Aronszajn trees and Kurepa trees

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Let λ be an infinite cardinal.

- A λ-tree is a tree $(T, <_T)$ of height λ all whose levels are smaller than λ.

Definition

1. *A λ-Aronszajn tree is a λ-tree which has no branch.*
2. *A λ^+-tree T is called special if there exists a function $f\colon T \to \lambda$ such that for $s, t \in T$, if $s <_T t$, then $f(s) \neq f(t)$.*
3. *A λ-Kurepa tree is a λ-tree which has more than λ many branches.*

Definition

1. *A λ-Aronszajn tree is a λ-tree which has no branch.*
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- i.e., a specializing function is injective on linearly ordered sets
- a branch would induce an injection from λ^+ to λ:

  special trees have no branches!
- a λ^+-tree is special $\iff$ it is the union of λ many antichains
  - one of the λ many antichains would have size λ^+
  - Therefore, a Suslin tree cannot be special
  - ...you can also view it in the following way: if there were a special Suslin tree, forcing with it would result in a special tree which has a branch, so λ^+ would be collapsed, contradicting the λ^+-c.c. of the Suslin tree.

Definition

1. *A λ-Aronszajn tree is a λ-tree which has no branch.*
2. *A λ^+-tree T is called special if there exists a function $f: T \to \lambda$ such that for $s, t \in T$, if $s <_T t$, then $f(s) \neq f(t)$.*

Fact

1. *There are no $\aleph_0$-Aronszajn trees. (König's Lemma)*
2. *There are always $\aleph_1$-Aronszajn trees.*
3. *the existence of $\aleph_2$-Aronszajn trees is independent of ZFC:*
    - *under CH, there exists an $\aleph_2$-Aronszajn tree*
    - *in Mitchell's model, the tree property on $\aleph_2$ holds (needs a weakly compact cardinal)*
4. *If $2^\lambda = \lambda^+$, then there exists a special λ^{++}-Aronszajn tree.*
5. *If κ is inaccessible, then*

   there is a κ-Aronszajn tree $\iff$ κ is not weakly compact.

Definition

- *A λ-Kurepa tree is a λ-tree which has more than λ many branches.*

Fact

1. *There is an $\aleph_0$-Kurepa tree (namely $2^{<\omega}$).*
2. *the existence of $\aleph_1$-Kurepa trees is independent of ZFC:*
    - *under $\diamondsuit^+$ (i.e., in particular in $V = L$), there exists an $\aleph_1$-Kurepa tree*
    - *it is consistent that there exists no $\aleph_1$-Kurepa tree (needs an inaccessible cardinal)*

Main Theorem

Theorem

There exists a model of ZFC in which for all $0 < n \in \omega$

there exists an $\aleph_n$*-Aronszajn tree,*

all $\aleph_n$*-Aronszajn trees are special,*

and there exists no $\aleph_n$*-Kurepa tree.*

Theorem

There exists a model of ZFC in which

there exists an $\aleph_2$-Aronszajn tree,

all $\aleph_2$-Aronszajn trees are special,

and there exists no $\aleph_2$-Kurepa tree.

Theorem

There exists a model of ZFC in which

there exists an $\aleph_2$-Aronszajn tree,

all $\aleph_2$-Aronszajn trees are special,

and there exists no $\aleph_2$-Kurepa tree and no $\aleph_1$-Kurepa tree.

Theorem (Laver-Shelah)

There exists a model of ZFC in which

there exists an $\aleph_2$-Aronszajn tree,

all $\aleph_2$-Aronszajn trees are special.

Theorem (Baumgartner-Malitz-Reinhardt)

There exists a model of ZFC in which

there exists an $\aleph_1$-Aronszajn tree,

all $\aleph_1$-Aronszajn trees are special.

Theorem (Baumgartner-Malitz-Reinhardt)

There exists a model of ZFC in which

there exists an $\aleph_1$-Aronszajn tree, (always true)

all $\aleph_1$-Aronszajn trees are special.

Specializing $\aleph_1$-trees

Definition

Let T be an $\aleph_1$-Aronszajn tree. Let $\mathbb{S}(T)$ be the forcing consisting of conditions p satisfying the following:

1. *$p\colon T \to \omega$ is a finite partial function*
2. *if $s, t \in \mathrm{dom}(p)$ and $s <_T t$, then $p(s) \neq p(t)$.*

The order is given by $q \leq p$ if $q \supseteq p$.

