Coloring isosceles triangles countably in $\mathbb{R}^2$ but not in $\mathbb{R}^3$

Yuxin Zhou

University of Florida

Dec 14, 2021

0 Outline

1 Outline

1 Historical partition/coloring problems (ZFC)

Theorem

there is a partition of $\mathbb{R}^n$ into countably many sets no one of which contains the vertices of an isosceles triangle.

Given Axiom of Choice and the Continuum Hypothesis:

$n = 1$ case was proved by Erdős and Kakutani (1943).

$n = 2$ case was proved by by Davies (1972)

Any n case was proved by Kunen (1987)

Given Axiom of Choice only:

Any n case was proved by J. H. Schmerl (1996)

1 Historical partition/coloring problems (ZFC)

Theorem (Schuml, 1993)

There is a partition of $\mathbb{R}^n$ into countably many sets no one of which contains the vertices of an equilateral triangle.

Theorem (Erdős and Komjáth, 1990)

"There is a partition of the plane into countably many sets no one of which contains the vertices of an right-angled triangle" is equivalent to the Continuum Hypothesis.

Later, Schmerl provided a complete classification:

Theorem (Schmerl, 2000)

The chromatic number of any algebraic hypergraph Γ on an Euclidean space satisfies a trichotomy: ZFC proves

- *the chromatic number is countable, or*
- *the chromatic number is uncountable, or*
- *there is a natural number $m \in \omega$ such that the chromatic number is countable if and only if $2^{\aleph_0} \leq \aleph_m$;*

In addition, there is a computer program which, for a given algebraic equation, outputs the number m for the hypergraph given by the equation.

1 In the choiceless context

Trying to do a parallel categorization in terms of chromatic numbers is much more difficult without AC.

Theorem (Z., 2021)

It is consistent relative to an inaccessible cardinal that ZF+DC holds, Γ_2 has countable chromatic number while Γ_3 has uncountable chromatic number.

There are two main difficulties:

- "$+$" part: invent the suitable balanced analytic poset which will add a countable total coloring.
- "$-$" part: refine the poset to be fine enough so that it doesn't add a countable coloring for another hypergraph (often by investigating the geometric, algebraic, or combinatorial differences).

2 Outline

2 (A) Solovay model

Theorem (Solovay, 1970)

From a model V of ZFC with an inaccessible cardinal, one can build a model of ZF, DC and all sets of reals are Lebesgue measurable and have the Baire and perfect set properties.

Definition (Solovay, 1970)

Let κ be an inaccessible cardinal and let $P_C = \mathrm{Coll}(\omega, <\kappa)$. Let filter G be P_C-generic over V. The symmetric Solovay model, denoted as W, is $\mathrm{HOD}(V \cup \omega^\omega)$ as computed in $V[G]$.

2 (B) Balanced forcing

- The balanced forcings were invented in the new book *Geometric Set Theory* (2020) by Larson and Zapletal.
- It provides independence theorems in choiceless set theory in such a form "it is consistent relative to ZF+CD that ϕ holds while ψ does not hold" where ϕ, ψ are both Σ^2_1 sentences, typically consequences of AC or AC+CH.

Σ^2_1 sentences is of the form $\exists A \subset \ X\phi(A)$ where X is a Polish space and ϕ quantifies only over natural numbers and elements of X.

- If G is a Borel graph on X: the statement G has countable chromatic number;
- If X is a vector space over a countable field: X has a basis;
- the Continuum Hypothesis.

2 (B) Balanced forcing

Definition

Let P be a analytic forcing. P is balanced if for every condition $p \in P$ there is a balanced virtual condition below p.

Given some algebraic (hyper)graphs Γ, Γ' on $\mathbb{R}^n$, we want to find a (cofinally) balanced $\mathbb{R}$-analytic poset $(P, \leq)$ to force over the symmetric Solovay model W to get the model $W[G]$, in which Γ has countable chromatic number while Γ' doesn't.

