Custom-made Souslin trees

Ari Meir Brodsky ©

Bar-Ilan University

Thursday, January 21, 2016
Kurt Gödel Research Center, Vienna

This is joint work with Assaf Rinot, and still in progress.
This project is partially supported by German-Israeli Foundation for Scientific Research and Development,
Grant No. I-2354-304.6/2014.

Souslin trees — higher cardinals

Recall:

Definition

For any regular cardinal κ, a tree T is κ-Souslin if:

- it has height κ,
- it has no chain of size κ,
- it has no antichain of size κ.

Souslin trees — higher cardinals

Recall:

Definition

For any regular cardinal κ, a tree T is κ-Souslin if:

- it has height κ,
- it has no chain of size κ,
- it has no antichain of size κ.

What does it take to construct a κ-Souslin tree for arbitrary regular cardinal κ?

What does it take to build a κ-Souslin tree?

Theorem (Jensen, 1972)

- *Suppose λ is a regular cardinal.*
  Assuming $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(E^{\lambda^+}_{\lambda})$, there exists a λ^+-Souslin tree.

What does it take to build a κ-Souslin tree?

Theorem (Jensen, 1972)

- *Suppose* λ *is a regular cardinal.*
  Assuming $\lambda^{<\lambda} = \lambda$ *and* $\diamondsuit(E^{\lambda^+}_{\lambda})$*, there exists a* λ^+*-Souslin tree.*
- *Suppose* λ *is a singular cardinal.*
  Assuming GCH *and* $\square_\lambda$*, there exists a* λ^+*-Souslin tree.*

What does it take to build a κ-Souslin tree?

Theorem (Jensen, 1972)

- *Suppose* λ *is a regular cardinal.* *Assuming* $\lambda^{<\lambda} = \lambda$ *and* $\diamondsuit(E^{\lambda^+}_{\lambda})$, *there exists a* λ^+*-Souslin tree.*
- *Suppose* λ *is a singular cardinal.* *Assuming* GCH *and* $\square_\lambda$, *there exists a* λ^+*-Souslin tree.*
- *If* $V = L$, *then for every regular uncountable cardinal* κ *that is not weakly compact, there exists a* κ*-Souslin tree.*

Weakening the axioms

We write CH_λ for the assertion that $2^\lambda = \lambda^+$.

Theorem (Gregory, 1976)

If $\lambda^{<\lambda} = \lambda$, CH_λ, *and there exists a non-reflecting stationary subset of* $E^{\lambda^+}_{<\lambda}$, *then there exists a* λ^+*-Souslin tree.*

Weakening the axioms

We write CH_λ for the assertion that $2^\lambda = \lambda^+$.

Theorem (Gregory, 1976)

If $\lambda^{<\lambda} = \lambda$, CH_λ, *and there exists a non-reflecting stationary subset of* $E^{\lambda^+}_{<\lambda}$, *then there exists a* λ^+*-Souslin tree.*

Theorem (Shelah, 1984)

If CH *holds and* NS *is saturated, then there exists an* $\aleph_2$*-Souslin tree.*

Weakening the axioms

We write CH_λ for the assertion that $2^\lambda = \lambda^+$.

Theorem (Gregory, 1976)

If $\lambda^{<\lambda} = \lambda$, CH_λ, and there exists a non-reflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then there exists a λ^+-Souslin tree.

Theorem (Shelah, 1984)

If CH holds and NS is saturated, then there exists an $\aleph_2$-Souslin tree.

Theorem (Ben-David & Shelah, 1986)

If $\boxminus_{\lambda,\geq\chi}$ and CH_λ for cardinals $\chi < \lambda$ where λ is a singular strong limit cardinal, then there exists a λ^+-Souslin tree.

Weakening the axioms

We write CH_λ for the assertion that $2^\lambda = \lambda^+$.

Theorem (Gregory, 1976)

If $\lambda^{<\lambda} = \lambda$, CH_λ, and there exists a non-reflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then there exists a λ^+-Souslin tree.

Theorem (Shelah, 1984)

If CH holds and NS is saturated, then there exists an $\aleph_2$-Souslin tree.

Theorem (Ben-David & Shelah, 1986)

If $\boxminus_{\lambda,\geq\chi}$ and CH_λ for cardinals $\chi < \lambda$ where λ is a singular strong limit cardinal, then there exists a λ^+-Souslin tree.

Baumgartner proved that $\boxminus_{\lambda,\geq\chi}$ is consistent with the failure of $\square_\lambda$ and even $\square^*_\lambda$.

Weakening the axioms

Theorem (König, Larson & Yoshinobu, 2007)

If $\lambda^{<\lambda} = \lambda$, CH_λ, *and* $\curlywedge^*(\lambda, E^{\lambda^+}_\lambda)$ *holds for a regular uncountable cardinal* λ, *then there exists a* λ^+*-Souslin tree.*

Weakening the axioms

Theorem (König, Larson & Yoshinobu, 2007)

If $\lambda^{<\lambda} = \lambda$, CH_λ, and $\curlywedge^*(\lambda, E^{\lambda^+}_\lambda)$ holds for a regular uncountable cardinal λ, then there exists a λ^+-Souslin tree.

Theorem (Rinot, 2011)

If $\lambda^{<\lambda} = \lambda$ and $\langle\lambda\rangle^-_{E^{\lambda^+}_\lambda}$ holds for a regular uncountable cardinal λ, then there exists a λ^+-Souslin tree.

Weakening the axioms

Theorem (König, Larson & Yoshinobu, 2007)

If $\lambda^{<\lambda} = \lambda$, CH_λ, and $\curlywedge^*(\lambda, E^{\lambda^+}_\lambda)$ holds for a regular uncountable cardinal λ, then there exists a λ^+-Souslin tree.

Theorem (Rinot, 2011)

If $\lambda^{<\lambda} = \lambda$ and $\langle\lambda\rangle^-_{E^{\lambda^+}_\lambda}$ holds for a regular uncountable cardinal λ, then there exists a λ^+-Souslin tree.

These principles ($\curlywedge^*(\lambda, E^{\lambda^+}_\lambda)$, $\langle\lambda\rangle^-_{E^{\lambda^+}_\lambda}$) are consistent with the failure of $\diamondsuit(E^{\lambda^+}_\lambda)$.

Limitations of the classical constructions

- The classical constructions of κ-Souslin trees generally depend on the nature of κ: successor of regular, successor of singular, or inaccessible.

Limitations of the classical constructions

- The classical constructions of κ-Souslin trees generally depend on the nature of κ: successor of regular, successor of singular, or inaccessible.
- Constructions include extensive bookkeeping, counters, timers, coding and decoding whose particular nature makes it difficult to transfer the process from one cardinal to another.

Limitations of the classical constructions

- The classical constructions of κ-Souslin trees generally depend on the nature of κ: successor of regular, successor of singular, or inaccessible.
- Constructions include extensive bookkeeping, counters, timers, coding and decoding whose particular nature makes it difficult to transfer the process from one cardinal to another.
- For inaccessible κ, there is a dearth of axiom-based constructions.

Limitations of the classical constructions

- The classical constructions of κ-Souslin trees generally depend on the nature of κ: successor of regular, successor of singular, or inaccessible.
- Constructions include extensive bookkeeping, counters, timers, coding and decoding whose particular nature makes it difficult to transfer the process from one cardinal to another.
- For inaccessible κ, there is a dearth of axiom-based constructions.
- The classical $\diamondsuit$-based constructions all require $\diamondsuit$ to concentrate on a non-reflecting stationary set, in order to ensure that we don't get stuck when sealing antichains.

Limitations of the classical constructions

- The classical constructions of κ-Souslin trees generally depend on the nature of κ: successor of regular, successor of singular, or inaccessible.
- Constructions include extensive bookkeeping, counters, timers, coding and decoding whose particular nature makes it difficult to transfer the process from one cardinal to another.
- For inaccessible κ, there is a dearth of axiom-based constructions.
- The classical $\diamondsuit$-based constructions all require $\diamondsuit$ to concentrate on a non-reflecting stationary set, in order to ensure that we don't get stuck when sealing antichains.

Can we overcome these limitations?

Souslin trees with extra properties

What additional properties might a κ-Souslin tree satisfy?

Example: Free trees

Theorem (Kurepa)

The square of a κ-Souslin tree T cannot be κ-Souslin.

Example: Free trees

Theorem (Kurepa)

The square of a κ-Souslin tree T cannot be κ-Souslin.

How close can we get?

Example: Free trees

Theorem (Kurepa)

The square of a κ-Souslin tree T cannot be κ-Souslin.

How close can we get?

Definition

A κ-Souslin tree $\langle T, <_T\rangle$ is said to be free if for every nonzero $n < \omega$, any $\beta < \kappa$, and any sequence of distinct nodes $\langle w_0, \ldots, w_{n-1}\rangle \in {}^n T_\beta$, the derived tree $w_0{}^\uparrow \otimes \cdots \otimes w_{n-1}{}^\uparrow$ is again a κ-Souslin tree.

Example: Free trees

Theorem (Kurepa)

The square of a κ-Souslin tree T cannot be κ-Souslin.

How close can we get?

Definition

A κ-Souslin tree $\langle T, <_T\rangle$ is said to be free if for every nonzero $n < \omega$, any $\beta < \kappa$, and any sequence of distinct nodes $\langle w_0, \ldots, w_{n-1}\rangle \in {}^n T_\beta$, the derived tree $w_0{}^\uparrow \otimes \cdots \otimes w_{n-1}{}^\uparrow$ is again a κ-Souslin tree.

Here, the derived tree $w_0{}^\uparrow \otimes \cdots \otimes w_{n-1}{}^\uparrow$ stands for the tree $(\hat{T}, <_{\hat{T}})$, as follows:

- $\hat{T} = \{\langle z_0, \ldots, z_{n-1}\rangle \in {}^n T \mid \exists\delta < \kappa \forall i < n(z_i \in T_\delta$ and z_i is $<_T$-compatible with $w_i)\}$;
- $\vec{y} <_{\hat{T}} \vec{z}$ iff $y_i <_T z_i$ for all $i < n$.

Example: Free trees

Theorem (Kurepa)

The square of a κ-Souslin tree T cannot be κ-Souslin.

How close can we get?

Definition

A κ-Souslin tree $\langle T, <_T\rangle$ is said to be free if for every nonzero $n < \omega$, any $\beta < \kappa$, and any sequence of distinct nodes $\langle w_0, \ldots, w_{n-1}\rangle \in {}^nT_\beta$, the derived tree $w_0{}^\uparrow \otimes \cdots \otimes w_{n-1}{}^\uparrow$ is again a κ-Souslin tree.

