A GEOMETRIC INTRODUCTION TO FORKING AND THORN-FORKING

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A ternary relation $\mathop{\smile\hskip-0.9em ^| \ }$ between subsets of the big model of a complete first-order theory T is called an independence relation if it satisfies a certain set of axioms. The primary example is forking in a simple theory, but o-minimal theories are also known to have an interesting independence relation. Our approach in this paper is to treat independence relations as mathematical objects worth studying. The main application is a better understanding of thorn-forking, which turns out to be closely related to modular pairs in the lattice of algebraically closed sets.

Keywords: Independence relation; Forking; Thorn-forking.

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Introduction

This paper is based on the first chapter of my thesis under the supervision of Martin Ziegler [1]. The exposition has been adapted to the new medium, and some easy proofs have been omitted.

Saharon Shelah defined forking to study stable theories [2]. Later Byunghan Kim showed that even in the larger class of simple theories forking is very well-behaved in the sense that it defines what we will call an independence relation [3]. O-minimal theories, which are never simple, satisfy the exchange principle for algebraic closure [4]. Therefore they also have a nice independence relation. Alf Onshuus introduced thorn-forking, which generalises both forking in simple theories (with elimination of hyperimaginaries) and the natural independence relation in o-minimal theories [5, 6].

Here we will treat independence relations as mathematical objects. We will develop forking and thorn-forking from scratch and as much in parallel as possible. We will restrict ourselves to arguments of a very simple, general, geometric flavour, reaching some fundamental results on forking and thorn-forking very quickly. But some seem unattainable in this way.

There are good reasons why geometric stability theory is usually based on a combinatorial foundation. In a sequel to this paper we will see how the usual combinatorial approach can be adapted to give sharper results on forking and thorn-forking, again in a completely uniform way for both notions [7].

In Section 1 we introduce the axioms for independence relations and some additional properties that a relation $\mathop{\smile\hskip-0.9em ^| \ }$ may satisfy. We also meet the first important example of an independence relation, the relation $\mathop{\smile\hskip-0.9em ^| \ }^{\mathrm{a}}$ (sometimes called 'algebraic independence') in a theory in which the lattice of algebraically closed sets is modular. In Section 2 we use Morley sequences in order to prove that every independence relation is symmetric. In Section 3 we show that, in a certain sense, the extension axiom comes free. We use these results in Sections 4 and 5, where we define forking and thorn-forking, the two primary examples of independence relation. They can be characterised by dual universal properties. Section 6 finishes with a number of contrived examples of relations that are not independence relations.

We use the standard notations and conventions of stability theory such as working in a big saturated model. Tuples of elements $(\bar{a}, \bar{b}, \bar{c}, \dots)$ or of variables $(\bar{x}, \bar{y}, \bar{z}, \dots)$ are allowed to be infinite unless mentioned otherwise. When a formula is written $\varphi(\bar{x})$ it means that each of its free variables appears in the (possibly infinite) tuple $\bar{x}$. $\mathrm{S}^*(B)$ denotes the class of complete types over B in arbitrarily long sequences of distinct formal variables. AB is short for $A \cup B$, and for any tuple $\bar{a} = (a_0, a_1, \dots,)$ we will abuse notation by writing $\bar{a}$ for the set $\{a_0, a_1, \dots, \}$ as well. $A \equiv_B A'$ means that there is an automorphism of the big model that fixes B pointwise and maps the set A to the set A'. $\bar{a} \equiv_B \bar{a}'$ means that there is an automorphism fixing B pointwise and mapping the tuple $\bar{a}$ to the tuple $\bar{a}'$. $(A, B) \equiv_C (A', B')$ means that there is an automorphism fixing C pointwise that maps A to A' and B to B'.

1 Abstract Independence Relations

We will call a ternary relation $\mathop{\smile\!\!\!\!\!\mid}$ between (small) subsets of the big model an *independence relation* if it satisfies the following *axioms for independence relations.*

(invariance)
If $A \mathop{\smile\!\!\!\!\!\mid}_C B$ and $(A', B', C') \equiv (A, B, C)$, then $A' \mathop{\smile\!\!\!\!\!\mid}_{C'} B'$.

(monotonicity)
If $A \mathop{\smile\!\!\!\!\!\mid}_C B$, $A' \subseteq A$ and $B' \subseteq B$, then $A' \mathop{\smile\!\!\!\!\!\mid}_C B'$.

(base monotonicity)
Suppose $D \subseteq C \subseteq B$. If $A \mathop{\smile\!\!\!\!\!\mid}_D B$, then $A \mathop{\smile\!\!\!\!\!\mid}_C B$.

(transitivity)
Suppose $D \subseteq C \subseteq B$. If $B \mathop{\smile\!\!\!\!\!\mid}_C A$ and $C \mathop{\smile\!\!\!\!\!\mid}_D A$, then $B \mathop{\smile\!\!\!\!\!\mid}_D A$.

(normality)
$A \mathop{\smile\!\!\!\!\!\mid}_C B$ implies $AC \mathop{\smile\!\!\!\!\!\mid}_C B$.

(extension)
If $A \mathop{\smile\!\!\!\!\!\mid}_C B$ and $\hat{B} \supseteq B$, then there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\mid}_C \hat{B}$.

(finite character)
If $A_0 \mathop{\smile\!\!\!\!\!\mid}_C B$ for all finite $A_0 \subseteq A$, then $A \mathop{\smile\!\!\!\!\!\mid}_C B$.

(local character)
For every A there is a cardinal $\kappa(A)$ such that for any set B there is a subset $C \subseteq B$ of cardinality $|C| < \kappa(A)$ such that $A \mathop{\smile\!\!\!\!\!\mid}_C B$.

By invariance, in the extension axiom it would have been equivalent to require that there is $\hat{B}' \equiv_{BC} \hat{B}$ such that $A \mathop{\smile\!\!\!\!\!\mid}_C \hat{B}'$. 'By transitivity' will often refer to the following stronger property.

Remark 1.1. Let $\mathop{\smile\!\!\!\!\!\mid}$ be a relation satisfying monotonicity, transitivity and normality. Then $B \mathop{\smile\!\!\!\!\!\mid}_{CD} A$ and $C \mathop{\smile\!\!\!\!\!\mid}_D A$ together imply $BC \mathop{\smile\!\!\!\!\!\mid}_D A$.

Proof. If $B \mathop{\smile\!\!\!\!\!\mid}_{CD} A$ and $C \mathop{\smile\!\!\!\!\!\mid}_D A$, then $BCD \mathop{\smile\!\!\!\!\!\mid}_{CD} A$ and $CD \mathop{\smile\!\!\!\!\!\mid}_D A$ by normality. Hence we have $BCD \mathop{\smile\!\!\!\!\!\mid}_D A$ by transitivity, so $BC \mathop{\smile\!\!\!\!\!\mid}_D A$ by monotonicity. □

As a first example, we can take the trivial relation that holds for all triples. We say that a relation $\mathop{\smile\!\!\!\!\!\mid}$ is *stronger* than $\mathop{\smile\!\!\!\!\!\mid}'$, and $\mathop{\smile\!\!\!\!\!\mid}'$ *weaker* than $\mathop{\smile\!\!\!\!\!\mid}$, if $A \mathop{\smile\!\!\!\!\!\mid}_C B$ implies $A \mathop{\smile\!\!\!\!\!\mid}'_C B$. So the trivial relation is always the weakest independence relation. In fact, it is too weak to be useful, and so we will say that an independence relation is *strict* if it also satisfies the following condition.

(anti-reflexivity)
$a \mathop{\smile\!\!\!\!\!\mid}_B a$ implies $a \in \operatorname{acl} B$.

The trivial relation is, of course, never strict. We will also be interested in the following additional properties which a relation $\mathop{\smile\!\!\!\!\!\mid}$ may have.

(full existence)
For any A, B and C there is $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\mid}_C B$.

(symmetry)
$A \mathop{\smile\!\!\!\!\!\mid}_C B \iff B \mathop{\smile\!\!\!\!\!\mid}_C A$.

Remark 1.2. (1) Any relation that satisfies invariance, extension and symmetry also satisfies normality.

(2) Any relation that satisfies invariance, extension and local character also satisfies full existence.

(3) Any relation that satisfies invariance, monotonicity, transitivity, normality, full existence and symmetry also satisfies extension.

Remark 1.3. (1) Suppose the relation $\mathop{\smile\!\!\!\!\!\!|}$ satisfies invariance and the full existence condition. Let $\kappa(A) = (|T| + |A|)^+$. For any sets A and B there is $C_1 \subseteq B$ such that $A \mathop{\smile\!\!\!\!\!\!|}_{C_1} B$ and also a set C_2 such that $|C_2| \leq \kappa(A)$ and $A \mathop{\smile\!\!\!\!\!\!|}_{C_2} B$.

(2) Suppose $\mathop{\smile\!\!\!\!\!\!|}$ is an independence relation. Let $\mathcal{A}$ be a set of finite subsets of the big model such that for every finite subset A of the big model there is $A' \in \mathcal{A}$ such that $A \equiv A'$. Let $\kappa = \sup_{A \in \mathcal{A}} \kappa(A)$. Then for any sets A and B there is $C \subseteq B$ such that $|C| < (\kappa + |A|)^+$ and $A \mathop{\smile\!\!\!\!\!\!|}_C B$.

The reader may want to prove these easy remarks, in order to get a feeling for the axioms. Our first real example is a well-known stable theory in which forking has a nice graph theoretical interpretation [8].