2 (B) Balanced forcing

$(P, \leq)$ need to satisfy:

- P is a set of some countable approximations of a countable colorings of Γ, ordered by strengthened reverse extension $\leq$.
- P is transitive and σ-closed;
- P is analytic;
- $\forall p \in P$, $\mathrm{supp}(p) \subset \mathbb{R}$ is a countable real-closed subfield;
- $\forall p \in P, \forall x \in X$, $\exists q \in P$ that $x \in \mathrm{dom}(q)$ $q \leq p$;
- P is (cofinally) balanced:
  In some situations, $\forall p_1, p_2 \leq p_0 \in P$, $\exists q \in P$ that $q \leq p_1, p_2$;

2 (B) Balanced forcing

Theorem (Larson and Zapletal, 2020)

In all (cofinally) balanced extensions of the symmetric Solovay model W, every well-ordered sequence of elements of W belongs to W.

Balanced Suslin/analytic forcings have a limitation:

Theorem (Zapletal, 2020)

There is no (cofinally) balanced Suslin/analytic forcing which adds a total countable coloring for right-angled triangle hypergraphs.

2 (C) Real-closed fields

A Real-closed field: $\langle \mathbb{R}, \leq, +, \cdot \rangle$:

- it's a field
- $\leq$ is invarient under addition and multiplication by positive numbers
- -1 is not a sum of squares
- every odd polynomial has a root

Fact

- *it's a complete theory;*
- *it satisfies quantifier elimination*

Theorem (Marker)

if $F \subset \mathbb{R}$ is a real-closed subfield, then F is an elementrary submodel of $\mathbb{R}$.

3 Outline

3 Definition

Let Γ_2 be the hypergraph of isosceles triangles on $\mathbb{R}^2$.
A countable coloring for Γ_2 is a function from $\mathbb{R}^2$ to the natural numbers such that no vertices of any isosceles triangle all get the same color.

Fix an ideal $\mathcal{I}$ on ω that contains all finite sets and it is not generated by countably many sets.

Definition

Define P to be the poset of all conditions p such that

1. $\mathrm{supp}(p)$ is a countable real closed subfield of $\mathbb{R}$ and $\mathrm{supp}(p)^2 = \mathrm{dom}(p) \subset \mathbb{R}^2$;
2. $p : \mathrm{dom}(p) \longrightarrow \omega$ is a Γ_2-coloring;
3. (the symmetrical colors requirement) for each $l \in E^L(\mathrm{dom}(p))$, $s(p, l) \in \mathcal{I}$ where $s(p, l) = \{i \in \omega : \text{there are points } y_0, y_1 \text{ symmetrical with respect to } l \text{ and } p(y_0) = p(y_1) = i\}$;

Definition (continue)

And the ordering $\leq$ is defined by $p_1 \leq p_0$ if:

4. $\mathrm{dom}(p_1) \supset \mathrm{dom}(p_0)$;
5. $p_1|_{\mathrm{dom}(p_0)} = p_0$;
6. (the symmetrical colors invariant requirement) for each line $l \in E^L(\mathrm{dom}(p_0))$, $s(p_0, l) = s(p_1, l)$;
7. (the avoid center requirement) for each circle $e \in E^O(\mathrm{dom}(p_0))$, for each point $x \in e \cap (\mathrm{dom}(p_1) \setminus \mathrm{dom}(p_0))$, $p_1(x) \neq p_0(t)$ where t is the center of e;
8. (the algebraic points requirement) for every finite set $a \subset \mathrm{supp}(p_1)$, the p_1-image of the set $\{x \in \mathrm{dom}(p_1) \setminus \mathrm{dom}(p_0)\colon\ x \text{ is algebraic over } \mathrm{supp}(p_0) \cup a\}$ is in $\mathcal{I}$.

4 Outline

4 The Main Theorem

Theorem (Z., 2021)

Let κ be an inaccessible cardinal. There is a model of ZF + DC in which Γ_2 has countable chromatic number while for every non-meager set $A \subset \mathbb{R}^3$, A contains all vertices of an isosceles triangle.

