On regular countably compact $\mathbb{R}$-rigid spaces

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Definitions

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A space X is called regular if any finite subset of X is closed, and for any $x \in X$ and neighborhood U of x there exists a neighborhood V of x such that $\overline{V} \subset U$.

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A space X is called separable if it contains a dense countable subset.

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A space X is called countably compact if every countable subset of X has an accumulation point.

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Question (Urysohn)

Is it true that each regular space admits a non-constant continuous real-valued function?

Theorem (Hewitt)

There exists a regular $\mathbb{R}$-rigid space.

Theorem (Herrlich)

For any T_1 space Y there exists a regular Y-rigid space X.

A pseudocharacter of a space Y is the smallest cardinal $\psi(Y)$ such that for any $y \in Y$ there exists a family of open sets $\mathcal{U}$ of cardinality $\leq \psi(Y)$ such that $\{y\} = \cap\mathcal{U}$.

Theorem (Ciesielski, Wojciechowski)

There exists a regular space of arbitrary large cardinality which is Y-rigid with respect to any Hausdorff space Y of countable pseudocharacter.

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When regular spaces are not $\mathbb{R}$-rigid

General question

Under which conditions a regular space X admits a continuous nonconstant real-valued function?

Few answers

- if X is Lindelöf or second-countable or disconnected;
- if X is a countably compact semitopological group (folklore);
- if X is a paratopological group (Banakh, Ravsky).

Note that a space is compact if and only if it is Lindelöf and countably compact.

Natural question

What can we say about rigidity of countably compact regular spaces?

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Theorem (Vaughan)

There exists a regular first-countable countably compact space which is not Tychonoff.

Theorem (Tzannes)

There exists a Hausdorff countably compact $\mathbb{R}$-rigid space.

Nevertheless, the space of Tzannes is strongy nonregular in the sense that the closure of any two open sets in it has a nonempty intersection.

Problems (Tzannes, 2003)

1. Does there exist a regular (separable, first-countable) countably compact $\mathbb{R}$-rigid space?
2. Does there exist, for every Hausdorff space X, a regular (separable, first-countable) countably compact X-rigid space?

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Definition

A topological space X is called κ-bounded if the closure of any subset $A \subset X$ of cardinality $\leq \kappa$ is compact.

The following theorem gives an affirmative answer to the above problems of Tzannes (without additional properties in brackets) and extends the mentioned before results of Ciesielski, Wojciechowski and Herrlich.

Theorem (B., Osipov)

For any cardinal κ there exists a regular κ-bounded space (of arbitrary large cardinality) which is Y-rigid with respect to any T_1 space Y of pseudocharacter $\leq \kappa$.

However our examples are neither separable nor first-countable.

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Question

What about separable or first-countable examples?

Theorem (Ciesielski, Wojciechowski)

For any uncountable cardinal $\kappa \leq 2^{\mathfrak{c}}$ there exists a separable regular space of cardinality κ which is Y-rigid with respect to any Hausdorff space Y of countable pseudocharacter.

A topological space X is called totally countably compact, if each set $A \subset X$ contains an infinite subset with compact closure in X. Clearly,

ω-boundedness $\Rightarrow$ total countable compactness $\Rightarrow$ countable compactness.

Theorem (B., Zdomskyy)

There exists a regular **separable** totally countably compact $\mathbb{R}$-rigid space (of cardinality $2^{\mathfrak{c}}$).

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Consistent generalization

Theorem (B., Zdomskyy)

Let κ be any cardinal such that there exist a maximal tower $\mathcal{T} = \{T_\alpha : \alpha \in \lambda\}$ on ω such that the cardinal λ is regular and $\kappa^+ < \lambda$. Then there exists a regular separable totally countably compact space which is Y-rigid with respect to any T_1 space Y of pseudocharacter $\leq \kappa$.

Corollary (B., Zdomskyy)

It is consistent with ZFC that for each $\kappa < \mathfrak{c}$, there exists a regular separable totally countably compact space which is Y-rigid with respect to any T_1 space Y of pseudocharacter $\leq \kappa$.

Remark

The latter theorem doesn't hold for $\kappa \geq \mathfrak{c}$, because any regular separable space X has pseudocharacter $\leq \mathfrak{c}$. Hence the identity selfmap of X is not constant.

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A sketch of constructing compact-like $\mathbb{R}$-rigid spaces

- Step 1: take an appropriate regular non-normal space X;
- Step 2: construct a regular space $J(X)$ which contains two points a, b such that $f(a) = f(b)$ for any continuous real-valued function f;
- Step 3: construct a regular $\mathbb{R}$-rigid extension $D[J(X)]$ of the space $J(X)$;
- Step 4: densely embed $D[J(X)]$ into a regular compact-like space.

Natural question

Under which condition a regular space X can be (densely) embedded into a regular countably compact space Y?

Embedding into compact-like spaces

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Definition. The space X is called totally $\overline{\kappa}$-normal if we can separate (by disjoint open sets) any disjoint closed subsets $A, B \subset X$ providing one of them is contained in the closure of some set of cardinality $\leq \kappa$.

Definition. A space has property D if for any countable discrete subset $A \subset X$ there exists a locally finite family $\{U_a : a \in A\}$ of pairwise disjoint open sets such that $a \in U_a$ for any $a \in A$.

Theorem (Banakh, B., Ravsky)

For every infinite cardinal κ a regular totally $\overline{\kappa}$-normal space can be (densely) embedded into a regular κ-bounded space.