Example 1.4. (Everywhere infinite forest)
Let T be the theory, in a signature with one binary relation E, of a non-empty undirected tree that branches infinitely in every node. Then T is complete, and the models of T are precisely the non-empty forests that branch infinitely in every node. In this theory, $\operatorname{acl} A$ is the set of all nodes that lie on a path between two elements of A. Consider the following relation.

$$A \mathop{\smile\!\!\!\!\!\!|}_C B \iff \text{every path from } A \text{ to } B \text{ meets } \operatorname{acl} C.$$

It is easy to see that $A \mathop{\smile\!\!\!\!\!\!|}_C B$ implies $AC \mathop{\smile\!\!\!\!\!\!|}_C B$ and $\operatorname{acl} A \mathop{\smile\!\!\!\!\!\!|}_C B$. Using this, it is not hard to check that $\mathop{\smile\!\!\!\!\!\!|}$ is a strict independence relation.

Proposition 1.5. *Consider the following relation.*

$$A \mathop{\smile\!\!\!\!\!\!|}^{a}_C B \iff \operatorname{acl}(AC) \cap \operatorname{acl}(BC) = \operatorname{acl} C.$$

(1) The relation $\mathop{\smile\!\!\!\!\!\!|}^{a}$ satisfies the full existence condition.

(2) The relation $\mathop{\smile\!\!\!\!\!\!|}^{a}$ satisfies all axioms for strict independence relations except base monotonicity.

(3) The relation $\mathop{\smile\!\!\!\!\!\!|}^{a}$ satisfies base monotonicity (and is a strict independence relation) iff the lattice of algebraically closed subsets of the big model is modular, *i. e., whenever A, B, C are algebraically closed sets such that $B \supseteq C$, the equation $B \cap \operatorname{acl}(AC) = \operatorname{acl}((B \cap A)C)$ holds.*

(4) An independence relation is called perfectly trivial *if $A \mathop{\smile\!\!\!\!\!\!|}_C B$ implies $A \mathop{\smile\!\!\!\!\!\!|}_{C'} B$ for all $C' \supseteq C$. Suppose $\mathop{\smile\!\!\!\!\!\!|}^{a}$ is an independence relation. Then $\mathop{\smile\!\!\!\!\!\!|}^{a}$ is a perfectly trivial independence relation if and only if the lattice of algebraically closed sets is* distributive, *i. e., $\operatorname{acl}((A \cap B)C) = \operatorname{acl}((A \cap C)(B \cap C))$ holds for all algebraically closed sets A, B and C.*

Proof. (1) For completeness we include a proof of this well-known fact. It is sufficient to show: If A, B and $C = \operatorname{acl} C$ are such that $A \cap C = B \cap C = \emptyset$, then there is $A' \equiv_C A$ such that $A' \cap B = \emptyset$. If this were false, then by compactness there would be a counter-example with A and B finite. Towards a contradiction let A, B and $C = \operatorname{acl} C$ be such that $A \cap C = B \cap C = \emptyset$, $A' \cap B \neq \emptyset$ for all $A' \equiv_C A$, B finite and $|A|$ minimal for these properties. Let $A_* \subseteq A$ be a maximal subset of A such that $\{A' \mid A' \equiv_C A \text{ and } A_* \subseteq A'\}$ is infinite. By minimality of $|A|$ there is $A'_* \equiv_C A_*$ such that $A'_* \cap B = \emptyset$. We may assume $A'_* = A_*$, so $A_* \cap B = \emptyset$. For every $b \in B$, by maximality of A_* there are only finitely many $A' \equiv_C A$ such that $A_* \cup b \subseteq A'$. Hence there are only finitely many $A' \equiv_C A$ such that $A_* \subseteq A'$ and $A' \cap B \neq \emptyset$. Since $\{A' \mid A' \equiv_C A \text{ and } A_* \subseteq A'\}$ is infinite, there is $A' \equiv_C A$ such that $A_* \subseteq A$ and $A' \cap B = \emptyset$, a contradiction.

(2) *Invariance, monotonicity* and *normality* are obvious. *Finite character:* Suppose $d \in \operatorname{acl}(AC) \cap \operatorname{acl}(BC) \setminus \operatorname{acl} C$. Then d is already algebraic over a finite tuple $\bar{a}\bar{c}$ with $\bar{a} \in A$ and $\bar{c} \in C$, and also over a finite tuple $\bar{b}\bar{c}'$ with $\bar{b} \in B$ and $\bar{c}' \in C$. Hence $d \in \operatorname{acl}(C\bar{a}) \cap \operatorname{acl}(C\bar{b}) \setminus \operatorname{acl} C$. *Transitivity:* Suppose $D \subseteq C \subseteq B$. If $\operatorname{acl} B \cap \operatorname{acl}(AC) \subseteq \operatorname{acl} C$ and $\operatorname{acl} C \cap \operatorname{acl}(AD) \subseteq \operatorname{acl} D$, then $\operatorname{acl} B \cap \operatorname{acl}(AD) \subseteq \operatorname{acl} C \cap \operatorname{acl}(AD) \subseteq \operatorname{acl} D$. *Extension:* Using (3) of Remark 1.2 this follows from (1), symmetry and the other axioms already shown to hold. *Local character:* Given sets A and B, construct sets $C_i \subseteq B$ and D_i $(i < \omega)$ as follows: $C_0 = D_0 = \emptyset$. $D_{i+1} = \operatorname{acl}(AC_i) \cap \operatorname{acl} B$. For every $d \in D_{i+1}$ let $\bar{c}_d \in B$ be a finite tuple such that $d \in \operatorname{acl} \bar{c}_d$. Let C_{i+1} be the union over all tuples $\bar{c}_d$. Let $C = \bigcup_{i<\omega} C_i$. It is easy to see that $C \subseteq B$ and $|C| \leq |T| + |A|$. Moreover, if $d \in \operatorname{acl}(AC) \cap \operatorname{acl}(BC)$, then already $d \in \operatorname{acl}(AC_i) \cap \operatorname{acl}(BC) \subseteq D_{i+1}$ for some $i < \omega$, hence $d \in \operatorname{acl} C_{i+1} \subseteq \operatorname{acl} C$.

(3) Let A and $B \supseteq C$ be algebraically closed sets. First note that $B \cap \operatorname{acl}(AC) \supseteq \operatorname{acl}((B \cap A)C)$ holds without any further assumptions. Now suppose $\mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{a}}$ satisfies the base monotonicity axiom. Then $A \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{a}}_{A\cap B} B$ implies $A \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{a}}_{(A\cap B)C} B$. Hence $B \cap \operatorname{acl}(AC) \subseteq \operatorname{acl}((B \cap A)C)$. Conversely, suppose the modular law holds, $A \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{a}}_{C} B$ and $C \subseteq C' \subseteq B$. Then $\operatorname{acl} B \cap \operatorname{acl}(AC') \subseteq \operatorname{acl}((\operatorname{acl} B \cap \operatorname{acl} A)C')$ by modularity and $C' \subseteq B$. Note that $\operatorname{acl} B \cap \operatorname{acl} A \subseteq C \subseteq C'$, so $\operatorname{acl} B \cap \operatorname{acl}(AC') \subseteq \operatorname{acl} C'$. Hence $A \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{a}}_{C'} B$.

(4) Let A, B, C be algebraically closed sets. First note that $\operatorname{acl}((A \cap B)C) \subseteq \operatorname{acl}(AC) \cap \operatorname{acl}(BC)$ holds without any further assumptions. Now suppose $\mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{a}}$ is perfectly trivial. Since $A \mathop{\smile\hskip -0.9em ^| \ }_{A\cap B} B$ it follows that $A \mathop{\smile\hskip -0.9em ^| \ }_{(A\cap B)C} B$ as well, hence $AC \mathop{\smile\hskip -0.9em ^| \ }_{(A\cap B)C} BC$ (by base monotonicity, which can be applied on both sides due to symmetry), hence $\operatorname{acl}(AC) \cap \operatorname{acl}(BC) \subseteq \operatorname{acl}((A \cap B)C)$. Conversely, suppose the lattice is distributive, $A \mathop{\smile\hskip -0.9em ^| \ }_{C} B$ and $C' \supseteq C$. Then $\operatorname{acl}(AC') \cap \operatorname{acl}(BC') = \operatorname{acl}((\operatorname{acl} A \cap \operatorname{acl} B)C') \subseteq \operatorname{acl}(CC') = \operatorname{acl} C'$, hence $A \mathop{\smile\hskip -0.9em ^| \ }_{C'} B$. □

Proposition 1.5 is from my diploma thesis [9]. It was inspired by Lee Fong Low's thesis [10], but its roots are much older. In fact, if we apply John von Neumann's definition of $A \perp B$ in his 1930s work on lattice theory to the lattice of algebraically closed sets it means $A \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{a}}_{\emptyset} B$ [11]. The original context was a very special type of lattices which were, in particular, modular. Similarities between von Neumann's reconstruction of a ring from its lattice of ideals and Hrushovski's model theoretical group constructions were also noted by Ivan Tomašić [12].

Example 1.6. (Theories with many strict independence relations)
For any cardinal κ consider the free theory of κ equivalence relations E_i $(i < \kappa)$. Then for any $S \subseteq \kappa$ the following relation is a strict independence relation.

$$A \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{S}}_{C} B \iff A/E_i \cap B/E_i \subseteq C/E_i \text{ for all } i \in S.$$

So this theory admits at least 2^κ different strict independence relations.

On the other hand, we will soon encounter a theory with no strict independence relations at all.

Question 1.7. Is there a complete theory T such that T^{eq} has more than one strict independence relation?