1. For $t \in T$, the set of conditions p with $t \in \mathrm{dom}(p)$ is dense
   - the generic function $f\colon T \to \omega$ is total,
   - the generic function is a specializing function.
2. if T has a branch b, then $\mathbb{S}(T)$ adds an injection $f\colon b \to \omega$
   - ω_1 is collapsed to ω.
3. If T is Aronszajn, then $\mathbb{S}(T)$ does not collapse cardinals
   - it has the c.c.c. (difficult)

Theorem (Baumgartner-Malitz-Reinhardt)

There exists a model of ZFC in which

there exists an $\aleph_1$-Aronszajn tree, (always true)

all $\aleph_1$-Aronszajn trees are special.

1. Start with a model of $2^{\aleph_1} = \aleph_2$.
2. Use a finite support iteration of length ω_2 to specialize $\aleph_1$-Aronszajn trees.
3. The iteration is c.c.c. (f.s.i. of c.c.c. forcings is c.c.c.)
4. $2^{\aleph_1} = \aleph_2$ stays true during the iteration.
5. Since $\aleph_1$-Aronszajn trees have size $\aleph_1$, there are only $\aleph_2$ many.
6. Use a bookkeeping to specialize all $\aleph_1$-Aronszajn trees.
7. If T is special, then it stays special.

Theorem (Baumgartner-Malitz-Reinhardt)

There exists a model of ZFC in which

there exists an $\aleph_1$-Aronszajn tree, (always true)

all $\aleph_1$-Aronszajn trees are special.

Back to $\aleph_2$-trees:

Theorem (Laver-Shelah)

There exists a model of ZFC in which

- *there exists an $\aleph_2$-Aronszajn tree,*
- *all $\aleph_2$-Aronszajn trees are special.*

In fact, their aim was to get a model of

$$CH + \text{no } \aleph_2\text{-Suslin tree.}$$

Theorem (Laver-Shelah)

There exists a model of ZFC in which

> *there exists an $\aleph_2$-Aronszajn tree,*
>
> *all $\aleph_2$-Aronszajn trees are special.*

Specializing $\aleph_1$-Aronszajn trees:

Definition

Let T be an $\aleph_1$-Aronszajn tree. Let $\mathbb{S}(T)$ be the forcing consisting of conditions p satisfying the following:

1. *$p\colon T \to \omega$ is a finite partial function*
2. *if $s, t \in \mathrm{dom}(p)$ and $s <_T t$, then $p(s) \neq p(t)$.*

The order is given by $q \leq p$ if $q \supseteq p$.

- $\mathbb{S}(T)$ has the c.c.c.

Theorem (Laver-Shelah)

There exists a model of ZFC in which

there exists an $\aleph_2$-Aronszajn tree,

all $\aleph_2$-Aronszajn trees are special.

Specializing $\aleph_2$-Aronszajn trees:

Definition

Let T be an $\aleph_2$-Aronszajn tree. Let $\mathbb{S}(T)$ be the forcing consisting of conditions p satisfying the following:

1. $p\colon T \to \omega_1$ *is a countable partial function*
2. *if* $s, t \in \mathrm{dom}(p)$ *and* $s <_T t$, *then* $p(s) \neq p(t)$.

The order is given by $q \leq p$ if $q \supseteq p$.

- Want $\mathbb{S}(T)$ to have the $\aleph_2$-c.c. (needs a weakly compact)

Theorem (Laver-Shelah)

There exists a model of ZFC in which

there exists an $\aleph_2$-Aronszajn tree,

all $\aleph_2$-Aronszajn trees are special.

Lemma (Laver-Shelah)

If κ_2 is weakly compact, then in the extension by $\mathrm{col}(\aleph_1, {<}\kappa_2)$, $\kappa_2 = \aleph_2$ and $\mathbb{S}(T)$ has the $\aleph_2$-c.c. for each $\aleph_2$-Aronszajn tree T.

1. Use a countable support iteration of length ω_3 and bookkeeping to specialize all $\aleph_2$-Aronszajn trees.
2. It is shown "by hand" that also the iteration has the $\aleph_2$-c.c.
3. Warning: there is no iteration theorem for countable support iterations with $\aleph_2$-c.c. iterands.

Theorem (Laver-Shelah)

There exists a model of ZFC in which

there exists an $\aleph_2$-Aronszajn tree,

all $\aleph_2$-Aronszajn trees are special.