Here, the derived tree $w_0{}^\uparrow \otimes \cdots \otimes w_{n-1}{}^\uparrow$ stands for the tree $(\hat{T}, <_{\hat{T}})$, as follows:

- $\hat{T} = \{\langle z_0, \ldots, z_{n-1}\rangle \in {}^nT \mid \exists\delta < \kappa \forall i < n(z_i \in T_\delta$ and z_i is $<_T$-compatible with $w_i)\}$;
- $\vec{y} <_{\hat{T}} \vec{z}$ iff $y_i <_T z_i$ for all $i < n$.

Can we construct a free κ-Souslin tree?

Example: Constructing free Souslin trees

Theorem (Jensen)

Assuming $\lambda^{<\lambda} = \lambda$ *and* $\diamondsuit(E^{\lambda^+}_{\lambda})$, *there exists a free* λ^+*-Souslin tree.*

Example: Constructing free Souslin trees

Theorem (Jensen)

Assuming $\lambda^{<\lambda} = \lambda$ *and* $\diamondsuit(E^{\lambda^+}_{\lambda})$*, there exists a* *free* λ^+*-Souslin tree.*

This is the only classical construction of a free Souslin tree.

Example: Constructing free Souslin trees

Theorem (Jensen)

Assuming $\lambda^{<\lambda} = \lambda$ *and* $\diamondsuit(E^{\lambda^+}_{\lambda})$, *there exists a free* λ^+*-Souslin tree.*

This is the only classical construction of a free Souslin tree.
What if we replace $\diamondsuit(E^{\lambda^+}_{\lambda})$ by the axiom $\curlywedge^*(\lambda, E^{\lambda^+}_{\lambda})$, which was sufficient to construct a λ^+-Souslin tree for regular λ?

Example: Constructing free Souslin trees

Theorem (Jensen)

Assuming $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(E^{\lambda^+}_\lambda)$, there exists a free λ^+-Souslin tree.

This is the only classical construction of a free Souslin tree.
What if we replace $\diamondsuit(E^{\lambda^+}_\lambda)$ by the axiom $\curlywedge^*(\lambda, E^{\lambda^+}_\lambda)$, which was sufficient to construct a λ^+-Souslin tree for regular λ?
Can we construct a κ-Souslin tree for κ inaccessible?

Example: Constructing free Souslin trees

Theorem (Jensen)

Assuming $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(E^{\lambda^+}_{\lambda})$, there exists a free λ^+-Souslin tree.

This is the only classical construction of a free Souslin tree.
What if we replace $\diamondsuit(E^{\lambda^+}_{\lambda})$ by the axiom $\curlywedge^*(\lambda, E^{\lambda^+}_{\lambda})$, which was sufficient to construct a λ^+-Souslin tree for regular λ?
Can we construct a κ-Souslin tree for κ inaccessible?
What about κ successor of singular?

Example: Constructing free Souslin trees

Theorem (Jensen)

Assuming $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(E^{\lambda^+}_\lambda)$, there exists a free λ^+-Souslin tree.

This is the only classical construction of a free Souslin tree.
What if we replace $\diamondsuit(E^{\lambda^+}_\lambda)$ by the axiom $\curlywedge^*(\lambda, E^{\lambda^+}_\lambda)$, which was sufficient to construct a λ^+-Souslin tree for regular λ?
Can we construct a κ-Souslin tree for κ inaccessible?
What about κ successor of singular?
What about successor of singular when $\square^*_\lambda$ fails (as in the Ben-David & Shelah scenario).

Example: Constructing free Souslin trees

Theorem (Jensen)

Assuming $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(E^{\lambda^+}_{\lambda})$, there exists a free λ^+-Souslin tree.

This is the only classical construction of a free Souslin tree.
What if we replace $\diamondsuit(E^{\lambda^+}_{\lambda})$ by the axiom $\curlywedge^*(\lambda, E^{\lambda^+}_{\lambda})$, which was sufficient to construct a λ^+-Souslin tree for regular λ?
Can we construct a κ-Souslin tree for κ inaccessible?
What about κ successor of singular?
What about successor of singular when $\square^*_{\lambda}$ fails (as in the Ben-David & Shelah scenario).
Are we going to go over each of these models and tailor each of these particular constructions in order to get a free Souslin tree?

Order in the jungle?

We have a zoo of consistent constructions of κ-Souslin trees! Construction of a κ-Souslin tree with any desired property seems to depend on the nature of κ, and in some cases even depends on whether κ is successor of a singular of countable or of uncountable cofinality.

Order in the jungle?

We have a zoo of consistent constructions of κ-Souslin trees!
Construction of a κ-Souslin tree with any desired property seems to depend on the nature of κ, and in some cases even depends on whether κ is successor of a singular of countable or of uncountable cofinality.
Does it have to be this way?

Looking for an alternative to $\square$

Notation

For any set of ordinals D:

$$\begin{aligned} \mathrm{acc}(D) &= \{\alpha \in D \mid \sup(D \cap \alpha) = \alpha > 0\}; \text{ and} \\ \mathrm{nacc}(D) &= D \setminus \mathrm{acc}(D). \end{aligned}$$

Looking for an alternative to $\square$

Notation

For any set of ordinals D:

$$\mathrm{acc}(D) = \{\alpha \in D \mid \sup(D \cap \alpha) = \alpha > 0\}; \text{ and}$$
$$\mathrm{nacc}(D) = D \setminus \mathrm{acc}(D).$$

Recall Jensen's square principle, designed to enable construction of λ^+-Souslin trees:

Definition (Jensen, 1972)

For an infinite cardinal λ, $\square_\lambda$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ such that:

- C_α is a club in α for all limit $\alpha < \lambda^+$;
- if $\bar\alpha \in \mathrm{acc}(C_\alpha)$, then $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$;

Looking for an alternative to $\square$

Notation

For any set of ordinals D:

$$\mathrm{acc}(D) = \{\alpha \in D \mid \sup(D \cap \alpha) = \alpha > 0\}; \text{ and}$$
$$\mathrm{nacc}(D) = D \setminus \mathrm{acc}(D).$$

Recall Jensen's square principle, designed to enable construction of λ^+-Souslin trees:

Definition (Jensen, 1972)

For an infinite cardinal λ, $\square_\lambda$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that:

- C_α is a club in α for all limit $\alpha < \lambda^+$;
- if $\bar{\alpha} \in \mathrm{acc}(C_\alpha)$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$;
- $\mathrm{otp}(C_\alpha) \leq \lambda$ for all $\alpha < \lambda^+$.

Looking for an alternative to $\square$

Why is $\square_\lambda$ not ideal for our purpose?

- It becomes trivial at the level of $\aleph_1$, that is, $\square_{\aleph_0}$ is always true, thus it provides no information to help us build $\aleph_1$-Souslin trees

Looking for an alternative to $\square$

Why is $\square_\lambda$ not ideal for our purpose?

- It becomes trivial at the level of $\aleph_1$, that is, $\square_{\aleph_0}$ is always true, thus it provides no information to help us build $\aleph_1$-Souslin trees
- It has no appropriate analogue for inaccessible cardinals

Looking for an alternative to $\square$

Why is $\square_\lambda$ not ideal for our purpose?

- It becomes trivial at the level of $\aleph_1$, that is, $\square_{\aleph_0}$ is always true, thus it provides no information to help us build $\aleph_1$-Souslin trees
- It has no appropriate analogue for inaccessible cardinals
- It is tied to non-reflecting stationary sets, which we want to be able to avoid

A new axiom: $\boxtimes^-(\kappa)$

Definition

Fix a regular uncountable cardinal κ. The principle $\boxtimes^-(\kappa)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that:

A new axiom: $\boxtimes^-(\kappa)$

Definition

Fix a regular uncountable cardinal κ. The principle $\boxtimes^-(\kappa)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa\rangle$ such that:

- C_α is a club subset of α for every limit ordinal $\alpha < \kappa$;

A new axiom: $\boxtimes^-(\kappa)$

Definition

Fix a regular uncountable cardinal κ. The principle $\boxtimes^-(\kappa)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that:

- C_α is a club subset of α for every limit ordinal $\alpha < \kappa$;
- $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ for all ordinals $\alpha < \kappa$ and $\bar\alpha \in \mathrm{acc}(C_\alpha)$;

A new axiom: $\boxtimes^-(\kappa)$

Definition

Fix a regular uncountable cardinal κ. The principle $\boxtimes^-(\kappa)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that:

- C_α is a club subset of α for every limit ordinal $\alpha < \kappa$;
- $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$ for all ordinals $\alpha < \kappa$ and $\bar{\alpha} \in \mathrm{acc}(C_\alpha)$;
- for every cofinal subset $B \subseteq \kappa$, there exist stationarily many $\alpha < \kappa$ satisfying

$$\sup(\mathrm{nacc}(C_\alpha) \cap B) = \alpha.$$

Building a κ-Souslin tree from $\diamondsuit(\kappa) + \boxtimes^-(\kappa)$

Theorem

For any regular uncountable cardinal κ, $\diamondsuit(\kappa) + \boxtimes^-(\kappa)$ implies the existence of a κ-Souslin tree.

Building a κ-Souslin tree from $\diamondsuit(\kappa) + \boxtimes^-(\kappa)$

Theorem

For any regular uncountable cardinal κ, $\diamondsuit(\kappa) + \boxtimes^-(\kappa)$ implies the existence of a κ-Souslin tree.

What will our tree $\langle T, <_T \rangle$ look like?

- $\langle T, <_T \rangle$ will be a normal downward-closed subtree of $\langle {}^{<\kappa}2, \subset \rangle$. In particular:

Building a κ-Souslin tree from $\diamondsuit(\kappa) + \boxtimes^-(\kappa)$

Theorem

For any regular uncountable cardinal κ, $\diamondsuit(\kappa) + \boxtimes^-(\kappa)$ implies the existence of a κ-Souslin tree.

What will our tree $\langle T, <_T \rangle$ look like?

- $\langle T, <_T \rangle$ will be a normal downward-closed subtree of $\langle {}^{<\kappa}2, \subset \rangle$. In particular:
- Each node $t \in T$ is a function $t : \alpha \to 2$ for some ordinal $\alpha < \kappa$;
- The tree order $<_T$ is simply extension of functions $\subset$;
- If $t : \alpha \to 2$ is in T, then $t \restriction \beta \in T$ for every $\beta < \alpha$.
- For all $t \in T$, $\mathrm{ht}(t) = \mathrm{dom}(t)$ and $t_\downarrow = \{t \restriction \beta \mid \beta < \mathrm{dom}(t)\}$.
- For all $\alpha < \kappa$, the level $T_\alpha = T \cap {}^{\alpha}2$.

Building a κ-Souslin tree from $\diamondsuit(\kappa) + \boxtimes^-(\kappa)$

Theorem

For any regular uncountable cardinal κ, $\diamondsuit(\kappa) + \boxtimes^-(\kappa)$ implies the existence of a κ-Souslin tree.

What will our tree $\langle T, <_T \rangle$ look like?