Notes on the choice of axioms

Using Theorem 2.5 below it is easy to see that an independence relation is the same thing as a notion of independence in the sense of Kim and Pillay [13]. It follows that if we add the independence theorem over models to the axioms, they characterise forking independence in simple theories.

The single most important point about these axioms is the fact that transitivity is stated on the left-hand side and not, as usual, on the right-hand side, where it can be combined with monotonicity and base monotonicity into an axiom that is sometimes called 'full transitivity'. A variant of transitivity on the left-hand side is sometimes called the 'pairs lemma'.

Moreover, it is important to distinguish extension and full existence. In early axiomatisations the term 'extension' sometimes referred to a single axiom which combined extension with full existence. Kim and Pillay call 'extension' what we call full existence [13]. But most people seem to agree with our use of the term 'extension', using the term 'existence' for the axiom saying $A \mathop{\smile\!\!\!\!\!\mid}_B B$ for all A, B.

Local character has been revised to make it useful without finite character. Note that Itay Ben-Yaacov uses yet another, more robust, version of local character, which is more suitable for certain generalisation [14].

2 Morley Sequences and Symmetry

We now prove that every independence relation is symmetric. But we need some preparations.

Remark 2.1. Let $\mathop{\smile\!\!\!\!\!\mid}$ be an independence relation.
If $A \mathop{\smile\!\!\!\!\!\mid}_D BC$ and $B \mathop{\smile\!\!\!\!\!\mid}_D C$, then $AB \mathop{\smile\!\!\!\!\!\mid}_D C$.
Actually, it is sufficient that $\mathop{\smile\!\!\!\!\!\mid}$ satisfies the first five axioms.

Proof. $A \mathop{\smile\!\!\!\!\!\mid}_D BC$ implies $A \mathop{\smile\!\!\!\!\!\mid}_D BCD$ by extension and invariance, hence $A \mathop{\smile\!\!\!\!\!\mid}_{BD} BCD$ by base monotonicity, hence $A \mathop{\smile\!\!\!\!\!\mid}_{BD} C$ by monotonicity. Combining this with $B \mathop{\smile\!\!\!\!\!\mid}_D C$, we get $AB \mathop{\smile\!\!\!\!\!\mid}_D C$ by transitivity. □

If we know that $\mathop{\smile\!\!\!\!\!\mid}$ is an independence relation, and that $\mathop{\smile\!\!\!\!\!\mid}$ is symmetric, then it is clear how to generalise the notion of a Morley sequence for $\mathop{\smile\!\!\!\!\!\mid}$. The following definition for the more general case is useful in this paper, but I do not claim that it is appropriate in different contexts.

Let $\mathop{\smile\!\!\!\!\!\mid}$ be a ternary relation. A $\mathop{\smile\!\!\!\!\!\mid}$*-Morley sequence in* a type $p(\bar{x}) \in S^*(B)$ over a set $C \subseteq B$ is a sequence of B-indiscernibles $(\bar{a}_i)_{i<\omega}$ such that $(\bar{a}_i)_{i<n} \mathop{\smile\!\!\!\!\!\mid}_C \bar{a}_n$ for every $n < \omega$, and every $\bar{a}_i$ realises $p(\bar{x})$. A $\mathop{\smile\!\!\!\!\!\mid}$*-Morley sequence for* a complete type $p(\bar{x}) \in S^*(B)$ is a $\mathop{\smile\!\!\!\!\!\mid}$-Morley sequence in $p(\bar{x})$ over B.

Recall our convention that tuples may be infinite. In most cases just a convenience, this is crucial in this section and the next one. The following consequence of the Erdős-Rado theorem is due to Saharon Shelah [15].

Fact 2.2. (Extracting a sequence of indiscernibles) Let B be a set of parameters and κ a cardinal. Then for any sequence $(\bar{a}_i)_{i<\beth_{(2^{|T|+|B|+\kappa})^+}}$ consisting of sequences of length $|\bar{a}_i| = \kappa$ there is a B-indiscernible sequence $(\bar{a}'_j)_{j<\omega}$ with the following property: For every $k < \omega$ there are $i_0 < i_1 < \ldots < i_k < \kappa$ such that $\bar{a}_{i_0}, \bar{a}_{i_1}, \ldots, \bar{a}_{i_k} \equiv_B \bar{a}'_0, \bar{a}'_1, \ldots, \bar{a}'_k$.

Proposition 2.3. *Suppose* $\mathop{\smile\!\!\!\!\!\mid}$ *is an independence relation and* $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_C B$*. Then there is a Morley sequence in* $\operatorname{tp}(\bar{a}/BC)$ *over* C*.*
Actually, it is sufficient that $\mathop{\smile\!\!\!\!\!\mid}$ *satisfies the first five axioms and extension.*

Proof. Let $\bar{a}_0 = \bar{a}$. We choose a very big cardinal κ and use extension and transfinite induction to construct a sequence $(\bar{a}_i)_{i<\kappa}$ satisfying $\bar{a}_\alpha \equiv_{BC} \bar{a}_0$ and $\bar{a}_\alpha \mathop{\smile\!\!\!\!\!\mid}_C (\bar{a}_i)_{i<\alpha}$ for all $\alpha < \kappa$. If κ has been chosen sufficiently big, we can extract a sequence of BC-indiscernibles $(\bar{a}'_i)_{i<\omega}$ such that for every $n < \omega$ there are indices $\alpha_0, \ldots, \alpha_n$ such that $\bar{a}'_0 \ldots \bar{a}'_n \equiv_{BC} \bar{a}_{\alpha_0} \ldots \bar{a}_{\alpha_n}$. Note that $\bar{a}'_n \mathop{\smile\!\!\!\!\!\mid}_C (\bar{a}'_i)_{i<n}$ by monotonicity and invariance.

By compactness we can 'invert' the sequence $(\bar{a}'_i)_{i<\omega}$, i. e., find a new sequence $(\bar{a}''_i)_{i<\omega}$ such that $\bar{a}''_0 \ldots \bar{a}''_n \equiv_{BC} \bar{a}'_n \ldots \bar{a}'_0$ holds for every $n < \omega$. In particular, the new sequence is also indiscernible over BC. This new sequence satisfies $\bar{a}''_0 \mathop{\smile\!\!\!\!\!\mid}_C (\bar{a}''_i)_{0<i<n}$ for all $n < \omega$. Hence $(\bar{a}''_i)_{i<n} \mathop{\smile\!\!\!\!\!\mid}_C \bar{a}''_n$ for all $n < \omega$ by repeated use of Remark 2.1.

Thus $(\bar{a}''_i)_{i<\omega}$ is a Morley sequence in $\operatorname{tp}(\bar{a}/BC)$ over C. □

The trick used to prove the following proposition is due to Byunghan Kim, from whose proof of forking symmetry in simple theories this section was derived [3].

Proposition 2.4. *Suppose $\mathop{\smile\!\!\!\!\!\!|}\,$ is an independence relation and there is a Morley sequence in* $\operatorname{tp}(\bar a/BC)$ *over* C*. Then* $B \mathop{\smile\!\!\!\!\!\!|}_C \bar a$*.*
Actually, it is sufficient that $\mathop{\smile\!\!\!\!\!\!|}\,$ satisfies the first five axioms, finite character and local character.

Proof. Let $(\bar a_i)_{i<\omega}$ be a Morley sequence in $\operatorname{tp}(\bar a/BC)$ over C.

Let κ be a regular cardinal number that is greater than or equal to $\kappa(B)$ as in the local character axiom. By compactness we can extend the sequence $(\bar a_i)_{i<\omega}$ to obtain a BC-indiscernible sequence $(\bar a_i)_{i<\kappa}$. Using finite character, we see that $(\bar a_i)_{i<\alpha} \mathop{\smile\!\!\!\!\!\!|}_C \bar a_\alpha$ for each $\alpha < \kappa$.

By local character there is a set $D \subseteq C(\bar a_i)_{i<\kappa}$ of cardinality $|D| < \kappa$ such that $B \mathop{\smile\!\!\!\!\!\!|}_D C(\bar a_i)_{i<\kappa}$. By regularity of κ there is an index $\alpha < \kappa$ such that already $D \subseteq C(\bar a_i)_{i<\alpha}$. Thus, by base monotonicity and monotonicity, $B \mathop{\smile\!\!\!\!\!\!|}_{C(\bar a_i)_{i<\alpha}} \bar a_\alpha$.

Combining the results of the last two paragraphs using transitivity (actually, Remark 1.1 and monotonicity), we get $B \mathop{\smile\!\!\!\!\!\!|}_C \bar a_\alpha$. Since $\bar a_\alpha \equiv_{BC} \bar a$ this implies $B \mathop{\smile\!\!\!\!\!\!|}_C \bar a$ by invariance. □

For later use I have noted carefully which axioms were actually needed to prove Propositions 2.3 and 2.4. For our immediate use it would not have been necessary.

Theorem 2.5. *Every independence relation $\mathop{\smile\!\!\!\!\!\!|}\,$ is symmetric.*

Proof. If $\bar a \mathop{\smile\!\!\!\!\!\!|}_C B$, then there is a Morley sequence in $\operatorname{tp}(\bar a/BC)$ over C by Proposition 2.3. Hence $B \mathop{\smile\!\!\!\!\!\!|}_C \bar a$ by Proposition 2.4. □

For this theorem it seems to be crucial that we postulated transitivity on the left-hand side. The usual practice is to postulate it on the right-hand side and combine it with monotonicity and base monotonicity into an axiom called 'full transitivity'. Left-hand side transitivity is sometimes called the 'pairs lemma'.

