsets of ω,
3. the set $I(s, B) = \left\{ i \in \omega : B \cap A^s_{e_\delta(i)} \setminus \bigcup_{j<i} A^s_{e_\delta(j)} \neq \emptyset \right\}$ is infinite.

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Let $g : \omega_1 \to \omega^\omega$ be a $\diamondsuit(\mathfrak{b})$-sequence for F. Without loss of generality, $g(\delta)$ is a strictly increasing function for every $\delta < \omega_1$.

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is finite for every $i \in \omega$. Therefore for $i \in \omega$, the intersection $A^s_{e_\delta(i)} \cap A \subseteq A^s_{e_\delta(i)} \cap \left(g(\delta)(i) \cup \bigcup_{j<i} A^s_{e_\delta(j)} \right)$ is finite.

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We show that this is a winning strategy. Let $s = \{A^s_\xi : \xi < \omega_1\}$ be a complete match where the Builder played according to the strategy defined by g. We show that s is maximal. Let $B \in [\omega]^\omega$. Consider $f \in 2^{\omega_1}$ coding (B, s), i.e. $f(n) = 1$ iff $n \in B$, and $f(\omega \cdot (1 + \xi) + n) = 1$ iff $n \in A^s_\xi$.

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Aiming towards a contradiction assume that it is not the case, that is $\{B\} \cup \{A^s_\xi : \xi < \omega_1\}$ is an AD family, and for every indecomposable ordinal δ (1)-(3) are satisfied. Let δ be an indecomposable ordinal where $g(\delta)$ guesses f, so in particular, $F(s, B) \not\geq^* g(\delta)$.

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Let $\{i_k = i_k^{s\restriction\delta, B} : k \in \omega\}$ be the increasing enumeration of $I(s, B)$.
For $k \in \omega$, let $l_k = F(s, B)(k)$, i.e.

$$l_k = \min\left(B \cap A^s_{e_\delta(i_k)} \setminus \bigcup_{j<i_k} A^s_{e_\delta(j)}\right).$$

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Observe that the family $\{A^s_{e_\delta(i)} \setminus \bigcup_{j<i} A^s_{e_\delta(j)} : i \in \omega\}$ is disjoint, so the application $k \mapsto l_k$ is injective.

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This implies that $l_k \in A$. In particular, A is infinite and $A^s_\delta = A$. Hence $X \subseteq A^s_\delta$ follows.

Open question

The Builder has a winning strategy in the almost disjoint game $G_{\mathfrak{a}}$ $\not\to \mathfrak{a} = \omega_1$?

Games and cardinal invariants

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Lemma

$\mathfrak{t}_{Builder}$ *is a regular cardinal.*

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However we do not know:

Open question

Is $\mathfrak{t} < \mathfrak{t}_{Spoiler}$ *consistent?*

Note that in general the Builder has a distinct advantage over the Spoiler in that her moves appear on a closed unbounded subset of ω_1 ($\mathrm{pair}(\omega_1) \in \mathrm{Club}(\omega_1)$, while $\mathrm{odd}(\omega_1)$ is not stationary).

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Define $\mathfrak{t}^*_{Builder}$ and $\mathfrak{t}^*_{Spoiler}$ similarly as the unstarred versions.

It can be shown that

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$$\mathfrak{h} \geq \mathfrak{t}^*_{Builder} \geq \max\{\mathfrak{t}^*_{Spoiler}, \mathfrak{t}_{Builder}\} \geq$$

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Theorem (Brendle, Hrusak, T. 2019)

$\mathfrak{t}_{Builder} > \mathfrak{t}^*_{Spoiler}$.

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It is consistent that $\diamondsuit(2, \neq)$ holds and the Builder has no winning strategy in $G^*_{\mathfrak{t}}$.

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We additionally have the consistency of $\mathfrak{t} < \mathfrak{t}_{Builder} = \mathrm{add}(\mathcal{M})$.

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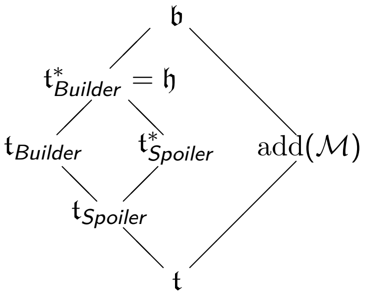

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Open question

- *Obviously* $\mathfrak{a} \leq \mathfrak{a}_{Spoiler} \leq \mathfrak{a}_{Builder}$. *Are these three numbers maybe equal?*

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1. *Is $\mathfrak{u} < \mathfrak{u}_{Spoiler}$ consistent? Is $\mathfrak{u}_{Spoiler} < \mathfrak{u}_{Builder}$ consistent?*
2. *Are $\mathfrak{u}_{Builder}$ and $\mathfrak{u}_{Spoiler}$ cardinals?*

Open question

- *Obviously $\mathfrak{a} \leq \mathfrak{a}_{Spoiler} \leq \mathfrak{a}_{Builder}$. Are these three numbers maybe equal?*
- *As for $\mathfrak{u}$, we even do not know whether $\mathfrak{a}_{Builder}$ and $\mathfrak{a}_{Spoiler}$ necessarily are cardinals.*

Thank you!