- $\langle T, <_T \rangle$ will be a normal downward-closed subtree of $\langle {}^{<\kappa}2, \subset \rangle$. In particular:
- Each node $t \in T$ is a function $t : \alpha \to 2$ for some ordinal $\alpha < \kappa$;
- The tree order $<_T$ is simply extension of functions $\subset$;
- If $t : \alpha \to 2$ is in T, then $t \restriction \beta \in T$ for every $\beta < \alpha$.
- For all $t \in T$, $\mathrm{ht}(t) = \mathrm{dom}(t)$ and $t_\downarrow = \{t \restriction \beta \mid \beta < \mathrm{dom}(t)\}$.
- For all $\alpha < \kappa$, the level $T_\alpha = T \cap {}^{\alpha}2$.
  Motivation: ease of completing a branch at a limit level.
  If $\langle t_\alpha \mid \alpha < \beta \rangle$ (for some $\beta < \kappa$) is a $\subseteq$-increasing sequence of nodes in T, then the (unique) limit of this sequence, which may or may not be a member of T, is simply $\bigcup_{\alpha<\beta} t_\alpha$.

Refining an old axiom: From $\diamondsuit(\kappa)$ to $\diamondsuit(H_\kappa)$

Fix a regular uncountable cardinal κ.

Definition (Jensen, 1972)

$\diamondsuit(\kappa)$ asserts the existence of a sequence $\langle Z_\beta \mid \beta < \kappa \rangle$ such that for every $Z \subseteq \kappa$, the set $\{\beta < \kappa \mid Z \cap \beta = Z_\beta\}$ is stationary in κ.

Refining an old axiom: From $\diamondsuit(\kappa)$ to $\diamondsuit(H_\kappa)$

Fix a regular uncountable cardinal κ.

Definition (Jensen, 1972)

$\diamondsuit(\kappa)$ asserts the existence of a sequence $\langle Z_\beta \mid \beta < \kappa\rangle$ such that for every $Z \subseteq \kappa$, the set $\{\beta < \kappa \mid Z \cap \beta = Z_\beta\}$ is stationary in κ.

Definition

$\diamondsuit(H_\kappa)$ asserts the existence of
a sequence $\langle \Omega_\beta \mid \beta < \kappa\rangle$ such that for every $p \in H_{\kappa^+}$ and $\Omega \subseteq H_\kappa$, there exists an elementary submodel $\mathcal{M} \prec H_{\kappa^+}$ such that

- $p \in \mathcal{M}$;
- $\mathcal{M} \cap \kappa \in \kappa$;
- $\mathcal{M} \cap \Omega = \Omega_{\mathcal{M}\cap\kappa}$.

Here, H_λ denotes the collection of all sets of hereditary cardinality less than λ.

Refining an old axiom: From $\diamondsuit(\kappa)$ to $\diamondsuit(H_\kappa)$

Fix a regular uncountable cardinal κ.

Definition (Jensen, 1972)

$\diamondsuit(\kappa)$ asserts the existence of a sequence $\langle Z_\beta \mid \beta < \kappa \rangle$ such that for every $Z \subseteq \kappa$, the set $\{\beta < \kappa \mid Z \cap \beta = Z_\beta\}$ is stationary in κ.

Definition

$\diamondsuit(H_\kappa)$ asserts the existence of a partition $\langle R_i \mid i < \kappa \rangle$ of κ and a sequence $\langle \Omega_\beta \mid \beta < \kappa \rangle$ such that for every $p \in H_{\kappa^+}$, $i < \kappa$, and $\Omega \subseteq H_\kappa$, there exists an elementary submodel $\mathcal{M} \prec H_{\kappa^+}$ such that

- $p \in \mathcal{M}$;
- $\mathcal{M} \cap \kappa \in R_i$;
- $\mathcal{M} \cap \Omega = \Omega_{\mathcal{M} \cap \kappa}$.

Here, H_λ denotes the collection of all sets of hereditary cardinality less than λ.

Refining an old axiom: From $\diamondsuit(\kappa)$ to $\diamondsuit(H_\kappa)$

Fix a regular uncountable cardinal κ.

Definition (Jensen, 1972)

$\diamondsuit(\kappa)$ asserts the existence of a sequence $\langle Z_\beta \mid \beta < \kappa \rangle$ such that for every $Z \subseteq \kappa$, the set $\{\beta < \kappa \mid Z \cap \beta = Z_\beta\}$ is stationary in κ.

Definition

$\diamondsuit(H_\kappa)$ asserts the existence of a partition $\langle R_i \mid i < \kappa \rangle$ of κ and a sequence $\langle \Omega_\beta \mid \beta < \kappa \rangle$ such that for every $p \in H_{\kappa^+}$, $i < \kappa$, and $\Omega \subseteq H_\kappa$, there exists an elementary submodel $\mathcal{M} \prec H_{\kappa^+}$ such that

- $p \in \mathcal{M}$;
- $\mathcal{M} \cap \kappa \in R_i$;
- $\mathcal{M} \cap \Omega = \Omega_{\mathcal{M} \cap \kappa}$.

Here, H_λ denotes the collection of all sets of hereditary cardinality less than λ.

Proposition

$\diamondsuit(\kappa)$ is equivalent to $\diamondsuit(H_\kappa)$.

Preliminaries

Let $\langle R_i \mid i < \kappa\rangle$ and $\langle \Omega_\beta \mid \beta < \kappa\rangle$ together witness $\diamondsuit(H_\kappa)$.
Fix a sequence $\langle C_\alpha \mid \alpha < \kappa\rangle$ witnessing $\boxtimes^-(\kappa)$.
Fix a well-ordering $\lhd$ on H_κ.

The easy part

Let $T_0 = \{\emptyset\}$.

The easy part

Let $T_0 = \{\emptyset\}$.
For every $\alpha < \kappa$, define

$$T_{\alpha+1} = \{s^\frown\langle i\rangle \mid s \in T_\alpha, i < 2\}.$$

The hard part

What do we do at limit levels?
Fix a limit ordinal $\alpha < \kappa$, and assume $T \restriction \alpha = \bigcup_{\beta<\alpha} T_\beta$ has already been defined.
We need to decide which branches through $T \restriction \alpha$ will have their limits placed in the level T_α of the tree.
We need T_α to contain enough nodes so that the tree is normal.
That is, for every $x \in T \restriction \alpha$, we need to place some node $\mathbf{b}_x^\alpha$ in T_α above x.
The node $\mathbf{b}_x^\alpha$ will be the limit of some sequence b_x^α in $T \restriction \alpha$.
But we have to choose these sequences carefully, so that the resulting tree doesn't have large antichains.

Identifying cofinal branches

Recall that C_α is a club subset of α.
For every $x \in T \restriction C_\alpha$, we will use C_α to identify a cofinal branch b^α_x through $\langle T \restriction \alpha, \subseteq \rangle$, containing x, as follows:

- b^α_x will be an increasing, continuous sequence of nodes.
- $\mathrm{dom}(b^\alpha_x) = C_\alpha \setminus \mathrm{ht}(x)$.
- $b^\alpha_x(\mathrm{ht}(x)) = x$.
- We will need to identify $b^\alpha_x(\beta) \in T_\beta$ for all $\beta \in C_\alpha$ with $\beta > \mathrm{ht}(x)$.
  We will do this by recursion over β, considering the cases $\beta \in \mathrm{nacc}(C_\alpha)$ and $\beta \in \mathrm{acc}(C_\alpha)$ in turn.

Intersecting a maximal antichain at levels in $\mathrm{nacc}(C_\alpha)$

Suppose $\beta \in \mathrm{nacc}(C_\alpha)$ with $\beta > \mathrm{ht}(x)$.
Denote $\beta^- = \max(C_\alpha \cap \beta)$.
This exists and is in $\mathrm{dom}(b^\alpha_x)$, so that $b^\alpha_x(\beta^-)$ has been defined.
We need to identify $b^\alpha_x(\beta) \in T_\beta$, extending $b^\alpha_x(\beta^-)$.
Consider two possibilities:

- If there is some $y \in \Omega_\beta$ and $z \in T_\beta$ such that $b^\alpha_x(\beta^-) \cup y \subseteq z$, then let $b^\alpha_x(\beta)$ be the $\lhd$-least such z.
- Otherwise, let $b^\alpha_x(\beta)$ be the $\lhd$-least element of T_β extending $b^\alpha_x(\beta^-)$. Such a node must exist, because we are ensuring that the tree is normal as we construct every level.

Notice that if Ω_β is a maximal antichain through $T \restriction \beta$, then in particular there is some $y \in \Omega_\beta \cap (T \restriction \beta)$ compatible with $b^\alpha_x(\beta^-)$, so that $b^\alpha_x(\beta^-) \cup y \in T \restriction \beta$, and then by normality there is $z \in T_\beta$ extending this, so that the first option applies.

Will we get stuck at levels in $\mathrm{acc}(C_\alpha)$?

Suppose $\beta \in \mathrm{acc}(C_\alpha)$ with $\beta > \mathrm{ht}(x)$.
We want b_x^α to be continuous, so the only possible definition is:

$$b_x^\alpha(\beta) = \bigcup_{\gamma \in \mathrm{dom}(b_x^\alpha) \cap \beta} b_x^\alpha(\gamma).$$

Clearly $b_x^\alpha(\beta) \in {}^\beta 2$, but how do we know that $b_x^\alpha(\beta) \in T_\beta$?
This question highlights the difference between the classical approach and our new framework.

Coherence to the rescue!

Since $\beta \in \mathrm{acc}(C_\alpha)$, our choice of the sequence satisfying $\boxtimes^-(\kappa)$ gives $C_\beta = C_\alpha \cap \beta$.

For every $\gamma \in \mathrm{dom}(b_x^\alpha) \cap \beta$, the value of $b_x^\beta(\gamma)$ was determined in exactly the same way as $b_x^\alpha(\gamma)$:

- starting with $b_x^\beta(\mathrm{ht}(x)) = x = b_x^\alpha(\mathrm{ht}(x))$;
- for $\gamma \in \mathrm{nacc}(C_\alpha)$: depending only on $b_x^\alpha(\gamma^-)$, Ω_γ, and T_γ;
- for $\gamma \in \mathrm{acc}(C_\alpha)$: taking limits.

It follows that

$$b_x^\alpha(\beta) = \bigcup_{\gamma \in \mathrm{dom}(b_x^\alpha) \cap \beta} b_x^\alpha(\gamma) = \bigcup_{\gamma \in \mathrm{dom}(b_x^\beta)} b_x^\beta(\gamma) = \mathbf{b}_x^\beta.$$

Since $\beta < \alpha$, the level T_β has already been constructed, and the construction guarantees that we have included the limit $\mathbf{b}_x^\beta$ of the sequence b_x^β into T_β. But we have just shown that this is exactly $b_x^\alpha(\beta)$, so that $b_x^\alpha(\beta) \in T_\beta$, as required.