Example 2.6. (A theory with no strict independence relation)
Consider the following theory T_0: T_0 has two sorts P and E, the elements of which are called 'points' and 'equivalence relations', and a single ternary relation $\sim \subseteq P \times P \times E$ written as $p \sim_e q$. The axioms of T_0 say that $\sim_e$ is an equivalence relation on the points for every $e \in E$.

Clearly every substructure of a model of T_0 is again a model of T_0. The signature of T_0 is finite and relational. Moreover, the class of finite models of T_0 satisfies the joint embedding property and the amalgamation property. So T_0 has a Fraïssé limit T^* which is ω-categorical, has quantifier elimination, and whose finite submodels are precisely the finite models of T_0^*.

Since $\operatorname{acl} A = A$ for all sets, $A \mathop{\smile\!\!\!\!\!\!|}_C B \iff A \cap B \subseteq C$ defines a strict independence relation for T. But there is no strict independence relation for T^{eq}:

Suppose $\mathop{\smile\!\!\!\!\!\!|}\,$ is an independence relation for T^{eq}. Let $a_0 \in P$ be a single point, and let $(a_i)_{i<\omega}$ be a Morley sequence for $\operatorname{tp}(a_0/\emptyset)$. Let $e \in E$ be an equivalence relation such that $a_i \sim_e a_j$ for any $i, j < \omega$. Then $(a_i)_{i<\omega}$ is indiscernible over e.

Note that $(a_{2i}a_{2i+1})_{i<\omega}$ is a Morley sequence for $\operatorname{tp}(a_0a_1/\emptyset)$ and is also indiscernible over e. By Proposition 2.4, $e \mathop{\smile\!\!\!\!\!\!|}_\emptyset a_0a_1$ holds, so by base monotonicity, $e \mathop{\smile\!\!\!\!\!\!|}_{a_0} a_1$. On the other hand, $a_0 \mathop{\smile\!\!\!\!\!\!|}_\emptyset a_1$ also holds. Applying transitivity we get $a_0e \mathop{\smile\!\!\!\!\!\!|}_\emptyset a_1$. Using symmetry and base monotonicity, we get $a_0 \mathop{\smile\!\!\!\!\!\!|}_e a_1$.

But the equivalence class c of a_0 and a_1 under $\sim_e$ is (an element of T^{eq} and) definable both over a_0e and over a_1e. So $c \mathop{\smile\!\!\!\!\!\!|}_e c$. Since c is clearly not algebraic over e this contradicts anti-reflexivity. So $\mathop{\smile\!\!\!\!\!\!|}\,$ is not a strict independence relation for T^{eq}.

This example is essentially Shelah's theory T^*_{feq} from [16].

3 Forcing the Extension Axiom

We now show that, in a certain sense, the extension axiom comes free. The idea is really due to Saharon Shelah, since we merely treat an arbitrary relation $\mathop{\smile\!\!\!\!\!\!|}\,$ as if it was dividing independence and then do the step of passing to forking independence. For any relation $\mathop{\smile\!\!\!\!\!\!|}$ we define a new relation $\mathop{\smile\!\!\!\!\!\!|}^{*}$ as follows.

$$A \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} B \iff \big(\textit{ for all } \hat{B} \supseteq B \textit{ there is } A' \underset{BC}{\equiv} A \textit{ s.t. } A' \mathop{\smile\!\!\!\!\!\!|}_{C} \hat{B}\big).$$

Note that $A \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} B$ implies $A \mathop{\smile\!\!\!\!\!\!|}_{C} B$. Also, $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^{*}$ iff $\mathop{\smile\!\!\!\!\!\!|}$ satisfies the extension axiom. In analogy to the classical situation one might call $\mathop{\smile\!\!\!\!\!\!|}^{*}$ the notion of forking derived from the abstract notion of dividing given by $\mathop{\smile\!\!\!\!\!\!|}$. If $\mathop{\smile\!\!\!\!\!\!|}$ already satisfies some of the axioms for independence relations, then there cannot be much harm (possibly losing finite character and local character) in passing from $\mathop{\smile\!\!\!\!\!\!|}$ to $\mathop{\smile\!\!\!\!\!\!|}^{*}$, but we get extension free.

Lemma 3.1.
If $\mathop{\smile\!\!\!\!\!\!|}$ is a relation satisfying invariance and montonicity, then $\mathop{\smile\!\!\!\!\!\!|}^{*}$ satisfies invariance, monotonicity and extension. If, moreover, $\mathop{\smile\!\!\!\!\!\!|}$ satisfies one of the following axioms and properties, then $\mathop{\smile\!\!\!\!\!\!|}^{*}$ also satisfies it: base monotonicity, transitivity, normality, anti-reflexivity, full existence.

Proof. *Invariance* of $\mathop{\smile\!\!\!\!\!\!|}^{*}$ is obvious. *Monotonicity:* Suppose $A \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} B$, $A_0 \subseteq A$ and $B_0 \subseteq B$. Then for all $\hat{B} \supseteq B$ there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_{C} \hat{B}C$. Let $A'_0 \subseteq A'$ correspond to $A_0 \subseteq A$. Then clearly $A'_0 \equiv_{B_0C} A_0$ and $A'_0 \mathop{\smile\!\!\!\!\!\!|}_{C} \hat{B}$. Thus $A_0 \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} B_0$ holds. *Extension:* Suppose $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} B$, where $\bar{a}$ is a possibly infinite tuple, and let $\hat{B} \supseteq B$ be any superset of B. We first claim that there is a type $p(\bar{x}) \in \mathrm{S}^*(\hat{B}C)$, extending $\mathrm{tp}(\bar{a}/BC)$, such that for all cardinals κ there is a κ-saturated model $M \supseteq \hat{B}C$ and $\bar{a}' \models p(\bar{x})$ such that $\bar{a}' \mathop{\smile\!\!\!\!\!\!|}_{C} M$. If not, then for each $p(\bar{x}) \in \mathrm{S}^*(\hat{B}C)$ extending $\mathrm{tp}(\bar{a}/BC)$ there is a cardinal $\kappa(p)$ such that for no $\kappa(p)$-saturated model $M \supseteq \hat{B}C$ is there a tuple $\bar{a}' \models p$ satisfying $\bar{a}' \mathop{\smile\!\!\!\!\!\!|}_{C} M$. Let κ be the supremum of the cardinals $\kappa(p)$, and let $M \supseteq \hat{B}C$ be κ-saturated. Then there is no $\bar{a}' \equiv_{BC} \bar{a}$ such that $\bar{a}' \mathop{\smile\!\!\!\!\!\!|}_{C} M$. So we have found a contradiction to the definition of $\mathop{\smile\!\!\!\!\!\!|}^{*}$ and thereby proved our claim. Now choose $\bar{a}' \models p(\bar{x})$, where $p(\bar{x})$ is as in the claim. Then clearly $\bar{a}' \equiv_{BC} \bar{a}$, and $\bar{a}' \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} \hat{B}$ by monotonicity of $\mathop{\smile\!\!\!\!\!\!|}$. *Base monotonicity:* Suppose $A \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} B$ and $C \subseteq C' \subseteq B$. So for any $\hat{B} \supseteq B$ there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_{C} \hat{B}C$. Base monotonicity of $\mathop{\smile\!\!\!\!\!\!|}$ yields $A' \mathop{\smile\!\!\!\!\!\!|}_{C'} \hat{B}C$, so $A' \mathop{\smile\!\!\!\!\!\!|}_{C'} \hat{B}$ by monotonicity of $\mathop{\smile\!\!\!\!\!\!|}$. Thus $A \mathop{\smile\!\!\!\!\!\!|}^{*}_{C'} B$. *Transitivity:* Here we work with an alternative definition of $\mathop{\smile\!\!\!\!\!\!|}^{*}$, which is equivalent to our definition by invariance of $\mathop{\smile\!\!\!\!\!\!|}$:

$$A \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} B \iff \big(\text{ for all } \hat{B} \supseteq B \text{ there is } \hat{B}' \underset{BC}{\equiv} \hat{B} \text{ s.t. } A \mathop{\smile\!\!\!\!\!\!|}_{C} \hat{B}'\big).$$

So suppose $D \subseteq C \subseteq B$, $B \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} A$ and $C \mathop{\smile\!\!\!\!\!\!|}^{*}_{D} A$ hold, and $\hat{A} \supseteq A$. We need to show that $B \mathop{\smile\!\!\!\!\!\!|}_{D} \hat{A}^*$ for some $\hat{A}^* \equiv_{AD} \hat{A}$. Let $\hat{A}' \equiv_{AD} \hat{A}$ be such that $C \mathop{\smile\!\!\!\!\!\!|}_{D} \hat{A}'$, and let $\hat{A}^* \equiv_{AC} \hat{A}'$ be such that $B \mathop{\smile\!\!\!\!\!\!|}_{C} \hat{A}^*$. Note that $\hat{A}^* \equiv_{AD} \hat{A}$ and $C \mathop{\smile\!\!\!\!\!\!|}_{D} \hat{A}^*$. By transitivity of $\mathop{\smile\!\!\!\!\!\!|}$ we get $B \mathop{\smile\!\!\!\!\!\!|}_{D} \hat{A}^*$. *Normality:* Suppose $A \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} B$ and $\hat{B} \supseteq B$. Let $A' \equiv_{BC} A$ be such that $A' \mathop{\smile\!\!\!\!\!\!|}_{C} \hat{B}$. Then also $A'C \mathop{\smile\!\!\!\!\!\!|}_{C} \hat{B}$ by normality of $\mathop{\smile\!\!\!\!\!\!|}$. *Anti-reflexivity:* Trivial, since $A \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} B$ implies $A \mathop{\smile\!\!\!\!\!\!|}_{C} B$. *Full existence:* Suppose we are given A, B, C. Since $\mathop{\smile\!\!\!\!\!\!|}$ satisfies full existence by assumption, we have $A \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} \emptyset$. Since $\mathop{\smile\!\!\!\!\!\!|}^{*}$ satisfies extension there is $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} B$. □

Theorem 3.2. *Suppose $\mathop{\smile\!\!\!\!\!\!|}$ satisfies the first five axioms for independence relations and also has finite character. Suppose the derived relation $\mathop{\smile\!\!\!\!\!\!|}^{*}$ has local character. Then $\mathop{\smile\!\!\!\!\!\!|}^{*}$ is an independence relation.*

Proof. It follows from Lemma 3.1 that $\mathop{\smile\!\!\!\!\!\!|}^{*}$ satisfies the first five axioms and extension. As local character holds by assumption, we need only prove that $\mathop{\smile\!\!\!\!\!\!|}^{*}$ satisfies finite character. We will prove some other facts on our way.

