Type Omission and Subcompact cardinals

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Definition

Strongly compact cardinals have many equivalent definitions:
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<table>
<thead>
<tr>
<th>Theorem</th>
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</table>

1. The $\kappa$-compactness theorem for $L_{\kappa, \kappa}$.
2. Every $\kappa$-complete filter can be extended to $\kappa$-complete ultrafilter.
3. For every $\lambda \geq \kappa$, there is a fine $\kappa$-complete ultrafilter on $P_{\kappa \lambda}$.
4. For every $\lambda$, there is an elementary embedding $j: V \to M$, $M$ is transitive, $\text{crit } j = \kappa$ and $j[\lambda] \subseteq s \in M$, $|s| < j(\kappa)$.
5. $\kappa$ is inaccessible for every $\lambda$, and every $P_{\kappa \lambda}$-tree has a branch.
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Local strong compactness

By localizing, we get:

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Let $\kappa \leq \lambda = \lambda^\kappa < \kappa$ be regular cardinals. The following are equivalent:

1. Compactness of $L_{\kappa,\kappa}$ for languages of size $\lambda$.
2. $\kappa$ is inaccessible and every $P_{\kappa,\lambda}$-tree has a branch.
3. If $M$ is a model of set theory of size $\lambda$, $M \subseteq M^*$, then there is a transitive model $N$ and an elementary embedding $j: M \to N$, with $\text{crit} j = \kappa$, $j[M] \subseteq s \in N$, $|s|_N < j(\kappa)$.

If $\lambda = 2^\mu$ we can add:

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We want to have a normal analogue to each of the other characterizations of strong compactness.
Type Omission

One of the classical theorems in first order logic is the type omission theorem:
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**Theorem (Henkin-Orey)**

Let $T$ be a consistent theory and let $p(x)$ be a complete type (over a countable language). If there is no $\varphi$ such that $T \vdash \exists x \varphi(x)$ and for all $\psi(x) \in p(x)$, $T \vdash \forall x (\varphi(x) \rightarrow \psi(x))$ then there is a model $M$ of $T$ that omits $p$. 
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What is the $\mathcal{L}_{\kappa,\kappa}$-analogue?
Compactness of type omission

Let $T$ be an $\mathcal{L}_{\kappa,\kappa}$-theory and let $p(x)$ be an $\mathcal{L}_{\kappa,\kappa}$-type with a single variable $x$. We say that $T$ can omit $p$ if there is a model of $T$ that omits $p$. 
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**Theorem (Benda, 1976)**

$\kappa$ is supercompact if and only if for every $\mathcal{L}_{\kappa,\kappa}$-theory $T$ and $\mathcal{L}_{\kappa,\kappa}$-type such that for club many $T' \cup p' \in P_\kappa(T \cup p)$, $T'$ can omit $p'$, then $T$ can omit $p$.

We call this property $\kappa$-compactness for type omission.
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**Theorem (H. and Magidor)**

Let $\kappa \leq \lambda = \lambda^{<\kappa}$ be regular cardinals. The following are equivalent:

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Supercompactness by omitting first order types and transitivity

If we further assume that $\lambda^{<\lambda} = \lambda$, then we get an equivalence to $\lambda$-$\Pi^1_1$-subcompactness.
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In particular, the supercompact analogue of $\omega_1$-compactness is simply supercompactness.
At the beginning, I cited Jech’s characterization of strong compactness using $\mathbb{P}_{\kappa\lambda}$-trees.
The strong tree property

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**Definition**

Let $\kappa$ be a regular cardinal, $\lambda \geq \kappa$. A $P_{\kappa} \lambda$-tree $\mathcal{T}$ is a function, with domain $P_{\kappa} \lambda$ and $\mathcal{T}(x) \subseteq \mathcal{P}(x)$, $|\mathcal{T}(x)| < \kappa$.

Moreover, for every $x$, $|\mathcal{T}(x)| \neq \emptyset$ and if $x \subseteq y$ and $z \in \mathcal{T}(y)$ then $z \cap x \in \mathcal{T}(x)$.

A cofinal branch in $\mathcal{T}$ is a set $b \subseteq \lambda$, such that $b \cap x \in \mathcal{T}(x)$ for all $x$. 
Ineffable Tree Property

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But this is not the right *normalized* version of the strong tree property, since when taking $\lambda = \kappa$, we get weakly compact on one hand and ineffable cardinal in the other.
The normalized strong tree property

Let $T$ be a $P_{\kappa}\lambda$ tree. We say that $L$ is a ladder system on $T$ if

- $\text{dom } L \subseteq P_{\kappa}\lambda$ and contains a club,
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- $L(x) \subseteq \mathcal{T}(x)$ non-empty, and
- for every $y \in L(x)$ such that $\text{cf}(|x \cap \kappa|) > \omega$ there is a club $E_{x,y} \subseteq P_{|x \cap \kappa|}x$, such that for all $z \in E_{x,y}$, $z$ belongs to the domain of $L$ and $y \cap z \in L(z)$. 
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**Definition**

Let $\kappa \leq \lambda$ be regular cardinals. We say that $\kappa$ has the $P_{\kappa}\lambda$-tree property with ladder systems catching if every $P_{\kappa}\lambda$-tree $\mathcal{T}$ and a ladder system $L$, there is a cofinal branch $b$ such that $\{x \in P_{\kappa}\lambda \mid b \cap x \in L(x)\}$ is cofinal.
Theorem (H. and Magidor)

Let $\kappa \leq \lambda = \lambda^{<\lambda}$ be regular cardinals. The following are equivalent:

- $\kappa$ is $\lambda$-$\Pi^1_1$-subcompact.
- $\kappa$ has the $P_{\kappa, \lambda}$-tree property with ladder systems catching.
The Subcompactness Hierarchy

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