THORN-FORKING AS LOCAL FORKING

HANS ADLER

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We introduce the notion of a preindependence relation between subsets of the big model of a complete first-order theory, an abstraction of the properties which numerous concrete notions such as forking, dividing, thorn-forking, thorn-dividing, splitting or finite satisfiability share in all complete theories. We examine the relation between four additional axioms (extension, local character, full existence and symmetry) that one expects of a good notion of independence.

We show that thorn-forking can be described in terms of local forking if we localise the number k in Kim's notion of 'dividing with respect to k' (using Ben-Yaacov's 'k-inconsistency witnesses') rather than the forking formulas. It follows that every theory with an M-symmetric lattice of algebraically closed sets (in T^{eq}) is rosy, with a simple lattice theoretical interpretation of thorn-forking.

Keywords: Preindependence relation; Local forking; Thorn-forking; M-symmetric lattice.

MSC2000: 03C45.

Introduction

This paper is based on the second chapter of my thesis under the supervision of Martin Ziegler [1]. It continues a paper based on the first chapter [2]. The treatment of local forking has been simplified compared to my thesis. It is now a bit less general, but should be much more comprehensible.

In the previous paper we tried to develop the basic theory of forking and thorn-forking using only geometric methods. This was partially successful. Central to this paper is a new notion of local forking that refines the usual one, and which allows us to improve our previous results on forking and thorn-forking. We will introduce the notion of potential inconsistency witnesses. If Ω is a set of potential inconsistency witnesses we get a relation $\mathop{\smile\!\!\!\!\!\mid}^{\Omega}$ such that $\mathop{\smile\!\!\!\!\!\mid}^{\Omega*}$ is a good candidate for being an independence relation. In particular, $\mathop{\smile\!\!\!\!\!\mid}^{\Omega*} = \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}$ (forking independence) or $\mathop{\smile\!\!\!\!\!\mid}^{\Omega*} = \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}$ (thorn-forking independence) for suitable choices of Ω. As an application we solve a question about M-symmetric theories from my diploma thesis.

Perhaps I should try to justify what may look like a strange obsession with the geometric foundations of stability theory in general, and with the lattice theoretic notion of M-symmetry in particular. When John von Neumann achieved the reconstruction of certain rings from their lattice of ideals in the 1930s, a short boom of lattice theory was initiated. His work had generalised classical results for finite semimodular lattices to certain infinite modular lattices. Researchers like Garrett Birkhoff, Saunders Mac Lane and Lee Roy Wilcox were looking for the right notion of semimodularity for infinite lattices. As a result of their creativity we now have many definitions of semimodularity, which are all equivalent on finite lattices [3]. The most natural generalisation of semimodularity to infinite lattices seems to be the notion of M-symmetry, due to Wilcox. Certain books from the 1960s (none of which I have understood yet) seem to foreshadow geometric stability theory in strange ways. The role of forking independence (over the empty set) is played by modular pairs of elements whose meet is the minimal element. While this looks very familiar, what seems to be a substitute for types—'perspectivity' [4] or 'normal automorphisms' [5]—appears outlandish when applied to the lattice of algebraically closed sets. It seems likely to me that some methods can be transferred between stability theory and the now unfashionable subject of continuous geometry.

In addition to the conventions of the previous paper, we write $\bar{a}_{<k}$ for the tuple $\bar{a}_0\bar{a}_1 \dots \bar{a}_{k-1}$, and $\bar{a}_{<\omega}$ for the sequence $(\bar{a}_i)_{i<\omega}$. We will often implicitly assume that certain tuples are compatible. For example when we write $\bar{y}_{<k}$ it goes without saying that the tuples $\bar{y}_0, \bar{y}_1, \dots, \bar{y}_{k-1}$ are pairwise compatible.

1 Preindependence Relations

We will call a ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ between (small) subsets of the big model a *preindependence relation* if it satisfies the following *axioms for preindependence relations.*

(invariance)
If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $(A', B', C') \equiv (A, B, C)$, then $A' \mathop{\smile\!\!\!\!\!\!|}_{C'} B'$.

(monotonicity)
If $A \mathop{\smile\!\!\!\!\!\!|}_C B$, $A' \subseteq A$ and $B' \subseteq B$, then $A' \mathop{\smile\!\!\!\!\!\!|}_C B'$.