First note that $A \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} B$ implies $B \mathop{\smile\!\!\!\!\!\!|}_{C} A$: If $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} B$, there is a $\mathop{\smile\!\!\!\!\!\!|}^{*}$-Morley sequence in $\mathrm{tp}(\bar{a}/BC)$ over C by Proposition 2.3 (applied to $\mathop{\smile\!\!\!\!\!\!|}^{*}$). This sequence is also a $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequence in $\mathrm{tp}(\bar{a}/BC)$ over C, hence $B \mathop{\smile\!\!\!\!\!\!|}_{C} \bar{a}$ by Proposition 2.4 (applied to $\mathop{\smile\!\!\!\!\!\!|}$).

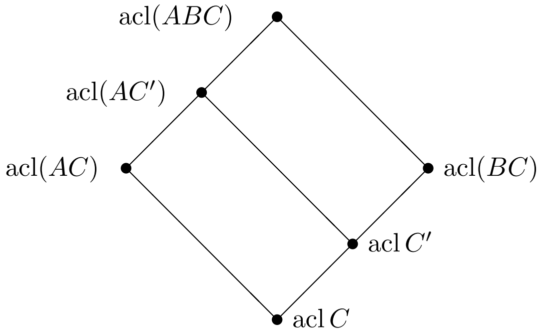

Figure 1: An illustration for the definition of $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ in the lattice of algebraically closed sets. We have a map $\operatorname{acl} C' \mapsto \operatorname{acl}(AC')$ from the sublattice between $\operatorname{acl} C$ and $\operatorname{acl}(BC)$ to the sublattice between $\operatorname{acl}(AC)$ and $\operatorname{acl}(ABC)$. $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}_C B$ says that the map $D \mapsto D \cap \operatorname{acl}(BC)$ takes $D = \operatorname{acl}(AC')$ back to $\operatorname{acl} C'$.

It follows that $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ implies $B \mathop{\smile\!\!\!\!\!\!|}^*_C A$: Suppose $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ and $\hat{A} \supseteq A$. Since $\hat{A} \mathop{\smile\!\!\!\!\!\!|}^*_{AC} AC$ by local character and base monotonicity, we can use extension to find $\hat{A}' \equiv_{AC} \hat{A}$ such that $\hat{A}' \mathop{\smile\!\!\!\!\!\!|}^*_{AC} ABC$, hence $\hat{A}' \mathop{\smile\!\!\!\!\!\!|}^*_{AC} B$ by monotonicity. Combining this with $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$, we get $\hat{A}' \mathop{\smile\!\!\!\!\!\!|}^*_C B$ by transitivity. This implies $B \mathop{\smile\!\!\!\!\!\!|}_C \hat{A}$. Thus $B \mathop{\smile\!\!\!\!\!\!|}^*_C A$.

Now we can prove that $\mathop{\smile\!\!\!\!\!\!|}^*$ has finite character. Suppose $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^*_C B$ holds for all finite $\bar{a} \in A$. We need to show that $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$. So suppose $\hat{B} \supseteq B$. Since $A \mathop{\smile\!\!\!\!\!\!|}^*_{BC} BC$ by local character and base monotonicity, we can obtain $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}^*_{BC} \hat{B}$ using full existence and monotonicity. By invariance, there is also $\hat{B}' \equiv_{BC} \hat{B}$ such that $A \mathop{\smile\!\!\!\!\!\!|}^*_{BC} \hat{B}'$. It suffices to show that $A \mathop{\smile\!\!\!\!\!\!|}_C \hat{B}'$. For every finite $\bar{a} \in A$ we have $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^*_C B$ by assumption, and $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^*_{BC} \hat{B}'$ by $A \mathop{\smile\!\!\!\!\!\!|}^*_{BC} \hat{B}'$ and monotonicity. Since $\mathop{\smile\!\!\!\!\!\!|}^*$ is symmetric we can combine these results using transitivity on the right-hand side. Thus we get $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^*_C \hat{B}'$ for all finite $\bar{a} \in A$. Hence $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C \hat{B}'$ for all finite $\bar{a} \in A$. Since $\mathop{\smile\!\!\!\!\!\!|}$ has finite character, this implies $A \mathop{\smile\!\!\!\!\!\!|}_C \hat{B}'$. ☐

Although it does not seem to happen in natural examples, we *can* lose local character when passing from $\mathop{\smile\!\!\!\!\!\!|}$ to $\mathop{\smile\!\!\!\!\!\!|}^*$, and indeed Example 6.4 below shows that it is not sufficient in Theorem 3.2 to assume local character of $\mathop{\smile\!\!\!\!\!\!|}$. But losing finite character, of all things, seems weird, and the way we recovered it using local character is a bit contrived. The reader might be happy to hear that a stronger version of finite character (which says roughly that dependence is caused by a 'forking' formula) is actually preserved when passing from $\mathop{\smile\!\!\!\!\!\!|}$ to $\mathop{\smile\!\!\!\!\!\!|}^*$ [7].

Question 3.3. Can there really be a relation $\mathop{\smile\!\!\!\!\!\!|}$ satisfying invariance, monotonicity and finite character, for which $\mathop{\smile\!\!\!\!\!\!|}^*$ does not have finite character?

4 Thorn-Forking and Rosy Theories

In Proposition 1.5 we have examined the relation $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}$, which turned out to satisfy all axioms for independence relations with the possible exception of base monotonicity. After our success with forcing the extension axiom, the obvious thing to do is to try forcing base monotonicity. To take all suspense out of this section: The only axioms that can break are local character and extension, and the result is known as thorn-forking.

The relation $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ *(M-dividing independence)* is defined by

$$A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}_C B \iff \begin{pmatrix} \text{for any } C' \text{ s. t. } C \subseteq C' \subseteq \operatorname{acl}(BC)\text{:} \\ \operatorname{acl}(AC') \cap \operatorname{acl}(BC') = \operatorname{acl} C' \end{pmatrix}.$$

We define the relation $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$ *(thorn-forking independence)* by $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}} = \mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}*}$, i.e.:

$$A \mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}_{C} B \iff \Big(\textit{for all } \hat{B} \supseteq B \textit{ there is } A' \underset{BC}{\equiv} A \textit{ s.t. } A' \mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}_{C} \hat{B} \Big).$$

We say that a complete first-order theory T is *rosy* if $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$ is an independence relation for T^{eq}.

For example if the lattice of algebraically closed sets in T^{eq} is modular, then $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{a}}$ is already an independence relation, and therefore $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{a}} = \mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}} = \mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$ and the theory is rosy. Like $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{a}}$, the relation $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}$ was already defined in my diploma thesis [9], and like $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{a}}$ it has roots that are indeed much older. Towards a generalisation of John von Neumann's work to certain non-modular lattices, Lee Roy Wilcox defined the notion of modular pairs. If we apply his definition for the notation $(A, B) \perp$ to the lattice of algebraically closed sets it means $A \mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}_{\emptyset} B$ [17]. There is more to be said on this connection, but not here [7].

Remark 4.1. If $\mathop{\smile\!\!\!\!\!\!\mid}$ is any strict independence relation, then $A \mathop{\smile\!\!\!\!\!\!\mid}_{C} B$ implies $A \mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}_{C} B$.