Completing the construction of T_α

The sequence b^α_x just identified determines a cofinal branch through $T \restriction \alpha$ containing x.
As promised, we take its limit

$$\mathbf{b}^\alpha_x = \bigcup_{\beta \in \mathrm{dom}(b^\alpha_x)} b^\alpha_x(\beta),$$

which is an element of ${}^\alpha 2$.

Completing the construction of T_α

The sequence b_x^α just identified determines a cofinal branch through $T \restriction \alpha$ containing x.
As promised, we take its limit

$$\mathbf{b}_x^\alpha = \bigcup_{\beta \in \mathrm{dom}(b_x^\alpha)} b_x^\alpha(\beta),$$

which is an element of ${}^\alpha 2$.
Finally, we collect all nodes constructed in this way, and let

$$T_\alpha = \{\mathbf{b}_x^\alpha \mid x \in T \restriction C_\alpha\}.$$

Completing the construction of T_α

The sequence b_x^α just identified determines a cofinal branch through $T \restriction \alpha$ containing x.
As promised, we take its limit

$$\mathbf{b}_x^\alpha = \bigcup_{\beta \in \mathrm{dom}(b_x^\alpha)} b_x^\alpha(\beta),$$

which is an element of ${}^\alpha 2$.
Finally, we collect all nodes constructed in this way, and let

$$T_\alpha = \{\mathbf{b}_x^\alpha \mid x \in T \restriction C_\alpha\}.$$

Having constructed all levels of the tree, we then let

$$T = \bigcup_{\alpha < \kappa} T_\alpha.$$

Here we use $\diamondsuit(H_\kappa)$

Claim

Suppose $A \subseteq T$ is a maximal antichain. Then the set

$$B = \{\beta < \kappa \mid A \cap (T \restriction \beta) = \Omega_\beta \text{ is a maximal antichain in } T \restriction \beta\}.$$

is a stationary subset of κ.

Proof.

Let $D \subseteq \kappa$ be an arbitrary club. We must show that $D \cap B \neq \emptyset$. Put $p = \{A, T, D\}$. Using the fact that the sequence $\langle \Omega_\beta \mid \beta < \kappa \rangle$ satisfies $\diamondsuit(H_\kappa)$, pick $\mathcal{M} \prec H_{\kappa^+}$ with $p \in \mathcal{M}$ such that $\beta = \mathcal{M} \cap \kappa$ is in κ and $\Omega_\beta = A \cap \mathcal{M}$. Since $D \in \mathcal{M}$ and D is club in κ, we have $\beta \in D$. We claim that $\beta \in B$. For all $\alpha < \beta$, by $\alpha, T \in \mathcal{M}$, we have $T_\alpha \in \mathcal{M}$, and by $\mathcal{M} \models |T_\alpha| < \kappa$, we have $T_\alpha \subseteq \mathcal{M}$. So $T \restriction \beta \subseteq \mathcal{M}$. As $\mathrm{dom}(z) \in \mathcal{M}$ for all $z \in T \cap \mathcal{M}$, we conclude that $T \cap \mathcal{M} = T \restriction \beta$. So, $\Omega_\beta = A \cap (T \restriction \beta)$. As $H_{\kappa^+} \models A$ is a maximal antichain in T and $T \cap \mathcal{M} = T \restriction \beta$, we get that $A \cap (T \restriction \beta)$ is maximal in $T \restriction \beta$. $\square$

Verifying that T is κ-Souslin

Claim

The tree $\langle T, \subset \rangle$ is a κ-Souslin tree.

Proof.

Let $A \subseteq T$ be a maximal antichain. From the previous claim,

$$B = \{\beta < \kappa \mid A \cap (T \restriction \beta) = \Omega_\beta \text{ is a maximal antichain in } T \restriction \beta\}$$

is a stationary subset of κ.
Thus we apply $\boxtimes^-(\kappa)$ to obtain a limit ordinal $\alpha < \kappa$ satisfying

$$\sup(\mathrm{nacc}(C_\alpha) \cap B) = \alpha.$$

Consider any $v \in T_\alpha$. By construction,
$v = \mathbf{b}_x^\alpha = \bigcup_{\beta \in \mathrm{dom}(b_x^\alpha)} b_x^\alpha(\beta)$ for some $x \in T \restriction C_\alpha$. Fix
$\beta \in \mathrm{nacc}(C_\alpha) \cap B$ with $\mathrm{ht}(x) < \beta < \alpha$. So $\Omega_\beta = A \cap (T \restriction \beta)$ is a maximal antichain in $T \restriction \beta$. Thus we chose $b_x^\alpha(\beta)$ to extend some $y \in \Omega_\beta$. Altogether, $y \subseteq b_x^\alpha(\beta) \subseteq \mathbf{b}_x^\alpha = v$, as required. $\square$

How does $\boxtimes^-(\kappa)$ fit with other axioms?

So we've built a κ-Souslin tree from $\diamondsuit(\kappa) + \boxtimes^-(\kappa)$, but how does this compare with other known axioms?

How does $\boxtimes^-(\kappa)$ fit with other axioms?

So we've built a κ-Souslin tree from $\diamondsuit(\kappa) + \boxtimes^-(\kappa)$, but how does this compare with other known axioms?

Theorem

$\diamondsuit(\kappa) + \boxtimes^-(\kappa)$ holds, assuming any of the following:

- $\kappa = \aleph_1$ *and* $\diamondsuit(\aleph_1)$ *holds;*
- $\kappa = \lambda^+$ *for* λ *uncountable, and* $\square_\lambda + \mathrm{CH}_\lambda$ *holds;*
- $\kappa = \lambda^+$, λ *is not subcompact, and* V *is a Jensen-type extender model of the form* $L[E]$*;*
- κ *is a regular uncountable cardinal that is not weakly compact, and* $V = L$.

How does $\boxtimes^-(\kappa)$ fit with other axioms?

So we've built a κ-Souslin tree from $\diamondsuit(\kappa) + \boxtimes^-(\kappa)$, but how does this compare with other known axioms?

Theorem

$\diamondsuit(\kappa) + \boxtimes^-(\kappa)$ holds, assuming any of the following:

- *$\kappa = \aleph_1$ and $\diamondsuit(\aleph_1)$ holds;*
- *$\kappa = \lambda^+$ for λ uncountable, and $\square_\lambda + \mathrm{CH}_\lambda$ holds;*
- *$\kappa = \lambda^+$, λ is not subcompact, and V is a Jensen-type extender model of the form $L[E]$;*
- *κ is a regular uncountable cardinal that is not weakly compact, and $V = L$.*

Thus, we get a κ-Souslin tree uniformly in all these scenarios!

Slim Souslin trees

In the tree we just built, what can we say about how fast the levels grow?

Slim Souslin trees

In the tree we just built, what can we say about how fast the levels grow?
At each limit level $\alpha < \kappa$, we put (at most) one node into T_α for every $x \in T \restriction \alpha$.
It follows that $|T_\alpha| \leq \max\{|\alpha|, \aleph_0\}$ for every $\alpha < \kappa$.
Thus we say the tree is slim.

Slim Souslin trees

In the tree we just built, what can we say about how fast the levels grow?
At each limit level $\alpha < \kappa$, we put (at most) one node into T_α for every $x \in T \restriction \alpha$.
It follows that $|T_\alpha| \leq \max\{|\alpha|, \aleph_0\}$ for every $\alpha < \kappa$.
Thus we say the tree is slim.
What if we consider an opposite property?

Complete Souslin trees

Definition

For cardinals $\chi < \kappa$, the κ-Souslin tree $\langle T, <_T \rangle$ is χ-complete if every $<_T$-increasing sequence of elements of T of length $< \chi$ has an upper bound in T.

Complete Souslin trees

Definition

For cardinals $\chi < \kappa$, the κ-Souslin tree $\langle T, <_T \rangle$ is χ-complete if every $<_T$-increasing sequence of elements of T of length $< \chi$ has an upper bound in T.

To get our κ-Souslin tree to be χ-complete, we would like to tweak our construction so that at levels α with $\mathrm{cf}(\alpha) < \chi$ we put the limits of all branches into T_α. Does this work?

Complete Souslin trees

Definition

For cardinals $\chi < \kappa$, the κ-Souslin tree $\langle T, <_T \rangle$ is χ-complete if every $<_T$-increasing sequence of elements of T of length $< \chi$ has an upper bound in T.

To get our κ-Souslin tree to be χ-complete, we would like to tweak our construction so that at levels α with $\mathrm{cf}(\alpha) < \chi$ we put the limits of all branches into T_α. Does this work?
For one thing, we can't let the tree get too wide. So we better assume $\lambda^{<\chi} < \kappa$ for all $\lambda < \kappa$.

Complete Souslin trees

Definition

For cardinals $\chi < \kappa$, the κ-Souslin tree $\langle T, <_T \rangle$ is χ-complete if every $<_T$-increasing sequence of elements of T of length $< \chi$ has an upper bound in T.

To get our κ-Souslin tree to be χ-complete, we would like to tweak our construction so that at levels α with $\mathrm{cf}(\alpha) < \chi$ we put the limits of all branches into T_α. Does this work?
For one thing, we can't let the tree get too wide. So we better assume $\lambda^{<\chi} < \kappa$ for all $\lambda < \kappa$.
But if there are elements of T_α that are not of the form $\mathbf{b}_x^\alpha$, then how do we kill the antichains using C_α?

Complete Souslin trees

Definition

For cardinals $\chi < \kappa$, the κ-Souslin tree $\langle T, <_T \rangle$ is χ-complete if every $<_T$-increasing sequence of elements of T of length $< \chi$ has an upper bound in T.

To get our κ-Souslin tree to be χ-complete, we would like to tweak our construction so that at levels α with $\mathrm{cf}(\alpha) < \chi$ we put the limits of all branches into T_α. Does this work?
For one thing, we can't let the tree get too wide. So we better assume $\lambda^{<\chi} < \kappa$ for all $\lambda < \kappa$.
But if there are elements of T_α that are not of the form $\mathbf{b}^\alpha_x$, then how do we kill the antichains using C_α?
We need to tweak an axiom to avoid such levels.

Complete Souslin trees

Definition

For cardinals $\chi < \kappa$, the κ-Souslin tree $\langle T, <_T \rangle$ is χ-complete if every $<_T$-increasing sequence of elements of T of length $< \chi$ has an upper bound in T.

To get our κ-Souslin tree to be χ-complete, we would like to tweak our construction so that at levels α with $\mathrm{cf}(\alpha) < \chi$ we put the limits of all branches into T_α. Does this work?
For one thing, we can't let the tree get too wide. So we better assume $\lambda^{<\chi} < \kappa$ for all $\lambda < \kappa$.
But if there are elements of T_α that are not of the form $\mathbf{b}_x^\alpha$, then how do we kill the antichains using C_α?
We need to tweak an axiom to avoid such levels.
Recall that the previous construction used $\diamondsuit(\kappa) + \boxtimes^-(\kappa)$.
Which axiom should we strengthen: $\diamondsuit(\kappa)$ or $\boxtimes^-(\kappa)$?