(base monotonicity)
Suppose $D \subseteq C \subseteq B$. If $A \mathop{\smile\!\!\!\!\!\!|}_D B$, then $A \mathop{\smile\!\!\!\!\!\!|}_C B$.

(transitivity)
Suppose $D \subseteq C \subseteq B$. If $B \mathop{\smile\!\!\!\!\!\!|}_C A$ and $C \mathop{\smile\!\!\!\!\!\!|}_D A$, then $B \mathop{\smile\!\!\!\!\!\!|}_D A$.

(normality)
$A \mathop{\smile\!\!\!\!\!\!|}_C B$ implies $AC \mathop{\smile\!\!\!\!\!\!|}_C B$.

(strong finite character)
If $A \mathop{\not\smile\!\!\!\!\!\!|}_C B$, then there are finite tuples $\bar{a} \in A$, $\bar{b} \in B$ and $\bar{c} \in C$ and a formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ without parameters such that $\models \varphi(\bar{a}, \bar{b}, \bar{c})$, and $\bar{a}' \mathop{\not\smile\!\!\!\!\!\!|}_C \bar{b}$ for all $\bar{a}'$ satisfying $\models \varphi(\bar{a}', \bar{b}, \bar{c})$.

In the previous paper we defined the notion of *independence relation.* We get the axioms of independence relations from the above axioms if we remove strong finite character and add the following axioms.

(extension)
If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $\hat{B} \supseteq B$, then there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_C \hat{B}$.

(finite character)
If $A_0 \mathop{\smile\!\!\!\!\!\!|}_C B$ for all finite $A_0 \subseteq A$, then $A \mathop{\smile\!\!\!\!\!\!|}_C B$.

(local character)
For every A there is a cardinal $\kappa(A)$ such that for any set B there is a subset $C \subseteq B$ of cardinality $|C| < \kappa(A)$ such that $A \mathop{\smile\!\!\!\!\!\!|}_C B$.

We have shown that an independence relation also satisfies the following rules.

(full existence)
For any A, B and C there is $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_C B$.

(symmetry)
$A \mathop{\smile\!\!\!\!\!\!|}_C B \iff B \mathop{\smile\!\!\!\!\!\!|}_C A$.

Question 1.1. Is there a complete theory with an independence relation which does not have strong finite character, or is every independence relation a preindependence relation?

Enrique Casanovas examined the relation between indiscernible sequences and relations $\mathop{\smile\!\!\!\!\!\!|}$ that satisfy certain axioms, including a strong form of finite character expressed as a topological condition on the type spaces [6]. Although it is not entirely obvious, the relations satisfying his axioms are precisely the independence relations with strong finite character [1].

Recall that if we have a relation satisfying some of the axioms for independence relations (but at least invariance and monotonicity), then we can pass to a relation $\mathop{\smile\!\!\!\!\!\!|}^*$ which still satisfies them, and also extension. Only for local character it is false in general, and for finite character we could not prove it. This is one respect in which strong finite character is better.

Lemma 1.2. *If $\mathop{\perp\!\!\!\perp}$ is a relation satisfying invariance, monotonicity and strong finite character, then $\mathop{\perp\!\!\!\perp}^{*}$ also satisfies strong finite character. Hence if $\mathop{\perp\!\!\!\perp}$ is a preindependence relation, then $\mathop{\perp\!\!\!\perp}^{*}$ is a preindependence relation satisfying extension.*

Proof. By Lemma 3.1 of the previous paper [2] all axioms other than (strong) finite character are preserved, and $\mathop{\perp\!\!\!\perp}^{*}$ satisfies extension. It remains to prove strong finite character of $\mathop{\perp\!\!\!\perp}^{*}$.