Proof. Suppose $\mathop{\smile\!\!\!\!\!\!\mid}$ is a strict independence relation, $A \mathop{\smile\!\!\!\!\!\!\mid}_{C} B$, and $\hat{B} \supseteq B$. We need to show that there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}_{C} \hat{B}$. So choose $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!\mid}_{C} \operatorname{acl}(\hat{B}C)$. For any D satisfying $C \subseteq D \subseteq \operatorname{acl}(\hat{B}C)$ we get $A' \mathop{\smile\!\!\!\!\!\!\mid}_{D} \operatorname{acl}(\hat{B}C)$ by base monotonicity of $\mathop{\smile\!\!\!\!\!\!\mid}$. By extension and symmetry of $\mathop{\smile\!\!\!\!\!\!\mid}$ there is a set $H \equiv_{A'D} \operatorname{acl}(A'D)$ that satisfies $H \mathop{\smile\!\!\!\!\!\!\mid}_{D} \operatorname{acl}(\hat{B}C)$. Clearly $H = \operatorname{acl}(A'D)$, so $\operatorname{acl}(A'D) \mathop{\smile\!\!\!\!\!\!\mid}_{D} \operatorname{acl}(\hat{B}C)$. Now by anti-reflexivity of $\mathop{\smile\!\!\!\!\!\!\mid}$, $\operatorname{acl}(A'D) \cap \operatorname{acl}(\hat{B}CD) \subseteq \operatorname{acl} D$, so $\operatorname{acl}(A'D) \cap \operatorname{acl}(\hat{B}CD) = \operatorname{acl} D$. □

Lemma 4.2. *The relation $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}$ of M-dividing independence always satisfies the first five axioms for independence relations and finite character. It also satisfies anti-reflexivity.*

Proof. Invariance, monotonicity, normality and *anti-reflexivity* are obvious. *Base monotonicity*: Suppose $A \mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}_{C} B$ and $C \subseteq D \subseteq B$. Then for any D' satisfying $D \subseteq D' \subseteq \operatorname{acl}(BD)$ we also have $C \subseteq D' \subseteq \operatorname{acl}(BC)$. So $A \mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}_{C} B$ implies $\operatorname{acl}(AD') \cap \operatorname{acl}(BD') = \operatorname{acl} D'$. Hence $A \mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}_{D'} B$. *Transitivity:* Suppose $D \subseteq C \subseteq B$, $B \mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}_{C} A$ and $C \mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}_{D} A$. Then for any D' such that $D \subseteq D' \subseteq \operatorname{acl}(AD)$ we can compute:

$$\begin{aligned} \operatorname{acl}(BD') \cap \operatorname{acl}(AD') &= \operatorname{acl}(BD') \cap \operatorname{acl}(ACD') \cap \operatorname{acl}(AD') \\ &= \operatorname{acl}(CD') \cap \operatorname{acl}(AD') && \big(\text{by } B \mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}_{C} A\big) \\ &= \operatorname{acl} D', && \big(\text{by } C \mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}_{D} A\big) \end{aligned}$$

so $B \mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}_{D} A$ holds. *Finite character:* Suppose $A \not\mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}_{C} B$. Let C' be such that $C \subseteq C' \subseteq \operatorname{acl}(BC)$ and $\operatorname{acl}(AC') \cap \operatorname{acl}(BC') \not\subseteq \operatorname{acl} C'$. Let $d \in (\operatorname{acl}(AC') \cap \operatorname{acl}(BC')) \setminus \operatorname{acl} C'$. Let $\bar{a} \in A$, finite, be such that $d \in \operatorname{acl}(\bar{a}C')$. Then clearly $\bar{a} \not\mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}_{C} B$. □

Theorem 4.3. *The relation $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$ of thorn-forking independence is a strict independence relation if and only if it has local character, if and only if there is any strict independence relation at all. If $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$ is a strict independence relation, then it is the weakest.*

Proof. To get the first equivalence, apply Theorem 3.2 to $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}$. If $\mathop{\smile\!\!\!\!\!\!\mid}$ is any strict independence relation, then, since $\mathop{\smile\!\!\!\!\!\!\mid}$ satisfies the local character axiom, so does $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$ by Remark 4.1. If $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$ is a strict independence relation, then it is the weakest, again by Remark 4.1. □

Note that the conditions of Theorem 4.3 do not imply $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}} = \mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$.

Example 4.4. (Everywhere infinite forest, continued from Example 1.4)
It follows from Theorem 4.3 that $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$ is also a strict independence relation, and that $A \mathop{\smile\!\!\!\!\!\!\mid}_{C} B \implies A \mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}_{C} B$. It is straightforward to check that the converse is also true, so $\mathop{\smile\!\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$. But it is not hard to see that $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}$ does not satisfy extension, so $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}} \neq \mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$: Let a and b be neighbours. Then $a \mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}} b$. However, there is no $c \equiv_b a$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}} bc$: Either $c = a$, or b lies between a and c. In the first case, $a = c \in (\operatorname{acl} a \cap \operatorname{acl}(bc)) \setminus \operatorname{acl} \emptyset$, so $a \not\mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}} bc$. In the second case, $b \in (\operatorname{acl}(ac) \cap \operatorname{acl}(bc)) \setminus \operatorname{acl} c$, so also $a \not\mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}} bc$. Thus $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{M}}$ does not satisfy the extension axiom.

In some cases (most notably strongly minimal and o-minimal theories), þ-forking as defined on the real elements of T is an important tool for understanding the structure of models of T. In these cases þ-forking on T agrees with the restriction to T of þ-forking in T^{eq}. This is not the case in general, and the existence of a strict independence relation on the real elements of T *per se* does not imply any 'structure' that is more than superficial:

Example 4.5. (Thorn-forking must be computed in T^{eq} in general)
Let T be any complete consistent theory in a relational language. Consider the following theory T': The language of T' is the language of T together with a new binary relation. The axioms of T' are the axioms of T, but with equality replaced by the new relation, together with axioms saying that the new relation is an equivalence relation with infinite classes. Then T' is a complete consistent theory satisfying $\operatorname{acl} A = A$ for every set A of real elements. Hence the lattice of algebraically closed sets is just the (distributive) lattice of subsets of the big model, and so the relation $A \mathop{\smile\mkern-11mu\mid}_C B \iff A \cap B \subseteq C$ is a strict independence relation for T (and agrees with $\mathop{\smile\mkern-11mu\mid}^{\text{þ}}$).

If T is rosy, then T' shows that in general $\mathop{\smile\mkern-11mu\mid}^{\text{þ}}$ as defined on T is not the restriction to T of $\mathop{\smile\mkern-11mu\mid}^{\text{þ}}$ as defined on T^{eq}, even though it must be an independence relation and in this case it is even strict.

The original definition of thorn-forking

We finish this section by relating $\mathop{\smile\mkern-11mu\mid}^{\text{þ}}$ to the original definition of thorn-forking, due to Alf Onshuus [5]. We will be glossing over some difficulties. Let $\varphi(\bar{x}, \bar{y})$ be a formula without parameters, let $\bar{b}$ be a tuple, and let C be a set. The formula $\varphi(\bar{x}, \bar{b})$ *strongly divides* over C if $\operatorname{tp}(\bar{b}/C)$ is not algebraic and $\left\{\varphi(\bar{x}, \bar{b}') \;\middle|\; \bar{b}' \models \operatorname{tp}(\bar{b}/C)\right\}$ is k-inconsistent for some natural number $k < \omega$. The formula $\varphi(\bar{x}, \bar{b})$ *þ-divides* over C if there is a tuple $\bar{c}$ such that $\varphi(\bar{x}, \bar{b})$ strongly divides over $C\bar{c}$. The formula $\varphi(\bar{x}, \bar{b})$ *þ-forks* over C if it implies a (finite) disjunction of formulas (with arbitrary parameters), each of which þ-divides over C.

Remark 4.6. If $p(\bar{x}) \in \mathrm{S}^*(B)$ does not thorn-fork over $C \subseteq B$, then for every set $\hat{B} \supseteq B$ there is an extension $\hat{p}$ of p to $\hat{B}$, $p(\bar{x}) \subseteq \hat{p}(\bar{x}) \in \mathrm{S}^*(\hat{B})$, such that $\hat{p}$ also does not thorn-fork over C.

Proof. The type p is consistent with the partial global type consisting of all negations of formulas that thorn-fork over C, and so it has a global extension that does not thorn-fork over C. □

Proposition 4.7. *For sets A, C and a $(|T| + |M|)^+$-saturated model $M \supseteq C$ the following conditions are equivalent.*

(1) $A \mathop{\smile\mkern-11mu\mid}^{\text{þ}}_C M$.

(2) $A \mathop{\smile\mkern-11mu\mid}^{\mathrm{M}}_C M$.

(3) For all sets C' such that $C \subseteq C' \subseteq M$ we have $\operatorname{acl}(AC') \cap M = \operatorname{acl} C' \cap M$.

(4) For all tuples $\bar{a} \in A$ and all formulas $\varphi(\bar{x})$ over M such that $\models \varphi(\bar{a})$ the following holds. $\varphi(\bar{x})$ does not strongly divide over any set C' such that $C \subseteq C' \subseteq M$.

(5) For all tuples $\bar{a} \in A$ and all formulas $\varphi(\bar{x})$ over M such that $\models \varphi(\bar{a})$ the following holds. $\varphi(\bar{x})$ does not thorn-divide over C.

(6) For all tuples $\bar{a} \in A$ and all formulas $\varphi(\bar{x})$ over M such that $\models \varphi(\bar{a})$ the following holds. $\varphi(\bar{x})$ does not thorn-fork over C.