Complete Souslin trees

Definition

For cardinals $\chi < \kappa$, the κ-Souslin tree $\langle T, <_T \rangle$ is χ-complete if every $<_T$-increasing sequence of elements of T of length $< \chi$ has an upper bound in T.

To get our κ-Souslin tree to be χ-complete, we would like to tweak our construction so that at levels α with $\mathrm{cf}(\alpha) < \chi$ we put the limits of all branches into T_α. Does this work?

For one thing, we can't let the tree get too wide. So we better assume $\lambda^{<\chi} < \kappa$ for all $\lambda < \kappa$.

But if there are elements of T_α that are not of the form $\mathbf{b}_x^\alpha$, then how do we kill the antichains using C_α?

We need to tweak an axiom to avoid such levels.

Recall that the previous construction used $\diamondsuit(\kappa) + \boxtimes^-(\kappa)$.

Which axiom should we strengthen: $\diamondsuit(\kappa)$ or $\boxtimes^-(\kappa)$?

Classical constructions of χ-complete Souslin trees would replace $\diamondsuit(\kappa)$ with $\diamondsuit(E^\kappa_{\geq\chi})$. But we'll try something different. . . .

A stronger parameter: $\boxtimes^-(S)$

Recall

Fix a regular uncountable cardinal κ.
The principle $\boxtimes^-(\kappa)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa\rangle$ such that:

- C_α is a club subset of α for every limit ordinal $\alpha < \kappa$;
- $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ for all ordinals $\alpha < \kappa$ and $\bar\alpha \in \mathrm{acc}(C_\alpha)$;
- for every cofinal subset $B \subseteq \kappa$, there exist stationarily many $\alpha < \kappa$ satisfying

$$\sup(\mathrm{nacc}(C_\alpha) \cap B) = \alpha.$$

A stronger parameter: $\boxtimes^-(S)$

Definition

Fix a regular uncountable cardinal κ and a stationary set $S \subseteq \kappa$.
The principle $\boxtimes^-(S)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that:

- C_α is a club subset of α for every limit ordinal $\alpha < \kappa$;
- $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ for all ordinals $\alpha < \kappa$ and $\bar\alpha \in \mathrm{acc}(C_\alpha)$;
- for every cofinal subset $B \subseteq \kappa$, there exist stationarily many $\alpha \in S$ satisfying

$$\sup(\mathrm{nacc}(C_\alpha) \cap B) = \alpha.$$

A stronger parameter: $\boxtimes^-(S)$

Definition

Fix a regular uncountable cardinal κ and a stationary set $S \subseteq \kappa$.
The principle $\boxtimes^-(S)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that:

- C_α is a club subset of α for every limit ordinal $\alpha < \kappa$;
- $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ for all ordinals $\alpha < \kappa$ and $\bar\alpha \in \mathrm{acc}(C_\alpha)$;
- for every cofinal subset $B \subseteq \kappa$, there exist stationarily many $\alpha \in S$ satisfying

$$\sup(\mathrm{nacc}(C_\alpha) \cap B) = \alpha.$$

Theorem

For any regular uncountable cardinal κ and any infinite $\chi < \kappa$ satisfying $\lambda^{<\chi} < \kappa$ for all $\lambda < \kappa$, $\diamondsuit(\kappa) + \boxtimes^-(E^\kappa_{\geq\chi})$ implies the existence of a χ-complete κ-Souslin tree.

A stronger parameter: $\boxtimes^-(S)$

There exist models satisfying $\diamondsuit(\kappa)$ and $\boxtimes^-(E^\kappa_{\geq\chi})$ in which $\diamondsuit(E^\kappa_{\geq\chi})$ fails. The preceding theorem shows that we can build a χ-complete κ-Souslin tree in such a model, despite the failure of $\diamondsuit(E^\kappa_{\geq\chi})$.

A stronger parameter: $\boxtimes^-(S)$

There exist models satisfying $\diamondsuit(\kappa)$ and $\boxtimes^-(E^\kappa_{\geq\chi})$ in which $\diamondsuit(E^\kappa_{\geq\chi})$ fails. The preceding theorem shows that we can build a χ-complete κ-Souslin tree in such a model, despite the failure of $\diamondsuit(E^\kappa_{\geq\chi})$. This is because the last clause of $\boxtimes^-(S)$ allows us to separate the stationary set of approximations to a maximal antichain from the stationary set of ordinals where we seal those antichains.

Coherent trees

Definition

A subtree $T \subseteq {}^{<\kappa}\kappa$ is coherent if for every $\alpha < \kappa$ and $s, t \in T \cap {}^{\alpha}\kappa$, the set $\{\beta < \alpha \mid s(\beta) \neq t(\beta)\}$ is finite.

Coherent trees

Definition

A subtree $T \subseteq {}^{<\kappa}\kappa$ is coherent if for every $\alpha < \kappa$ and $s, t \in T \cap {}^{\alpha}\kappa$, the set $\{\beta < \alpha \mid s(\beta) \neq t(\beta)\}$ is finite.

Jensen gave a consistent construction of a coherent $\aleph_1$-Souslin tree. Velickovic gave a consistent construction of a coherent $\aleph_2$-Souslin tree.

Coherent trees

Definition

A subtree $T \subseteq {}^{<\kappa}\kappa$ is coherent if for every $\alpha < \kappa$ and $s, t \in T \cap {}^{\alpha}\kappa$, the set $\{\beta < \alpha \mid s(\beta) \neq t(\beta)\}$ is finite.

Jensen gave a consistent construction of a coherent $\aleph_1$-Souslin tree. Velickovic gave a consistent construction of a coherent $\aleph_2$-Souslin tree.
What about inaccessible κ?

Strengthening $\boxtimes^-(S)$ to $\boxtimes(S)$

Recall

Fix a regular uncountable cardinal κ and a stationary set $S \subseteq \kappa$. The principle $\boxtimes^-(S)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that:

- C_α is a club subset of α for every limit ordinal $\alpha < \kappa$;
- $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ for all ordinals $\alpha < \kappa$ and $\bar\alpha \in \mathrm{acc}(C_\alpha)$;
- for every cofinal subset $B \subseteq \kappa$,
  there exist stationarily many $\alpha \in S$ satisfying

$$\sup(\mathrm{nacc}(C_\alpha) \cap B) = \alpha.$$

Strengthening $\boxtimes^-(S)$ to $\boxtimes(S)$

Definition

Fix a regular uncountable cardinal κ and a stationary set $S \subseteq \kappa$. The principle $\boxtimes(S)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that:

- C_α is a club subset of α for every limit ordinal $\alpha < \kappa$;
- $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$ for all ordinals $\alpha < \kappa$ and $\bar{\alpha} \in \mathrm{acc}(C_\alpha)$;
- for every sequence $\langle B_i \mid i < \kappa \rangle$ of cofinal subsets of κ, there exist stationarily many $\alpha \in S$ such that for all $i < \alpha$

$$\sup(\mathrm{nacc}(C_\alpha) \cap B_i) = \alpha.$$

Strengthening $\boxtimes^-(S)$ to $\boxtimes(S)$

Definition

Fix a regular uncountable cardinal κ and a stationary set $S \subseteq \kappa$. The principle $\boxtimes(S)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that:

- C_α is a club subset of α for every limit ordinal $\alpha < \kappa$;
- $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ for all ordinals $\alpha < \kappa$ and $\bar\alpha \in \mathrm{acc}(C_\alpha)$;
- for every sequence $\langle B_i \mid i < \kappa \rangle$ of cofinal subsets of κ, there exist stationarily many $\alpha \in S$ such that for all $i < \alpha$

$$\sup\{\beta < \alpha \mid \mathrm{succ}_\omega(C_\alpha \setminus \beta) \subseteq B_i\} = \alpha,$$

where

$$\mathrm{succ}_\omega(D) := \{\delta \in D \mid 0 < \mathrm{otp}(D \cap \delta) < \omega\}.$$

Construction of a coherent tree

Theorem

If κ is a regular uncountable cardinal and $\boxtimes(\kappa) + \diamondsuit(\kappa)$ holds, then there exists a coherent κ-Souslin tree.

Construction of a coherent tree

Let $\langle C_\alpha \mid \alpha < \kappa \rangle$ be a witness to $\boxtimes(\kappa)$.
WLOG, assume that $0 \in C_\alpha$ for all $\alpha < \kappa$.
Let $\langle R_i \mid i < \kappa \rangle$ and $\langle \Omega_\beta \mid \beta < \kappa \rangle$ together witness $\diamondsuit(H_\kappa)$.
Fix a well-ordering $\lhd$ on H_κ.
Let $\pi : \kappa \to \kappa$ be such that $\alpha \in R_{\pi(\alpha)}$ for all $\alpha < \kappa$.
By $\diamondsuit(\kappa)$, we have $2^{<\kappa} = \kappa$, thus let $\phi : \kappa \leftrightarrow {}^{<\kappa}2$ be some bijection.
Put $\psi := \phi \circ \pi$.

Construction of a coherent tree

Let $\langle C_\alpha \mid \alpha < \kappa\rangle$ be a witness to $\boxtimes(\kappa)$.
WLOG, assume that $0 \in C_\alpha$ for all $\alpha < \kappa$.
Let $\langle R_i \mid i < \kappa\rangle$ and $\langle \Omega_\beta \mid \beta < \kappa\rangle$ together witness $\diamondsuit(H_\kappa)$.
Fix a well-ordering $\lhd$ on H_κ.
Let $\pi : \kappa \to \kappa$ be such that $\alpha \in R_{\pi(\alpha)}$ for all $\alpha < \kappa$.
By $\diamondsuit(\kappa)$, we have $2^{<\kappa} = \kappa$, thus let $\phi : \kappa \leftrightarrow {}^{<\kappa}2$ be some bijection.
Put $\psi := \phi \circ \pi$.
For two elements of η, τ of H_κ, we define $\eta * \tau$ to be the emptyset, unless $\eta, \tau \in {}^{<\kappa}2$ with $\mathrm{dom}(\eta) < \mathrm{dom}(\tau)$, in which case $\eta * \tau : \mathrm{dom}(\tau) \to 2$ is defined by stipulating:

$$(\eta * \tau)(\beta) := \begin{cases} \eta(\beta), & \text{if } \beta \in \mathrm{dom}(\eta); \\ \tau(\beta), & \text{otherwise.} \end{cases}$$

Construction of a coherent tree (continued)

We shall now recursively construct a sequence $\langle T_\alpha \mid \alpha < \kappa \rangle$ of levels whose union will ultimately be the desired tree T.
Let $T_0 := \{\emptyset\}$.
For every $\alpha < \kappa$, define

$$T_{\alpha+1} = \{s^\frown\langle i\rangle \mid s \in T_\alpha, i < 2\}.$$

Construction of a coherent tree (continued)

Next, suppose that α is a nonzero limit ordinal, and that $\langle T_\beta \mid \beta < \alpha \rangle$ has already been defined.
As before, to each node $x \in T \restriction \alpha$ we shall associate some node $\mathbf{b}^\alpha_x : \alpha \to \kappa$ above x, and then let $T_\alpha := \{\mathbf{b}^\alpha_x \mid x \in T \restriction \alpha\}$.
Unlike the previous proof we first define $\mathbf{b}^\alpha_\emptyset$, and then use $\mathbf{b}^\alpha_\emptyset$ to define $\mathbf{b}^\alpha_x$ for $x \neq \emptyset$.
Define $b^\alpha_\emptyset \in \prod_{\beta \in C_\alpha} T_\beta$ by recursion. Let $b^\alpha_\emptyset(0) := \emptyset$.