There exists an $\aleph_2$-Aronszajn tree, because:

1. $\mathrm{col}(\aleph_1, <\kappa_2)$ collapses $2^{\aleph_0}$ to $\aleph_1$.
2. Both $\mathrm{col}(\aleph_1, <\kappa_2)$ and the subsequent iteration are σ-closed, so there are no new reals added.
3. So CH holds in the final model (which implies its existence).

Theorem

There exists a model of ZFC in which

there exists an $\aleph_2$-Aronszajn tree,

all $\aleph_2$-Aronszajn trees are special,

and there exists no $\aleph_2$-Kurepa tree and no $\aleph_1$-Kurepa tree.

How to avoid Kurepa trees?

Theorem (Silver)

Let λ be an inaccessible cardinal and $\mathbb{L} = \mathrm{col}(\aleph_n, <\lambda)$ with $n \geq 1$. There is no $\aleph_n$-Kurepa tree in the extension by $\mathbb{L}$.

Lemma (Silver)

Let $\mathbb{R}$ be a forcing which is $<\aleph_n$-closed. Then $\mathbb{R}$ does not add branches to $\aleph_n$-trees.

How to avoid Kurepa trees?

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Lemma (Silver)

Let $\mathbb{R}$ be a forcing which is $<\aleph_n$-closed. Then $\mathbb{R}$ does not add branches to $\aleph_n$-trees.

Lemma (Unger)

In V, let

- *$\mathbb{P}$ be a forcing which has the $\aleph_n$-c.c., and*
- *$\mathbb{R}$ be a forcing which is $<\aleph_n$-closed.*

In $V^{\mathbb{P}}$, let T be an $\aleph_n$-tree. Then forcing with $\mathbb{R}$ over $V^{\mathbb{P}}$ does not add branches to T.

How to avoid Kurepa trees?

Theorem (Silver)

Let λ be an inaccessible cardinal and $\mathbb{L} = \mathrm{col}(\aleph_n, <\lambda)$ with $n \geq 1$. There is no $\aleph_n$-Kurepa tree in the extension by $\mathbb{L}$.

The following is a generalization of Silver's Theorem:

Lemma

Let λ be an inaccessible cardinal and $\mathbb{L} = \mathrm{col}(\aleph_n, <\lambda)$ with $n \geq 1$. In $V^{\mathbb{L}}$, let $\mathbb{Q}$ be a forcing of size $\leq \aleph_n$ such that

either $\mathbb{Q}$ is $<\aleph_n$-distributive

or $\mathbb{Q}$ has the $\aleph_n$-c.c.

Then there is no $\aleph_n$-Kurepa tree in $V^{\mathbb{L}*\mathbb{Q}}$.

Lemma

Let λ be an inaccessible cardinal and $\mathbb{L} = \mathrm{col}(\aleph_n, {<}\lambda)$ with $n \geq 1$. In $V^{\mathbb{L}}$, let $\mathbb{Q}$ be a forcing of size $\leq \aleph_n$ such that

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Then there is no $\aleph_n$-Kurepa tree in $V^{\mathbb{L}*\mathbb{Q}}$.

- $\mathbb{L} = \mathrm{col}(\aleph_n, {<}\lambda) = \prod_{\alpha<\lambda} \mathrm{col}(\aleph_n, \alpha) = \mathbb{L}_{<\mu} * \mathbb{L}_{[\mu,\lambda)}$, with
  - $\mathbb{L}_{<\mu} = \mathrm{col}(\aleph_n, {<}\mu) = \prod_{\alpha<\mu} \mathrm{col}(\aleph_n, \alpha)$,
  - $\mathbb{L}_{[\mu,\lambda)} = \prod_{\mu\leq\alpha<\lambda} \mathrm{col}(\aleph_n, \alpha)$.
- Let T be an $\aleph_n$-tree in $V^{\mathbb{L}*\mathbb{Q}}$.
- We can fix $\mu < \lambda$ such that
  - $\mathbb{Q} \in V^{\mathbb{L}_{<\mu}}$, and
  - $T \in V^{\mathbb{L}_{<\mu}*\mathbb{Q}}$.
- Note that, in $V^{\mathbb{L}_{<\mu}*\mathbb{Q}}$, we have
  - $2^{\aleph_n} < \lambda$,
  - and hence $|[T]| < \lambda$.

Lemma

Let λ be an inaccessible cardinal and $\mathbb{L} = \mathrm{col}(\aleph_n, <\lambda)$ with $n \geq 1$. In $V^{\mathbb{L}}$, let $\mathbb{Q}$ be a forcing of size $\leq \aleph_n$ such that

either $\mathbb{Q}$ is $<\aleph_n$-distributive

or $\mathbb{Q}$ has the $\aleph_n$-c.c.