Proof. (1) ⇒ (2) ⇔ (3): trivial. (3) ⇒ (4): Suppose $\bar{a} \in A$, $\varphi(\bar{x}, \bar{b})$ ($\bar{b} \in M$) and C' ($C \subseteq C' \subseteq M$) are a counterexample to (4), i.e., $\models \varphi(\bar{a}, \bar{b})$ and $\varphi(\bar{x}, \bar{b})$ strongly divides over C'. Then $\bar{b} \in \operatorname{acl}(\bar{a}C') \setminus \operatorname{acl} C'$. (4) ⇒ (5) ⇒ (6): By saturation of M. (6) ⇒ (1): By the last Remark and the definition of $\mathop{\smile\mkern-11mu\mid}^{\mathrm{M}}$ it suffices to prove (2) from (6). Towards a contradiction, suppose (2) does not hold. Then there is a set C', $C \subseteq C' \subseteq M$ and an element $b \in \operatorname{acl}(AC') \cap M \setminus \operatorname{acl} C'$. Let $\bar{a} \in A$, $\bar{c} \in C'$, $\varphi(\bar{x}, y, \bar{z})$ and $k < \omega$ be such that $\models \varphi(\bar{a}, b, \bar{c})$ and $\models \exists_{\leq k} y' \varphi(\bar{a}, y, \bar{c})$. Define $\varphi(\bar{x}, y, \bar{z})$ as $\varphi(\bar{x}, y, \bar{z}) \wedge \exists_{\leq k} y' \varphi(\bar{x}, y', \bar{z})$. Then $\models \varphi'(\bar{a}, b, \bar{c})$, and $\varphi'(\bar{x}, b, \bar{c})$ strongly divides over $C\bar{c}$. Hence $\varphi'(\bar{x}, b, \bar{c})$ thorn-forks over C. □

Corollary 4.8. *If we write* $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{m}}_C B$ *if* $\operatorname{acl}(ACB_0) \cap \operatorname{acl}(BC) = \operatorname{acl}(CB_0)$ *for all* $B_0 \subseteq B$, *and* $A \mathop{\smile\!\!\!\!\!\!|}^{\text{þ-d}}_C B$ *if for all* $\bar{a} \in A$ *the type* $\operatorname{tp}(\bar{a}/BC)$ *does not thorn-fork over* C, *then* $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{m}*} = \mathop{\smile\!\!\!\!\!\!|}^{\text{þ-d}*} = \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$.

The following remark says that we could just as well have based our treatment of thorn-forking on $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{m}}$ or $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ-d}}$. We omit the proof because it is straightforward and we are not using this here.

Remark 4.9. The relations $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{m}}$ and $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ-d}}$ always satisfy the first five axioms for independence relations and finite character.

5 Shelah's Forking and Simple Theories

The relation $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$ *(dividing independence)* is defined by

$$A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C B \iff \begin{pmatrix} \text{for any sequence of } C\text{-indiscernibles } (\bar{b}_i)_{i<\omega} \text{ s. t. } \bar{b}_0 \in BC\text{:} \\ \exists\, A' \equiv_{BC} A \text{ s. t. the sequence is } A'C\text{-indiscernible} \end{pmatrix}.$$

The relation $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ *(forking independence)* is defined by $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}*}$, i.e.:

$$A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C B \iff \left(\text{for all } \hat{B} \supseteq B \text{ there is } A' \underset{BC}{\equiv} A \text{ s.t. } A' \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C \hat{B} \right).$$

These definitions are equivalent to dividing and forking as they were originally defined by Saharon Shelah to study stable theories (see Remark 5.5 below). Here, however, we are re-inventing the theory, so we motivate the definition of $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$ with the following easy remark (due to Kim and Pillay [13]).

Remark 5.1. If $\mathop{\smile\!\!\!\!\!\!|}$ is any independence relation, then $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C B$ implies $A \mathop{\smile\!\!\!\!\!\!|}_C B$.

Proof. Suppose $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C \bar{b}$. Let $(\bar{b}_i)_{i<\omega}$ be a $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequence for $\operatorname{tp}(\bar{b}/C)$. This exists by Proposition 2.3, since $\bar{b} \mathop{\smile\!\!\!\!\!\!|}_C C$. We may assume that $\bar{b}_0 = \bar{b}$. Since $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C \bar{b}$ there is $A' \equiv_{\bar{b}C} A$ such that the sequence $(\bar{b}_i)_{i<\omega}$ is $A'C$-indiscernible. By Proposition 2.4, it now follows that $A \mathop{\smile\!\!\!\!\!\!|}_C \bar{b}$. □

Of course this remark implies that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ whenever $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ is an independence relation. But we still need $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ for technical reasons.

We will call a complete first-order theory T *simple* if $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ is an independence relation for T. The greatest part of the following lemma can be found in Shelah's book [2]. Transitivity of $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$ in the general case is implicit in Kim's thesis [3],

Lemma 5.2. *The relation* $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$ *of dividing independence always satisfies the first five axioms for independence relations and finite character. It also satisfies anti-reflexivity.*

Proof. Invariance and *monotonicity* are obvious. *Base monotonicity:* Suppose $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C B$ and $C \subseteq C' \subseteq B$. Let $(\bar{b}_i)_{i<\omega}$ be a sequence of C'-indiscernibles with $\bar{b}_0 \in B = BC$. Let $\bar{c}'$ be an enumeration of C'. Then also $\bar{b}_0\bar{c}' \in BC$, and the sequence $(\bar{b}_i\bar{c}')_{i<\omega}$ is also C-indiscernible. Hence there is $A' \equiv_{\bar{b}_0\bar{c}'} A$ such that $(\bar{b}_i\bar{c}')_{i<\omega}$ is $A'C$-indiscernible. Thus $(\bar{b}_i)_{i<\omega}$ is $A'C'$-indiscernible. *Transitivity:* Suppose $D \subseteq C \subseteq B$, $B \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C A$ and $C \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_D A$. Let $(\bar{a}_i)_{i<\omega}$ be any sequence of D-indiscernibles with $\bar{a}_0 \in AD$. By $C \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_D A$ there is $C' \equiv_{AD} C$ such that the sequence $(\bar{a}_i)_{i<\omega}$ is indiscernible over C'. Choose any set B' such that $(B', C') \equiv_{AD} (B, C)$. Then $B' \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C'} A$ holds by invariance. Hence there is $B'' \equiv_{AC'} B'$ such that the sequence is B''-indiscernible. And really, $B'' \equiv_{AD} B$. *Normality:* Suppose $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C B$. Let $(\bar{b}_i)_{i<\omega}$ be a sequence of C-indiscernibles such that $\bar{b}_0 \in BC$. By definition there is $A' \equiv_{BC} A$ such that the sequence is $A'C$-indiscernible. But then also $A'C \equiv_{BC} AC$. *Finite character:* Let $\bar{a}$ be a possibly infinite tuple such that $\bar{a} \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C B$. Let $p(\bar{x}) = \operatorname{tp}(\bar{a}/BC)$. Then there is a sequence $(\bar{b}_i)_{i<\omega}$ with $\bar{b}_0 \in BC$ such that the type extending $p(\bar{x})$ and the theory of the big model with constants for $BC(\bar{b}_i)_{i<\omega}$ and expressing that $(\bar{b}_i)_{i<\omega}$ is $\bar{a}C$-indiscernible is inconsistent. By compactness, a finite sub-tuple $\bar{a}_0$ of $\bar{a}$ is sufficient for this, so $\bar{a}_0 \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C B$. For *anti-reflexivity* suppose $a \notin \operatorname{acl} B$. Then there is a B-indiscernible sequence $(a_i)_{i<\omega}$ of distinct elements, with $a_0 = a$. This sequence witnesses that $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_B a$. □

Theorem 5.3. *A theory T is simple if and only if $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ has local character. If T is simple, $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$, and this is the strongest independence relation for T.*

Proof. For the equivalence, just apply Theorem 3.2 to $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$. Now suppose T is simple. While $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_{C} B$ always implies $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C} B$, the converse is true by Remark 5.1. Hence $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$. Since $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ is an independence relation and $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$ is stronger than every independence relation by Remark 5.1, $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$ is the strongest. □

By comparing Remarks 4.1 and 5.1 one easily sees that $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_{C} B$ implies $A \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{C} B$, provided that a strict independence relation exists. The following remark shows that even without this assumption a stronger statement is true: $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C} B$ always implies $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}_{C} B$, hence $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_{C} B$ always implies $A \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{C} B$.

Remark 5.4. (1) Every sequence of B-indiscernibles is also indiscernible over $\operatorname{acl} B$.

(2) $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C} B$ implies $\operatorname{acl}(AC) \cap B \subseteq \operatorname{acl} C$.

(3) $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C} B$ implies $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C} \operatorname{acl}(BC)$. So $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$ always satisfies a weak variant of the extension axiom.

(4) If $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C} B$ and $C \subseteq C' \subseteq \operatorname{acl}(BC)$, then $\operatorname{acl}(AC') \cap \operatorname{acl}(BC) = \operatorname{acl} C'$. Hence $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C} B$ implies $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}_{C} B$.

This was a quick and cheap introduction to forking in simple theories, but some things are missing. For example, we would like to know that T is simple if and only if T^{eq} is simple. Or that simplicity is equivalent to $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$ having local character, or to not having the tree property. To get these results we need to get our hands dirty with formulas, and this is what we will do in the next episode [7]. As a first step in this direction, the reader might want to prove the following remark using only the results of this section.

Remark 5.5. A formula $\varphi(\bar{x};\bar{b})$ *divides* over a set C if there is a finite number $k < \omega$ and a sequence $(\bar{b}_i)_{i<\omega}$ such that $\bar{b}_i \equiv_C \bar{b}$ holds for all $i < \omega$ and $\{\varphi(\bar{x};\bar{b}_i) \mid i < \omega\}$ is k-inconsistent. A formula *forks* over C if it implies a finite disjunction of formulas that divide over C [2].

(1) $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{C} B$ iff there is a formula $\varphi(\bar{x};\bar{b}) \in \operatorname{tp}(\bar{a}/BC)$ which divides over C.

(2) $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_{C} B$ iff there is a formula $\varphi(\bar{x};\bar{b}) \in \operatorname{tp}(\bar{a}/BC)$ which forks over C.

(3) For simple T, a formula $\varphi(\bar{x};\bar{b})$ forks over a set C iff $\varphi(\bar{x};\bar{b})$ divides over C.