Theorem (B., Zdomskyy)

Let X be a regular space which has property D and the family of all countable closed discrete subsets of X forms a P-ideal. Then X can be embedded into a regular countably compact space.

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Embedding into compact-like spaces

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Definition. An ultrafilter u on ω is called a simple $P_{\mathfrak{c}}$-point if u possesses a base which is a tower of length $\mathfrak{c}$.

Theorem (B., Zdomskyy)

$([\omega_1 < \mathfrak{b} = \mathfrak{c}] \wedge$[exists a simple $P_{\mathfrak{c}}$-point]) Every regular (separable) **first-countable** space of cardinality $< \mathfrak{c}$ can be embedded into regular (separable) **first-countable** countably compact space.

The next examples show that the latter theorem cannot be proved within ZFC.

Example (Banakh, B., Ravsky)

There exists a regular separable first-countable scattered space X of cardinality $\mathfrak{d}$ which cannot be embedded into Urysohn countably compact spaces.

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Any Mrowka-Isbell space $\psi(\mathcal{A})$ over a MAD family $\mathcal{A}$ is an example of a separable first-countable zero-dimensional (and hence regular) space which cannot be embedded into any Hausdorff countably compact space of character $< \mathfrak{b}$.

Corollary

The latter theorem doesn't hold in any model of $\min\{\mathfrak{a}, \mathfrak{d}\} < \mathfrak{c}$.

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On the problem of Nyikos

Problem (Unofficially – early 70-th, officially – Nyikos, 1986)

Does there exist in ZFC a regular separable first-countable countably compact non-compact space?

Definition. A regular separable first-countable countably compact space is called a Nyikos space.

Theorem (Franklin, Rajagopalan)

$(\mathfrak{t} = \omega_1)$ There exists a non-compact Nyikos space.

Definition. A space X is called perfectly normal if for every disjoint closed sets $A, B \subset X$ there exists a real-valued continuous function f such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

Definition. A space is hereditary separable if every its subspace is separable.

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On the problem of Nyikos

Theorem (Ostaszewski)

($\diamondsuit$) There exists a hereditary separable perfectly normal non-compact Nyikos space.

Observation (van Douwen)

It is enough to assume $\mathfrak{b} = \mathfrak{c}$ to get a non-compact Nyikos space using Ostaszewski's construction.

Theorem (Nyikos)

It is consistent that $\omega_1 < \mathfrak{t} = \mathfrak{b} < \mathfrak{c}$ and there exists a non-compact Nyikos space.

Theorem (Nyikos, Zdomskyy)

PFA implies that every first-countable **normal** countably compact space is ω-bounded.

Consequently, PFA implies that every normal Nyikos space is compact.

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Some remarks

The problem is still open. It was reposed in: Open Problems in Topology I, II, and Open Problems from Topology Proceedings. In particular, the problem is included in the list of "Twenty problems in set-theoretic topology" by Hrušák and Moore.

A weaker version of the Nyikos problem

Definition. A space X is called sequentially compact if every sequence in X has a convergent subsequence.

$$\begin{array}{ccccc} \omega\text{-bounded} & \Longrightarrow & \text{totally countably compact} & \Longleftarrow & \text{sequentially compact} \\ & & \Downarrow & & \\ & & \text{countably compact} & & \end{array}$$

Every countably compact space of character $< \mathfrak{t}$ is sequentially compact.

While working on embedding into compact-like space we jointly with Banakh and Ravsky came across the following question: Does every regular separable sequentially compact space embeds into a compact space? We constructed a consistent non-Tychonoff counterexample. This motivates us to pose the following question.

Question (Banakh, B., Ravsky)

Does there exist in ZFC a regular separable sequentially compact non-Tychonoff space?

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It remains to consider the following question. What can we say about separable first-countable regular $\mathbb{R}$-rigid spaces?

Theorem (Ciesielski, Wojciechowski)

For any uncountable cardinal $\kappa \leq \mathfrak{c}$ there exists a separable first-countable regular $\mathbb{R}$-rigid space of cardinality κ.

The only known example of a non-normal Nyikos space was constructed by Nyikos and Vaughan under $\mathfrak{t} = \omega_1$.

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The next theorem gives an affirmative answer to the strongest version of the Tzannes' problem.

Theorem (B., Zdomskyy)

($[\omega_1 < \mathfrak{b} = \mathfrak{c}] \wedge$ [exists a simple $P_{\mathfrak{c}}$-point]) There exists an $\mathbb{R}$-rigid Nyikos space.

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Theorem

There exists a regular separable sequentially compact non-Tychonoff space.

Fix any increasing maximal tower $\mathcal{T} = \{T_\alpha \mid \alpha \in \kappa\}$. Consider the space $Y = \mathcal{T} \cup \omega$ which is topologized as follows. Points of ω are isolated and a basic open neighborhood of $T \in \mathcal{T}$ has the form

$$B(S,T,F) = \{P \in \mathcal{T} \mid S \subset^* P \subseteq^* T\} \cup ((T \setminus S) \setminus F),$$

where $S \in \mathcal{T} \cup \{\emptyset\}$ satisfies $S \subset^* T$ and F is a finite subset of ω. Observe that Y is separable normal locally compact and sequentially compact. By Y^* we denote the one point compactification of Y. Observe that the Tychonoff product of $Z = Y \times Y^*$ is separable regular and sequentially compact.

Claim

The space Z is not normal.

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Thank You for attention!