Construction of a coherent tree (continued)

Next, suppose $\beta^- < \beta$ are successive points of C_α, and $b^\alpha_\emptyset(\beta^-)$ has already been defined. In order to decide $b^\alpha_\emptyset(\beta)$, we advise with the following set:

$$Q^{\alpha,\beta} := \{t \in T_\beta \mid \exists s \in \Omega_\beta[(s \cup (\psi(\beta) * b^\alpha_\emptyset(\beta^-))) \subseteq t]\}.$$

Now, consider the two possibilities:

- If $Q^{\alpha,\beta} \neq \emptyset$, let t denote its $\lhd$-least element, and put $b^\alpha_\emptyset(\beta) := b^\alpha_\emptyset(\beta^-) * t$;
- Otherwise, let $b^\alpha_\emptyset(\beta)$ be the $\lhd$-least element of T_β that extends $b^\alpha_\emptyset(\beta^-)$.

Note that $Q^{\alpha,\beta}$ depends only on $T_\beta, \Omega_\beta, \psi(\beta)$ and $b^\alpha_\emptyset(\beta^-)$, and hence for every ordinal $\gamma < \kappa$, if $C_\alpha \cap (\beta+1) = C_\gamma \cap (\beta+1)$, then $b^\alpha_\emptyset \restriction (\beta+1) = b^\gamma_\emptyset \restriction (\beta+1)$. It follows that for all $\beta \in \mathrm{acc}(C_\alpha)$ such that $b^\alpha_\emptyset \restriction \beta$ has already been defined, we may let $b^\alpha_\emptyset(\beta) := \bigcup \mathrm{Im}(b^\alpha_\emptyset \restriction \beta)$ and infer that $b^\alpha_\emptyset(\beta) = \mathbf{b}^\beta_\emptyset \in T_\beta$. This completes the definition of $b^\alpha_\emptyset$ and its limit $\mathbf{b}^\alpha_\emptyset = \bigcup \mathrm{Im}(b^\alpha_\emptyset)$.

Construction of a coherent tree (continued)

Next, for each $x \in T \restriction \alpha$, let $\mathbf{b}_x^\alpha := x * \mathbf{b}_\emptyset^\alpha$. This completes the definition of the level T_α.

Having constructed all levels of the tree, we then let

$$T := \bigcup_{\alpha<\kappa} T_\alpha.$$

Construction of a coherent tree (continued)

Next, for each $x \in T \restriction \alpha$, let $\mathbf{b}^\alpha_x := x * \mathbf{b}^\alpha_\emptyset$. This completes the definition of the level T_α.
Having constructed all levels of the tree, we then let

$$T := \bigcup_{\alpha<\kappa} T_\alpha.$$

Claim

For every $\alpha < \kappa$, every two nodes of T_α differ on a finite set.

Proof.

Suppose not, and let α be the least counterexample. Clearly, α must be a limit nonzero ordinal. Pick $x, y \in T \restriction \alpha$ such that $\mathbf{b}^\alpha_x$ differs from $\mathbf{b}^\alpha_y$ on an infinite set. As $\mathbf{b}^\alpha_x = x * \mathbf{b}^\alpha_\emptyset$ and $\mathbf{b}^\alpha_y = y * \mathbf{b}^\alpha_\emptyset$, it follows that x and y differ on an infinite set, contradicting the minimality of α. $\square$

Thus, we are left with verifying that $(T, \subset)$ is κ-Souslin.

Construction of a coherent tree (continued)

Claim

Suppose $A \subseteq T$ is a maximal antichain. Then $|A| < \kappa$.

Construction of a coherent tree (continued)

Claim

Suppose $A \subseteq T$ is a maximal antichain. Then $|A| < \kappa$.

PROOF:

As in the previous theorem but this time making use of the sequence $\langle R_i \mid i < \kappa \rangle$, for every $i < \kappa$, the set

$$B_i := \{\beta \in R_i \mid A \cap (T \restriction \beta) = \Omega_\beta \text{ is a maximal antichain in } T \restriction \beta\}$$

is stationary. Thus, we apply $\boxtimes(\kappa)$ to the sequence $\langle B_i \mid i < \kappa \rangle$, and the club $D := \{\alpha < \kappa \mid T \restriction \alpha \subseteq \phi[\alpha]\}$ to obtain an ordinal $\alpha \in D$ such that for all $i < \alpha$:

$$\sup(\mathrm{nacc}(C_\alpha) \cap B_i) = \alpha.$$

Construction of a coherent tree (continued)

To see that $A \subseteq T \restriction \alpha$, consider any $z \in T \restriction (\kappa \setminus \alpha)$. Let $y := z \restriction \alpha \in T_\alpha$. By construction, $y = \mathbf{b}^\alpha_x = x * \mathbf{b}^\alpha_\emptyset$ for some $x \in T \restriction \alpha$. As $\alpha \in D$ and $x \in T \restriction \alpha$, we can fix $i < \alpha$ such that $\phi(i) = x$.
Fix $\beta \in \mathrm{nacc}(C_\alpha) \cap B_i$ with $\mathrm{ht}(x) < \beta < \alpha$. Clearly, $\psi(\beta) = \phi(\pi(\beta)) = \phi(i) = x$. Since $\beta \in B_i$, we know that $\Omega_\beta = A \cap (T \restriction \beta)$ is a maximal antichain in $T \restriction \beta$, and hence $Q^{\alpha,\beta} \neq \emptyset$. Let $t := \min(Q^{\alpha,\beta}, \lhd)$ and $\beta^- := \sup(C_\alpha \cap \beta)$. Then $b^\alpha_\emptyset(\beta) = b^\alpha_\emptyset(\beta^-) * t$, and there exists some $s \in \Omega_\beta$ such that $(s \cup (x * b^\alpha_\emptyset(\beta^-))) \subseteq t$. In particular, $x * b^\alpha_\emptyset(\beta)$ extends an element of Ω_β. Altogether, there exists some $s \in A \cap (T \restriction \beta)$ such that $s \subseteq x * b^\alpha_\emptyset(\beta) \subseteq x * \mathbf{b}^\alpha_\emptyset = \mathbf{b}^\alpha_x = y \subseteq z$, and hence $z \notin A$.

How does $\boxtimes(\kappa)$ fit with other axioms?

Now we've built a coherent κ-Souslin tree from $\diamondsuit(\kappa) + \boxtimes(\kappa)$. How does this compare with other known axioms?

How does $\boxtimes(\kappa)$ fit with other axioms?

Now we've built a coherent κ-Souslin tree from $\diamondsuit(\kappa) + \boxtimes(\kappa)$. How does this compare with other known axioms?

Theorem

$\diamondsuit(\kappa) + \boxtimes(\kappa)$ *holds, assuming any of the following:*

- $\kappa = \aleph_1$ *and* $\diamondsuit(\aleph_1)$ *holds;*
- $\kappa = \lambda^+$ *for* λ *singular, and* $\square_\lambda + \mathrm{CH}_\lambda$ *holds;*
- $\kappa = \lambda^+$ *for* λ *regular uncountable, and* $\boxed{\diamondsuit}_\lambda$ *holds;*
- $\kappa = \lambda^+$, λ *is not subcompact, and* V *is a Jensen-type extender model of the form* $L[E]$*;*
- κ *is a regular uncountable cardinal that is not weakly compact, and* $V = L$*;*
- $\kappa = \lambda^+$ *for* λ *regular uncountable and* $V = W^{\mathrm{Add}(\lambda,1)}$, *where*

$$W \models \mathrm{ZFC} + \square_\lambda + \mathrm{CH}_\lambda + \lambda^{<\lambda} = \lambda.$$

Unified result

Thus, we get a coherent κ-Souslin tree uniformly in all these scenarios!

Unified result

Thus, we get a coherent κ-Souslin tree uniformly in all these scenarios!
In fact, we can construct a free κ-Souslin tree from $\diamondsuit(\kappa) + \boxtimes(\kappa)$.
Thus there exists a free κ-Souslin tree in all of these scenarios as well!

Using the full strength of $\boxtimes(\kappa)$

The construction of the coherent and free trees does not use the full force of the axiom $\boxtimes(\kappa)$: We needed only

$$\sup(\mathrm{nacc}(C_\alpha) \cap B_i) = \alpha,$$

which is equivalent to

$$\sup\{\beta < \alpha \mid \mathrm{succ}_{1}(C_\alpha \setminus \beta) \subseteq B_i\} = \alpha,$$

while $\boxtimes(\kappa)$ provides

$$\sup\{\beta < \alpha \mid \mathrm{succ}_{\omega}(C_\alpha \setminus \beta) \subseteq B_i\} = \alpha.$$

Using the full strength of $\boxtimes(\kappa)$

The construction of the coherent and free trees does not use the full force of the axiom $\boxtimes(\kappa)$: We needed only

$$\sup(\mathrm{nacc}(C_\alpha) \cap B_i) = \alpha,$$

which is equivalent to

$$\sup\{\beta < \alpha \mid \mathrm{succ}_1(C_\alpha \setminus \beta) \subseteq B_i\} = \alpha,$$

while $\boxtimes(\kappa)$ provides

$$\sup\{\beta < \alpha \mid \mathrm{succ}_\omega(C_\alpha \setminus \beta) \subseteq B_i\} = \alpha.$$

Why do we need the stronger condition?

Using the full strength of $\boxtimes(\kappa)$: Ascent paths

Using $\boxtimes(\kappa)$, we can construct a κ-Souslin tree with a θ-ascent path, for every cardinal $\theta < \kappa$.

Using the full strength of $\boxtimes(\kappa)$: Ascent paths

Using $\boxtimes(\kappa)$, we can construct a κ-Souslin tree with a θ-ascent path, for every cardinal $\theta < \kappa$.
What is an ascent path?