Then there is no $\aleph_n$-Kurepa tree in $V^{\mathbb{L}*\mathbb{Q}}$.

- $V^{\mathbb{L}_{<\mu}*\mathbb{Q}} \models |[T]| < \lambda$.
- $\mathbb{L} = \mathbb{L}_{<\mu} * \mathbb{L}_{[\mu,\lambda)}$.
- Now $\mathbb{L} * \mathbb{Q}$ is forcing equivalent to $\mathbb{L}_{<\mu} * \mathbb{Q} * \check{\mathbb{L}}_{[\mu,\lambda)}$.
  1. If $\mathbb{Q}$ is $<\aleph_n$-distributive, $\mathbb{L}_{[\mu,\lambda)}$ is $<\aleph_n$-closed in $V^{\mathbb{L}_{<\mu}*\mathbb{Q}}$. So, by Silver's Lemma, it does not add branches to T.
  2. If $\mathbb{Q}$ has the $\aleph_n$-c.c., Unger's Lemma implies that $\mathbb{L}_{[\mu,\lambda)}$ does not add branches to T.
- Thus T has less than $\lambda = \aleph_{n+1}$ many branches in $V^{\mathbb{L}*\mathbb{Q}}$, so T is not a Kurepa tree.

Theorem

There exists a model of ZFC in which

there exists an $\aleph_2$-Aronszajn tree,

all $\aleph_2$-Aronszajn trees are special,

and there exists no $\aleph_2$-Kurepa tree and no $\aleph_1$-Kurepa tree.

- Start with a ground model in which
  - κ_2 is supercompact, and
  - κ_3 is weakly compact, with $\kappa_3 > \kappa_2$.
- Force with $\mathbb{L}_2 * \mathbb{L}_3 := \mathrm{col}(\aleph_1, <\kappa_2) * \mathrm{col}(\kappa_2, <\kappa_3)$, so κ_2 becomes $\aleph_2$, and κ_3 becomes $\aleph_3$.
- Then use forcings $\mathbb{S}(T)$ to specialize all $\aleph_2$-Aronszajn trees.
  - Use a countable support iteration of length ω_3 and bookkeeping to get a model in which all of them are special.
  - Denote the specializing iteration by $\mathbb{S}_{\omega_3}$
- CH holds in final model: so there exists an $\aleph_2$-Aronszajn tree.

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- All iterands $\mathbb{S}(T)$ and the whole iteration $\mathbb{S}_{\omega_3}$ have the $\aleph_2$-c.c.
  - Use the supercompact embedding of κ_2 together with a combinatorial argument (quite involved).
- It remains to show that in the final model $V^{\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\omega_3}}$,
  1. there are no $\aleph_1$-Kurepa trees, and
  2. there are no $\aleph_2$-Kurepa trees.

No $\aleph_1$-Kurepa trees

Lemma (Capturing $\aleph_1$-trees by suitable subforcings)

Let T be an $\aleph_1$-tree in $V^{\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\omega_3}}$. Then, in $V^{\mathbb{L}_2}$, there exists $\mathbb{Q}$ such that

1. *$\mathbb{Q}$ is a regular subforcing of $\mathbb{L}_3 * \mathbb{S}_{\omega_3}$ such that $T \in V^{\mathbb{L}_2 * \mathbb{Q}}$,*
2. *$|\mathbb{Q}| < \aleph_2$,*
3. *$\mathbb{Q}$ is σ-closed, and*
4. *$(\mathbb{L}_3 * \mathbb{S}_{\omega_3})/\mathbb{Q}$ is σ-closed.*

Lemma

Let λ be an inaccessible cardinal and $\mathbb{L} = \mathrm{col}(\aleph_n, {<}\lambda)$ with $n \geq 1$. In $V^{\mathbb{L}}$, let $\mathbb{Q}$ be a forcing of size $\leq \aleph_n$ such that

(either) $\mathbb{Q}$ is ${<}\aleph_n$-distributive

Then there is no $\aleph_n$-Kurepa tree in $V^{\mathbb{L} * \mathbb{Q}}$.