If there is any strict independence relation for T^{eq}, then T is rosy. If we accept the (as yet unproved) fact that T is simple if and only if T^{eq} is simple, it follows that every simple theory is rosy. Thus if T is simple, $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ is the weakest strict independence relation on T^{eq} while $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ is the strongest. It should be mentioned that in simple theories with elimination of hyperimaginaries we have $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ (by an unpublished result of Clifton Ealy), so this is the *only* strict independence relation in such a theory.

6 Some Relations that are Not Independence Relations

Let us call the first five axioms for independence relations and finite character the *basic axioms for independence relations* because they are satisfied by $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$, $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$, $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ and $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ in arbitrary theories. If we are interested in relaxing the notion of an independence relation in such a way that it becomes useful, for example, in all theories without the independence property, then it is important to know all the implications between the non-basic axioms (extension and local character) and additional properties such as full existence and symmetry. In this section we will make a first attempt in this direction, by proving the following result.

Proposition 6.1. *We consider relations $\mathop{\smile\!\!\!\!\!\!|}$ that satisfy the basic axioms for independence relations (i. e. invariance, monotonicity, base monotonicity, transitivity, normality, and finite character).*

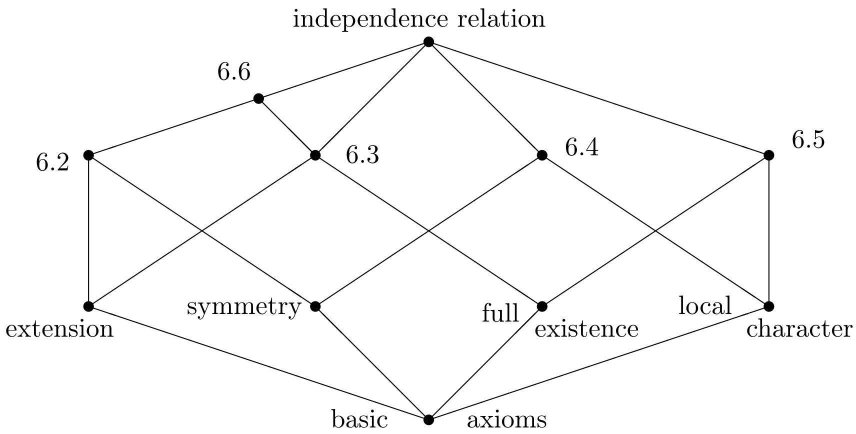

Figure 2: Classification of relations satisfying the basic axioms of independence relations, according to which of four remaining properties hold. For five nodes of this lattice diagram there is an example in Section 6. These can be used to assemble examples for the lower nodes.

(1) If $\mathop{\smile\hspace{-0.9em}|}$ satisfies extension and local character, then $\mathop{\smile\hspace{-0.9em}|}$ also satisfies symmetry and full existence.

(2) If $\mathop{\smile\hspace{-0.9em}|}$ satisfies symmetry and full existence, then $\mathop{\smile\hspace{-0.9em}|}$ also satisfies extension.

No other relations between extension, local character, symmetry and full existence hold in general. More precisely, there are examples of complete theories with relations $\mathop{\smile\hspace{-0.9em}|}$ satisfying the basic axioms which show that:

(3) $\mathop{\smile\hspace{-0.9em}|}$ may satisfy extension and symmetry, but neither full existence nor local character.

(4) $\mathop{\smile\hspace{-0.9em}|}$ may satisfy extension and full existence, but neither symmetry nor local character.

(5) $\mathop{\smile\hspace{-0.9em}|}$ may satisfy symmetry and local character, but neither extension nor full existence.

(6) $\mathop{\smile\hspace{-0.9em}|}$ may satisfy full existence and local character, but neither extension nor symmetry.

(7) $\mathop{\smile\hspace{-0.9em}|}$ may satisfy symmetry and full existence, but not local character.

Proof. (1) and (2) are by Remark 1.2 and Theorem 2.5. The rest is shown by the series of examples below. (3) Example 6.2. (4) Example 6.3. (5) Example 6.4. (6) Example 6.5. (7) Example 6.6. □

Example 6.2. (no local character, no full existence)
The empty ternary relation satisfies all axioms for strict independence relations except local character. It also satisfies symmetry, but not full existence.

Example 6.3. (no local character, no symmetry)
We will extend the theories T_0 and T from Example 2.6. First we describe the signature of the respective extensions T_0^* and T^*: It has the sorts P ('points') and E ('equivalence relations') as well as a new sort Γ ('equivalence classes'). The functions and relations of T_0^* consist of the relation $p \sim_e q$ (for $p, q \in P$ and $e \in E$), a new relation written (slightly abusing notation) as $p/e = c$ for $p \in P$, $e \in E$ and $c \in \Gamma$, and a function $\epsilon : \Gamma \to E$.

The axioms of T_0^* include those of T_0, i. e., $\sim_e$ is an equivalence relation for every $e \in E$. They also say that $\exists_{\leq 1} c(p/e = c)$, so it makes sense to regard p/e as a partial function $P \times E \to \Gamma$ which we will use informally in the following. The other axioms say $\epsilon(p/e) = e$ (if p/e exists) and $p \sim_e q \leftrightarrow p/e = q/e$ (also if p/e exists).

Clearly every model of T_0 is also a model of T_0^*, and if we restrict a model of T_0^* to the sorts P and E we get a model of T_0. By the same arguments as for T_0 we can find an ω-categorical theory T^* with elimination of quantifiers which is the Fraïssé limit of the finite models of T_0^*. So T^* extends both T and T_0^*.

For any subset A of the big model of T^* we write $P(A) = A \cap P$, $E(A) = (A \cap E) \cup \epsilon(A \cap \Gamma)$ and $\Gamma(A) = (A \cap \Gamma) \cup \{p/e \mid p \in P(A), e \in E(A)\}$. It is not hard to check that $\operatorname{acl} A = \operatorname{dcl} A =$

$P(A) \cup E(A) \cup \Gamma(A)$. It easily follows that

$$A \mathop{\perp}^{\mathrm{M}}_{C} B \iff \begin{pmatrix} P(A) \cap P(B) \subseteq P(C) \text{ and} \\ E(A) \cap E(B) \subseteq E(C) \text{ and} \\ A/e \cap B/e \subseteq C/e \text{ for all } e \in E(BC) \end{pmatrix},$$

where A/e is defined as $P(A)/e \cup (\Gamma(A) \cap P/e)$. Using this, it is not hard to check that $A \mathop{\perp}^{\mathrm{M}}_{C} B \implies A \mathop{\perp}^{\text{þ}}_{C} B$ and that $A \mathop{\perp}^{\mathrm{M}}_{C} B \implies A \mathop{\perp}^{\mathrm{d}}_{C} B$, from which it easily follows that $\mathop{\perp}^{\mathrm{M}} = \mathop{\perp}^{\text{þ}} = \mathop{\perp}^{\mathrm{d}} = \mathop{\perp}^{\mathrm{f}}$. Hence $\mathop{\perp}^{\mathrm{f}}$ satisfies all axioms for strict independence relations except local character (which would imply that there is a strict independence relation for T^*). It also satisfies full existence, though not symmetry.

Example 6.4. (no extension, no full existence)
Let T be the theory of an infinite set. Define $\mathop{\perp}$ as follows:

$$A \mathop{\perp}_{C} B \quad \iff \quad \big(|C| \geq \aleph_0 \text{ and } A \cap B \subseteq C\big) \text{ or } A \subseteq C \text{ or } B \subseteq C.$$

This relation satisfies all axioms for strict independence relations except extension. It also satisfies symmetry, but not full existence. Note that even though $\mathop{\perp}$ has local character, $\mathop{\perp}^{*}$ does not.

Example 6.5. (no extension, no symmetry)
Consider the theory from Examples 1.4 and 4.4. By Lemma 4.2 the relation $\mathop{\perp}^{\mathrm{M}}$ satisfies the axioms of strict independence relations except extension and local character. $\mathop{\perp}^{\mathrm{M}}$ also satisfies local character and full existence because $\mathop{\perp}^{\text{þ}}$ does. We have already seen that $\mathop{\perp}^{\mathrm{M}}$ does not satisfy extension.

$\mathop{\perp}^{\mathrm{M}}$ is not symmetric either: Suppose b lies between a and c. Then $a \mathop{\not\perp}^{\mathrm{M}} bc$ as we have just seen. But it is easy to see that $bc \mathop{\perp}^{\mathrm{M}} a$.

Example 6.6. (no local character)
Consider the theory of the random graph (the Fraïssé limit of the finite undirected graphs) with the following relation:

$$A \mathop{\perp}_{C} B \iff A \cap B \subseteq C \text{ and there is no edge from } A \setminus C \text{ to } B \setminus C.$$

The relation $\mathop{\perp}$ satisfies all axioms for strict independence relations except local character. It also satisfies full existence and symmetry.

In fact, for relations $\mathop{\perp}$ which satisfy invariance and monotonicity the only implications between the remaining axioms are those given by Remark 1.2 and Theorem 2.5 [1]. Of course, for some specific relations $\mathop{\perp}$ more can be said. For example, Byunghan Kim has shown that T is simple iff $\mathop{\perp}^{\mathrm{f}}$ (or $\mathop{\perp}^{\mathrm{d}}$) has local character, iff $\mathop{\perp}^{\mathrm{f}}$ (or $\mathop{\perp}^{\mathrm{d}}$) is symmetric, iff $\mathop{\perp}^{\mathrm{f}}$ (or $\mathop{\perp}^{\mathrm{d}}$) satisfies the transitivity axiom on the right hand side [18].

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