Using the full strength of $\boxtimes(\kappa)$: Ascent paths

Using $\boxtimes(\kappa)$, we can construct a κ-Souslin tree with a θ-ascent path, for every cardinal $\theta < \kappa$.
What is an ascent path?
Instead of defining it, let's look at its consequences.

Ascent paths make a tree non-specializable

An ascent path makes a tree hard to specialize.
Recall: Baumgartner, Malitz & Reinhardt (1970) proved that every $\aleph_1$-Aronszajn tree can be made special in some cofinality-preserving extension. The next example is of a λ^+-Souslin tree that cannot be specialized without reducing it to the BMR scenario.

Theorem

Assume $\square_\lambda + \mathrm{CH}_\lambda$ for a given singular cardinal λ of countable cofinality.
Then there exists a λ^+-Souslin tree $\langle T, <_T\rangle$ satisfying the following. If W is a ZFC *extension of the universe in which $\langle T, <_T\rangle$ is a special $|\lambda|^+$-tree, then $W \models |\lambda| = \aleph_0$.*

Free trees with ascent paths

Theorem

For any regular uncountable cardinal κ and any infinite cardinal $\theta < \kappa$, $\diamondsuit(\kappa) + \boxtimes(\kappa)$ implies that there exists a free κ-Souslin tree with a θ-ascent path.

Reduced-power trees

Ascent paths provide a branch through the reduced-power tree, while freeness can prevent such branches from existing. With careful control over both, we obtain:

Theorem

Assume $V = L$.
Then there exist trees T_0, T_1, T_2, T_3, and selective ultrafilters $\mathcal{U}_0$ over ω and $\mathcal{U}_1$ over ω_1, such that:

	T	$T^\omega/\mathcal{U}_0$	$T^{\omega_1}/\mathcal{U}_1$
T_0	$\aleph_3$-*Souslin*	$\aleph_3$-*Aronszajn*	$\aleph_3$-*Aronszajn*
T_1	$\aleph_3$-*Souslin*	$\aleph_3$-*Kurepa*	$\aleph_3$-*Kurepa*
T_2	$\aleph_3$-*Souslin*	$\aleph_3$-*Aronszajn*	$\aleph_3$-*Kurepa*
T_3	$\aleph_3$-*Souslin*	$\neg\aleph_3$-*Aronszajn*	$\aleph_3$-*Aronszajn*

This is new: Previous results addressed θ-power trees with respect to a single power θ, but here we control different powers simultaneously and independently.

The proxy principle

The axioms we have defined so far, $\boxtimes^-(S)$ and $\boxtimes(S)$, are special cases of a parametrized proxy principle.

The proxy principle

The axioms we have defined so far, $\boxtimes^-(S)$ and $\boxtimes(S)$, are special cases of a parametrized proxy principle.

Definition (Proxy principle)

The principle $\mathrm{P}^-(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \mathcal{E})$ asserts the existence of a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ such that:

- for every limit $\alpha < \kappa$, $\mathcal{C}_\alpha$ is a collection of club subsets of α;
- for every ordinal $\alpha < \kappa$, $0 < |\mathcal{C}_\alpha| < \mu$, and $C \mathrel{\mathcal{E}} D$ for all $C, D \in \mathcal{C}_\alpha$;
- for every ordinal $\alpha < \kappa$, every $C \in \mathcal{C}_\alpha$, and every $\bar{\alpha} \in \mathrm{acc}(C)$, there exists $D \in \mathcal{C}_{\bar{\alpha}}$ such that $D \mathrel{\mathcal{R}} C$;
- for every sequence $\langle A_i \mid i < \theta \rangle$ of cofinal subsets of κ, and every $S \in \mathcal{S}$, there exist stationarily many $\alpha \in S$ for which:
  - $|\mathcal{C}_\alpha| < \nu$; and
  - for every $C \in \mathcal{C}_\alpha$ and $i < \min\{\alpha, \theta\}$:

$$\sup\{\beta \in C \mid \mathrm{succ}_\sigma(C \setminus \beta) \subseteq A_i\} = \alpha.$$

The proxy principle

Definition

$\mathsf{P}(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \mathcal{E})$ asserts that both $\mathsf{P}^{-}(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \mathcal{E})$ and $\diamondsuit(\kappa)$ hold.

The proxy principle

Definition

$\mathsf{P}(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \mathcal{E})$ asserts that both $\mathsf{P}^-(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \mathcal{E})$ and $\diamondsuit(\kappa)$ hold.

Why so many parameters?

The proxy principle

Definition
$\mathsf{P}(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \mathcal{E})$ asserts that both $\mathsf{P}^-(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \mathcal{E})$ and $\diamondsuit(\kappa)$ hold.

Why so many parameters?

- To calibrate various properties of Souslin trees, by identifying the weakest vector of parameters necessary to construct a tree satisfying any desired property;

The proxy principle

Definition

$\mathsf{P}(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \mathcal{E})$ asserts that both $\mathsf{P}^-(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \mathcal{E})$ and $\diamondsuit(\kappa)$ hold.

Why so many parameters?

- To calibrate various properties of Souslin trees, by identifying the weakest vector of parameters necessary to construct a tree satisfying any desired property;
- To capture all of the axioms used in classical $\diamondsuit$-based constructions of Souslin trees.

The proxy principle

Recall:
We constructed a slim κ-Souslin tree from $\boxtimes^-(\kappa) + \diamondsuit(\kappa)$, and a χ-complete κ-Souslin tree from $\boxtimes^-(E^\kappa_{\geq\chi}) + \diamondsuit(\kappa)$.
We can restate $\boxtimes^-$ in terms of the proxy principle as follows:

$$\boxtimes^-(S) \iff \mathrm{P}^-(\kappa, 2, \sqsubseteq, 1, \{S\}, 2, 1, \mathcal{E}_\kappa),$$

so that we get a slim κ-Souslin tree from

$$\mathrm{P}(\kappa, 2, \sqsubseteq, 1, \{\kappa\}, 2, 1, \mathcal{E}_\kappa),$$

and a χ-complete κ-Souslin tree (assuming $\lambda^{<\chi} < \kappa$ for all $\lambda < \kappa$) from

$$\mathrm{P}(\kappa, 2, \sqsubseteq, 1, \{E^\kappa_{\geq\chi}\}, 2, 1, \mathcal{E}_\kappa).$$

Recovering the classical axioms

For any regular uncountable cardinal κ and any stationary $S \subseteq \kappa$:

$$\begin{aligned}
\clubsuit_w(S) &\iff \mathrm{P}^-(\kappa, 2, {}_\kappa\sqsubseteq, 1, \{S\}, 2, \kappa, \mathcal{E}_\kappa) \\
\diamondsuit(S) &\iff \mathrm{P}(\kappa, 2, {}_\kappa\sqsubseteq, 1, \{S\}, 2, \kappa, \mathcal{E}_\kappa)
\end{aligned}$$

For any infinite cardinal λ and any stationary $S \subseteq \lambda^+$:

$$\begin{aligned}
\square_\lambda &\iff \mathrm{P}^-(\lambda^+, 2, \sqsubseteq, 1, \{\lambda^+\}, 2, 0, \mathcal{E}_\lambda) \\
\square_\lambda + \mathrm{CH}_\lambda &\iff \mathrm{P}(\lambda^+, 2, \sqsubseteq, 1, \{\lambda^+\}, 2, 0, \mathcal{E}_\lambda) && (\lambda > \aleph_0) \\
\boxed{\diamondsuit}_\lambda &\iff \mathrm{P}(\lambda^+, 2, \sqsubseteq, 1, \{E^{\lambda^+}_{\mathrm{cf}(\lambda)}\}, 2, \lambda^+, \mathcal{E}_\lambda) \\
\boxminus_{\lambda, \geq\chi} &\iff \mathrm{P}^-(\lambda^+, 2, \sqsubseteq_\chi, 1, \{\lambda^+\}, 2, 0, \mathcal{E}_\lambda) && \lambda > \aleph_0; \lambda \geq \chi \geq \aleph_0 \\
\langle\lambda\rangle^-_S &\iff \mathrm{P}(\lambda^+, 2, {}_\lambda\sqsubseteq, 1, \{S\}, 2, 1, \mathcal{E}_\lambda)
\end{aligned}$$

Recovering the classical axioms

For any regular uncountable cardinal κ and any stationary $S \subseteq \kappa$:

$$\begin{aligned} \clubsuit_w(S) &\iff \mathrm{P}^-(\kappa, 2, {}_\kappa{\sqsubseteq}, 1, \{S\}, 2, \kappa, \mathcal{E}_\kappa) \\ \diamondsuit(S) &\iff \mathrm{P}(\kappa, 2, {}_\kappa{\sqsubseteq}, 1, \{S\}, 2, \kappa, \mathcal{E}_\kappa) \end{aligned}$$

For any infinite cardinal λ and any stationary $S \subseteq \lambda^+$:

$$\begin{aligned} \square_\lambda &\iff \mathrm{P}^-(\lambda^+, 2, \sqsubseteq, 1, \{\lambda^+\}, 2, 0, \mathcal{E}_\lambda) & \\ \square_\lambda + \mathrm{CH}_\lambda &\iff \mathrm{P}(\lambda^+, 2, \sqsubseteq, 1, \{\lambda^+\}, 2, 0, \mathcal{E}_\lambda) & (\lambda > \aleph_0) \\ \boxed{\diamondsuit}_\lambda &\iff \mathrm{P}(\lambda^+, 2, \sqsubseteq, 1, \{E^{\lambda^+}_{\mathrm{cf}(\lambda)}\}, 2, \lambda^+, \mathcal{E}_\lambda) & \\ \boxminus_{\lambda,\geq\chi} &\iff \mathrm{P}^-(\lambda^+, 2, \sqsubseteq_\chi, 1, \{\lambda^+\}, 2, 0, \mathcal{E}_\lambda) & \lambda > \aleph_0; \lambda \geq \chi \geq \aleph_0 \\ \langle\lambda\rangle^-_S &\iff \mathrm{P}(\lambda^+, 2, {}_\lambda{\sqsubseteq}, 1, \{S\}, 2, 1, \mathcal{E}_\lambda) & \end{aligned}$$

Thus any time we carry out a construction from the proxy principle, we can tell immediately which of the classical axioms are sufficient for the construction.

More examples

Theorem

Assuming $P(\kappa, \kappa, {}_{\chi}\sqsubseteq, \kappa, \{E^{\kappa}_{\geq\chi}\}, 2, 1, \mathcal{E}_{\kappa})$ *and* $\lambda^{<\chi} < \kappa$ *for all* $\lambda < \kappa$*, there exists a* χ*-complete, free* κ*-Souslin tree.*

More examples

Theorem

Assuming $P(\kappa, \kappa, {}_{\chi}\sqsubseteq, \kappa, \{E^{\kappa}_{\geq\chi}\}, 2, 1, \mathcal{E}_{\kappa})$ *and* $\lambda^{<\chi} < \kappa$ *for all* $\lambda < \kappa$*, there exists a* χ*-complete, free* κ*-Souslin tree.*

Theorem

Assuming GCH $+$ $P(\lambda^+, \lambda^+, {}_{\chi}\sqsubseteq^*, 1, \{E^{\lambda^+}_{\lambda}\}, \lambda^+, 1, =^*)$*, there exists a* λ*-complete specializable* λ^+*-Souslin tree.*

Still more

By further tweaking the parameters and varying the construction slightly, we can construct a Souslin tree from weaker axioms than those mentioned earlier.