No $\aleph_1$-Kurepa trees

Lemma (Capturing $\aleph_1$-trees by suitable subforcings)

Let T be an $\aleph_1$-tree in $V^{\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\omega_3}}$. Then, in $V^{\mathbb{L}_2}$, there exists $\mathbb{Q}$ such that

1. *$\mathbb{Q}$ is a regular subforcing of $\mathbb{L}_3 * \mathbb{S}_{\omega_3}$ such that $T \in V^{\mathbb{L}_2 * \mathbb{Q}}$,*
2. *$|\mathbb{Q}| < \aleph_2$,*
3. *$\mathbb{Q}$ is σ-closed, and*
4. *$(\mathbb{L}_3 * \mathbb{S}_{\omega_3})/\mathbb{Q}$ is σ-closed.*

- Apply the generalization of Silver's Theorem: T is not an $\aleph_1$-Kurepa tree in $V^{\mathbb{L}_2 * \mathbb{Q}}$.
- By (4), using Silver's Lemma, $(\mathbb{L}_3 * \mathbb{S}_{\omega_3})/\mathbb{Q}$ does not add branches to T.
- So T is not an $\aleph_1$-Kurepa tree in $V^{\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\omega_3}}$.

No $\aleph_2$-Kurepa trees

We use the following lemma:

Lemma (Mitchell)

Let $\mathbb{P}$ be a forcing where $\mathbb{P} \times \mathbb{P}$ has the $\aleph_n$-c.c.. Then $\mathbb{P}$ does not add branches to trees of height $\aleph_n$.

- Let $T \in V^{\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\omega_3}}$ be an $\aleph_2$-tree.
- Since $\mathbb{S}_{\omega_3}$ has the $\aleph_2$-c.c., there exists $\alpha < \omega_3$ such that $T \in V^{\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha}$.

Lemma

Let λ be an inaccessible cardinal and $\mathbb{L} = \mathrm{col}(\aleph_n, {<}\lambda)$ with $n \geq 1$. In $V^{\mathbb{L}}$, let $\mathbb{Q}$ be a forcing of size $\leq \aleph_n$ such that

(or) $\mathbb{Q}$ has the $\aleph_n$-c.c.

Then there is no $\aleph_n$-Kurepa tree in $V^{\mathbb{L} * \mathbb{Q}}$.

No $\aleph_2$-Kurepa trees

We use the following lemma:

Lemma (Mitchell)

Let $\mathbb{P}$ be a forcing where $\mathbb{P} \times \mathbb{P}$ has the $\aleph_n$-c.c.. Then $\mathbb{P}$ does not add branches to trees of height $\aleph_n$.

- Let $T \in V^{\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\omega_3}}$ be an $\aleph_2$-tree.
- Since $\mathbb{S}_{\omega_3}$ has the $\aleph_2$-c.c., there exists $\alpha < \omega_3$ such that $T \in V^{\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha}$.
- Since $|\mathbb{S}_\alpha| \leq \aleph_2$, by the generalization of Silver's Theorem, T is not an $\aleph_2$-Kurepa tree in $V^{\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha}$.
- $\mathbb{S}_{\omega_3}/\mathbb{S}_\alpha$ has the $\aleph_2$-c.c., and even $\mathbb{S}_{\omega_3}/\mathbb{S}_\alpha \times \mathbb{S}_{\omega_3}/\mathbb{S}_\alpha$ has the $\aleph_2$-c.c., so $\mathbb{S}_{\omega_3}/\mathbb{S}_\alpha$ does not add branches to T.
- So T is not an $\aleph_2$-Kurepa tree in $V^{\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\omega_3}}$.

Theorem

It follows from ω-many supercompact cardinals that there exists a model of ZFC in which for all $0 < n \in \omega$

there exists an $\aleph_n$-Aronszajn tree,

all $\aleph_n$-Aronszajn trees are special,

and there exists no $\aleph_n$-Kurepa tree.

- Use an iteration of Lévy collapses to make the supercompact cardinals become the $\aleph_n$'s.
- Use forcings to specialize all $\aleph_n$-Aronszajn trees in a mixed support iteration.
- The iteration can be factorized into a forcing which is $<\aleph_n$-closed, followed by a forcing which has the $\aleph_n$-c.c.
- Capture an $\aleph_n$-tree with a subforcing of size at most $\aleph_n$.
- Show that it is not an $\aleph_n$-Kurepa tree here.
- Show that the quotient forcing does not add branches.

Theorem

It follows from a proper class of supercompact cardinals that there exists a model of ZFC in which for all regular cardinals κ

> *there exists a κ^+-Aronszajn tree,*
>
> *all κ^+-Aronszajn trees are special,*
>
> *and there exists no κ^+-Kurepa tree.*

- Use an Easton support iteration to combine the forcings which work for ω-many successive regular cardinals.

Thank you!