4 Proof

Let W be the symmetric Solovay model derived from κ, and the model we need is the P extension over W, and call it $W[G]$, in which:

- "$+$": Γ_2 has countable chromatic number by our construction of P.
- "$-$": for every non-meager set $A \subset \mathbb{R}^3$, A contains all vertices of an isosceles triangle.

4 Proof

Work in W. Suppose $p \in P$ is a condition and let τ be a P-name for a nonmeager subset of $\mathbb{R}^3$. We will find a Γ_3-hyperedge e and a strengthening of the condition p which forces e to be a subset of τ.

By the definition of the symmetric Solovay model W, the condition p as well as the name τ must be definable from ground model parameters and an additional parameter $z \in 2^\omega$.

Let $V[K]$ be an intermediate extension obtained by a forcing of cardinality smaller than κ such that $z, p \in V[K]$ and that the Continuum Hypothesis holds.

(P is balanced in $V[K]$ by CH).

4 Proof

Work in the model $V[K]$. Let $\bar{p} \leq p$ be a balanced virtual condition in the poset P. Consider the Cohen poset $P_{\mathbb{R}^3}$ consisting of nonempty open subsets of $\mathbb{R}^3$ ordered by inclusion, with its name $\dot{x}_{gen}$ for a generic element of the space $\mathbb{R}^3$.

Claim

there must be a condition $O \in P_{\mathbb{R}^3}$, a poset Q of cardinality smaller than κ, and a $P_{\mathbb{R}^3} \times Q$-name σ for a condition in the poset P stronger than $\bar{p}$ such that $O \Vdash_{P_{\mathbb{R}^3}} Q \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \dot{x}_{gen} \in \tau$

Otherwise, in the model W , the virtual condition $\bar{p}$ would force τ to be disjoint from the comeager set of points in $\mathbb{R}^3$ which are Cohen generic over the model $V[K]$, contradicting the initial assumption that the name τ forced to be a non-meager set.

4 Proof

Consider the Cohen forcing poset $P_{\bar{\Gamma}_3}$ associated with the Polish space $\bar{\Gamma}_3$, and observe that O^3 is a condition in $P_{\bar{\Gamma}_3}$. Let e to be the a generic point of $P_{\bar{\Gamma}_3}$ that meets O^3. Observe that for each vertex $x \in e$, x is $P_{\mathbb{R}^3}$-generic, and each pair of vertices in e is mutually generic, by the openness of some projection maps.

Let $H_x \subset Q$ for $x \in e$ be filters mutually generic over the model $V[K][e]$. Work in the model $V[K][e][H_x : x \in e]$.

Observe that

- the models $V[K][x][H_x]$ for $x \in e$ form a Δ-system with the root $V[K]$;
- there is no isosceles triangle in $\mathbb{R}^2$ with one vertex in each model $V[K][x][H_x]$ for $x \in e$ respectively.

Claim

Whenever for each vertex $x \in e$, p_x is a condition in $V[K][e][H_x]$ such that $p_x \leq \bar{p}$, the conditions p_x for $x \in e$ have a common lower bound.

For each $x \in e$, write $p_x = \sigma/x$, which is a condition in $V[K][x][H_x]$.
By forcing theorem applied in the model $V[K][x][H_x]$, $p_x \leq \bar{p}$ is a condition forcing $x \in \tau$. By Claim, let q be a lower bound of the conditions p_x for $x \in e$.
We have produced a Γ_3-hyperedge e and a condition q stronger than p which forces all vertices of e to be in τ, as required.

5 Outline

5 "+": Amalgamation and Balance

Claim

P has amalgamation property; therefore P is balanced under CH.

5 “−”: Open maps and hyperedge forcing

Let $\bar{\Gamma}_3 \subset (\mathbb{R}^3)^3$ be an ordered hypergraph for Γ_3 with each middle coordinate to the the pivot.
It is G_δ subset of $(\mathbb{R}^3)^3$, hence a Polish space.
Consider the Cohen forcing poset $P_{\bar{\Gamma}_3}$ adding a generic ordered hyperedge for $\bar{\Gamma}_3$.