Theorem

A form of the proxy principle $P(\dots)$ holds enabling the construction of a λ^+-Souslin tree for uncountable λ, assuming any of the following:

- $\lambda^{<\lambda} = \lambda + \diamondsuit(E^{\lambda^+}_{\lambda})$;
- $V = W^{\mathrm{Add}(\lambda,1)}$, *where* $W \models \mathrm{ZFC} + \mathrm{CH}_\lambda + \lambda^{<\lambda} = \lambda$;
- $V = W^{\mathrm{Prikry}(\lambda)}$, *where* $W \models \mathrm{ZFC} + \mathrm{CH}_\lambda + \lambda$ *is measurable;*
- $\lambda^{<\lambda} = \lambda + \mathrm{CH}_\lambda + \mathrm{NS} \restriction E^{\lambda}_{\theta}$ *is saturated where* $\lambda = \theta^+$ *for regular* θ;
- $\lambda^{<\lambda} = \lambda + \mathrm{CH}_\lambda + \exists$ *a non-reflecting stationary set of* $E^{\lambda^+}_{<\lambda}$.

Still more

By further tweaking the parameters and varying the construction slightly, we can construct a Souslin tree from weaker axioms than those mentioned earlier.

Theorem

A form of the proxy principle $P(\dots)$ holds enabling the construction of a λ^+-Souslin tree for uncountable λ, assuming any of the following:

- $\lambda^{<\lambda} = \lambda + \diamondsuit(E^{\lambda^+}_{\lambda})$;
- $V = W^{\mathrm{Add}(\lambda,1)}$, *where* $W \models \mathrm{ZFC} + \mathrm{CH}_\lambda + \lambda^{<\lambda} = \lambda$;
- $V = W^{\mathrm{Prikry}(\lambda)}$, *where* $W \models \mathrm{ZFC} + \mathrm{CH}_\lambda + \lambda$ *is measurable;*
- $\lambda^{<\lambda} = \lambda + \mathrm{CH}_\lambda + \mathrm{NS} \restriction E^{\lambda}_{\theta}$ *is saturated where* $\lambda = \theta^+$ *for regular* θ;
- $\lambda^{<\lambda} = \lambda + \mathrm{CH}_\lambda + \exists$ *a non-reflecting stationary set of* $E^{\lambda^+}_{<\lambda}$.
- $2^{<\lambda} = \lambda + \square^*_\lambda + \mathrm{CH}_\lambda + \exists$ *a non-reflecting stationary subset of* $E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$.

Returning to some earlier scenarios

Corollary

If $\lambda^{<\lambda} = \lambda$, CH_λ, *and* $\curlywedge^*(\lambda, E^{\lambda^+}_\lambda)$ *holds for a regular uncountable cardinal* λ, *then there exists a free* λ^+*-Souslin tree.*

Returning to some earlier scenarios

Corollary

If $\lambda^{<\lambda} = \lambda$, CH_λ, *and* $\text{⅄}^*(\lambda, E^{\lambda^+}_\lambda)$ *holds for a regular uncountable cardinal* λ, *then there exists a free* λ^+*-Souslin tree.*

Corollary

If $\lambda^{<\lambda} = \lambda$ *and* $\langle\lambda\rangle^-_{E^{\lambda^+}_\lambda}$ *holds for a regular uncountable cardinal* λ, *then there exists a free* λ^+*-Souslin tree.*

Returning to some earlier scenarios

Corollary

If $\lambda^{<\lambda} = \lambda$, CH_λ, and $\text{⅄}^*(\lambda, E^{\lambda^+}_{\lambda})$ holds for a regular uncountable cardinal λ, then there exists a free λ^+-Souslin tree.

Corollary

If $\lambda^{<\lambda} = \lambda$ and $\langle\lambda\rangle^-_{E^{\lambda^+}_{\lambda}}$ holds for a regular uncountable cardinal λ, then there exists a free λ^+-Souslin tree.

Corollary

If $\boxminus_{\lambda,\geq\chi}$ and CH_λ for cardinals $\chi < \lambda$ where λ is a singular strong limit cardinal, then there exists a free λ^+-Souslin tree.

Returning to some earlier scenarios

Corollary

If $\lambda^{<\lambda} = \lambda$, CH_λ, and $\text{⅄}^*(\lambda, E^{\lambda^+}_\lambda)$ holds for a regular uncountable cardinal λ, then there exists a free λ^+-Souslin tree.

Corollary

If $\lambda^{<\lambda} = \lambda$ and $\langle\lambda\rangle^-_{E^{\lambda^+}_\lambda}$ holds for a regular uncountable cardinal λ, then there exists a free λ^+-Souslin tree.

Corollary

If $\boxminus_{\lambda,\geq\chi}$ and CH_λ for cardinals $\chi < \lambda$ where λ is a singular strong limit cardinal, then there exists a free λ^+-Souslin tree.

The last theorem uses the ascent path to ensure that the construction goes through despite the possible failure of $\square^*_\lambda$ in this case.

Even more

Theorem

Assuming the consistency of a supercompact cardinal, there is a model of ZFC that satisfies:

1. *Martin's Maximum holds, and hence:*
   1.1 $\square^*_\lambda$ *fails for every singular cardinal* λ *of countable cofinality;*
   1.2 $\square_{\lambda,\aleph_1}$ *fails for every regular uncountable cardinal* λ*;*
   1.3 *There does not exist any* $\aleph_1$*-Souslin or* $\aleph_2$*-Souslin tree.*
2. $P(\lambda^+, 2, \sqsubseteq_{\aleph_2}, \lambda^+, \{E^{\lambda^+}_{\mathrm{cf}(\lambda)}\}, 2, \omega, \mathcal{E}_\lambda)$ *holds for every singular cardinal* λ*;*
3. $P(\lambda^+, 2, {}_\lambda{\sqsubseteq}, \lambda^+, \{E^{\lambda^+}_\lambda\}, 2, \omega, \mathcal{E}_\lambda)$ *holds for every regular uncountable cardinal* λ*.*

Even more

Theorem

Assuming the consistency of a supercompact cardinal, there is a model of ZFC *that satisfies:*

1. *Martin's Maximum holds, and hence:*
   1.1 $\square^*_\lambda$ *fails for every singular cardinal* λ *of countable cofinality;*
   1.2 $\square_{\lambda,\aleph_1}$ *fails for every regular uncountable cardinal* λ*;*
   1.3 *There does not exist any* $\aleph_1$*-Souslin or* $\aleph_2$*-Souslin tree.*
2. $P(\lambda^+, 2, \sqsubseteq_{\aleph_2}, \lambda^+, \{E^{\lambda^+}_{\mathrm{cf}(\lambda)}\}, 2, \omega, \mathcal{E}_\lambda)$ *holds for every singular cardinal* λ*;*
3. $P(\lambda^+, 2, {}_\lambda\sqsubseteq, \lambda^+, \{E^{\lambda^+}_\lambda\}, 2, \omega, \mathcal{E}_\lambda)$ *holds for every regular uncountable cardinal* λ*.*
4. *There are no inaccessible cardinals;*

Even more

Theorem

Assuming the consistency of a supercompact cardinal, there is a model of ZFC *that satisfies:*

1. *Martin's Maximum holds, and hence:*
   1.1 $\square^*_\lambda$ *fails for every singular cardinal* λ *of countable cofinality;*
   1.2 $\square_{\lambda,\aleph_1}$ *fails for every regular uncountable cardinal* λ*;*
   1.3 *There does not exist any* $\aleph_1$*-Souslin or* $\aleph_2$*-Souslin tree.*
2. $P(\lambda^+, 2, \sqsubseteq_{\aleph_2}, \lambda^+, \{E^{\lambda^+}_{\mathrm{cf}(\lambda)}\}, 2, \omega, \mathcal{E}_\lambda)$ *holds for every singular cardinal* λ*;*
3. $P(\lambda^+, 2, {}_\lambda\sqsubseteq, \lambda^+, \{E^{\lambda^+}_\lambda\}, 2, \omega, \mathcal{E}_\lambda)$ *holds for every regular uncountable cardinal* λ.
4. *There are no inaccessible cardinals;*

From (2), (3) and (4), it follows that there exists a free κ*-Souslin tree for every regular cardinal* $\kappa > \aleph_2$.

Even more

Theorem

Assuming the consistency of a supercompact cardinal, there is a model of ZFC that satisfies:

1. *Martin's Maximum holds, and hence:*
   1.1 $\square^*_\lambda$ *fails for every singular cardinal* λ *of countable cofinality;*
   1.2 $\square_{\lambda,\aleph_1}$ *fails for every regular uncountable cardinal* λ*;*
   1.3 *There does not exist any* $\aleph_1$*-Souslin or* $\aleph_2$*-Souslin tree.*
2. $P(\lambda^+, 2, \sqsubseteq_{\aleph_2}, \lambda^+, \{E^{\lambda^+}_{\mathrm{cf}(\lambda)}\}, 2, \omega, \mathcal{E}_\lambda)$ *holds for every singular cardinal* λ*;*
3. $P(\lambda^+, 2, {}_\lambda\sqsubseteq, \lambda^+, \{E^{\lambda^+}_\lambda\}, 2, \omega, \mathcal{E}_\lambda)$ *holds for every regular uncountable cardinal* λ.
4. *There are no inaccessible cardinals;*

From (2), (3) and (4), it follows that there exists a free κ*-Souslin tree for every regular cardinal* $\kappa > \aleph_2$.
For $\lambda > \mathrm{cf}(\lambda) = \omega$*, we seal the antichains at points of* $E^{\lambda^+}_\omega$*, even though* MM *implies that every stationary subset of* $E^{\lambda^+}_\omega$ *reflects!*

The end?

We've reached the end of today's presentation.
But the story doesn't end here.
Would you like to join our tree-building adventure?

Bibliography

Ari Meir Brodsky and Assaf Rinot.
Reduced powers of Souslin trees.
arXiv preprint arXiv:1507.05651, 2015.

Ari M. Brodsky and Assaf Rinot.
A microscopic approach to Souslin-tree constructions. Part I.
arXiv preprint arXiv:1601.01821, 2015.

Ari M. Brodsky and Assaf Rinot.
A microscopic approach to Souslin-tree constructions. Part II.
In preparation, 2016.