Proposition

Let $G \subset (\mathbb{R}^n)^m$ be G_δ and invariant under similarities of $\mathbb{R}^n$. Then the projection of G into any two coordinates is an open map to $(\mathbb{R}^n)^2$.

5 "—": Open maps and hyperedge forcing

Corollary

The projection of $\bar{\Gamma}_3$ to any two coordinates is an open map.

Corollary

Each pair of points in e are mutually generic over V for the product forcing $P_{\mathbb{R}^3} \times P_{\mathbb{R}^3}$.

5 “−”: Open maps and duplicated hyperedge forcing

This is for proving the claim in the main proof:

Claim

There is no isosceles triangle in $\mathbb{R}^2$ with one vertex in each model $V[K][x][H_x]$ for $x \in e$ respectively.

Definition

Define $G^k \subset (\mathbb{R}^n)^k \times (\mathbb{R}^n)^{m-1} = (\mathbb{R}^n)^{m+k-1}$ to be the ordered hypergraph of arity $m+k-1$ such that $(x_{0i}, x_j)_{i \in k, j \in m-1} \in G^k$ if $(\forall i \in k \ \ (x_{0i}, x_j)_{j \in m-1} \in G$ and all points $x_{0i}, i \in k$ are all in general position). We say that the hypergraph G^k is obtained from G by the duplication of its 0th coordinate k times.

5 "$-$": Open maps and duplicated hyperedge forcing

Proposition

The projection of $\bar{\Gamma}_3^4$ to the first four coordinates is an open map to $(\mathbb{R}^3)^4$, since G is stable against 4 vertices;
the projection to the last three coordinates is an open map to $\bar{\Gamma}_3$;
the projection to any two coordinates is an open map to $(\mathbb{R}^3)^2$;

Corollary

Let $e^4 = (x_{0i}, x_1, x_2 : i \in 4)$ be generic over V for the poset $P_{\bar{\Gamma}_3^4}$. Then

- *the first four points in e^4 are mutually generic over V for the product forcing $P^4_{\mathbb{R}^3}$.*
- *any two points in e^4 are mutually generic over V for the product forcing $P_{\mathbb{R}^3} \times P_{\mathbb{R}^3}$.*
- *$\forall i \in 4$, (x_{0i}, x_1, x_2) is generic over V for the poset $P_{\bar{\Gamma}_3}$.*

5 "−": Open maps and duplicated hyperedge forcing

Definition

Let $G \subset (\mathbb{R}^n)^m$. Let us say that G is stable against k vertices if the following holds. Whenever $x = \{x_i\}_{i \in k} \subset \mathbb{R}^n$ is a set of points in general position, there is an open neighborhood $U \subset (\mathbb{R}^n)^k$ of the origin and a continuous functional $\Psi : U \to (\mathbb{R}^n)$ such that for each $u = (u_i)_{i \in k} \in U$:

1. $\Psi(u)$ is a similarity of $\mathbb{R}^n$;
2. $\Psi(0)$ is the identity map;
3. $\forall y_1, y_2, \cdots, y_{m-1} \in \mathbb{R}^n$, if $G(-, y_1, y_2, \cdots, y_{m-1}) \supset x$, then
   - $\Psi(u)$ moves the section $G(-, y_1, y_2, \cdots, y_{m-1})$ to the section $G(-, \Psi(u)(y_1), \Psi(u)(y_2), \cdots, \Psi(u)(y_{m-1}))$;
   - $G(-, \Psi(u)(y_1), \Psi(u)(y_2), \cdots, \Psi(u)(y_{m-1})) \supset \{x_i + u_i\}_{i \in k}$.

We call Ψ a *similarity functional transferring points* $x_i, i \in k$ *for* G.

5 The paper

(submitted to JSL):
https://people.clas.ufl.edu/yuxinzhou/publications/

My email:
yuxinzhou@ufl.edu