Suppose $\bar{a} \not\mathop{\perp\!\!\!\perp}^{*}_{C} B$ ($\bar{a}$ being a sequence of arbitrary length), and let this be witnessed by $\hat{B} \supseteq B$ such that $\bar{a}' \not\mathop{\perp\!\!\!\perp}_{C} \hat{B}$ for all $\bar{a}' \equiv_{BC} \bar{a}$. Let $\bar{x}$ be a sequence of the same length as $\bar{a}$, and let $p(\bar{x})$ be the set of formulas over $\hat{B}C$ consisting of the negations of all those formulas $\varphi_i(\bar{x}, \bar{b}_i, \bar{c}_i)$ with parameters $\bar{b}_i \in \hat{B}$ and $\bar{c}_i \in C$ that have the property that $\bar{a}' \not\mathop{\perp\!\!\!\perp}_{C} \bar{b}_i$ for all $\bar{a}'$ satisfying $\models \varphi_i(\bar{a}', \bar{b}_i, \bar{c}_i)$. By choice of $\hat{B}$ and strong finite character of $\mathop{\perp\!\!\!\perp}$, $p(\bar{x}) \cup \operatorname{tp}(\bar{a}/BC)$ is inconsistent. So by compactness there is a formula $\psi(\bar{x}, \bar{b}, \bar{c}) \in \operatorname{tp}(\bar{a}/BC)$ such that $p(\bar{x}) \cup \{\psi(\bar{x}, \bar{b}, \bar{c})\}$ is inconsistent.

Now suppose $\bar{a}'$ satisfies $\models \psi(\bar{a}', \bar{b}, \bar{c})$. To finish our proof we claim that $\bar{a}' \not\mathop{\perp\!\!\!\perp}^{*}_{C} \bar{b}$. Otherwise there would be $\bar{a}^* \equiv_{C\bar{b}} \bar{a}'$ such that $\bar{a}^* \mathop{\perp\!\!\!\perp}_{C} \hat{B}$. But then $\models \psi(\bar{a}^*, \bar{b}, \bar{c})$ would also hold. On the other hand, $\bar{a}^*$ would realise $p(\bar{x})$, in contradiction to inconsistency of $p(\bar{x}) \cup \{\psi(\bar{x}, \bar{b}, \bar{c})\}$. $\square$

The recipe for obtaining independence relations which we used in the previous paper can now be described as follows. 1. Define a preindependence relation $\mathop{\perp\!\!\!\perp}$ (they seem to be fairly ubiquitous). 2. Pass from $\mathop{\perp\!\!\!\perp}$ to $\mathop{\perp\!\!\!\perp}^{*}$ to get a preindependence relation that also satisfies extension. 3. Work in the case when $\mathop{\perp\!\!\!\perp}^{*}$ has local character.

Proposition 1.3. *The relations $\mathop{\perp\!\!\!\perp}^{\mathrm{M}}$, $\mathop{\perp\!\!\!\perp}^{\text{þ}}$, $\mathop{\perp\!\!\!\perp}^{\mathrm{d}}$ and $\mathop{\perp\!\!\!\perp}^{\mathrm{f}}$ are preindependence relations.*

Proof. We only need to show that $\mathop{\perp\!\!\!\perp}^{\mathrm{M}}$ and $\mathop{\perp\!\!\!\perp}^{\mathrm{d}}$ are preindependence relations, the rest follows with Lemma 1.2. We have already checked all axioms except strong finite character. In the case of $\mathop{\perp\!\!\!\perp}^{\mathrm{d}}$ strong finite character is obvious from Remark 5.5 in the previous paper. So it remains to show that $\mathop{\perp\!\!\!\perp}^{\mathrm{M}}$ satisfies strong finite character.

Suppose $A \not\mathop{\perp\!\!\!\perp}^{\mathrm{M}}_{C} B$. Then there is D such that $C \subseteq D \subseteq \operatorname{acl}(BC)$ and an element $e \in \big(\operatorname{acl}(AD) \cap \operatorname{acl}(BD)\big) \setminus \operatorname{acl} D$. Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be enumerations of A, B and C, respectively.

Since $e \in \operatorname{acl}(AD)$, we can find a finite tuple $\bar{d} \in D$ and an algebraic formula $\alpha(u, \bar{a}, \bar{d})$ such that $\models \alpha(d, \bar{a}, \bar{d})$. Then for appropriate $k < \omega$, e satisfies the formula $\alpha'(u, \bar{a}, \bar{d})$ defined as $\alpha(u, \bar{a}, \bar{d}) \wedge \exists_{\leq k} u' \alpha(u', \bar{a}, \bar{d})$.

Since $e \in \operatorname{acl}(BD) = \operatorname{acl}(BC)$, there is an algebraic formula $\beta(u, \bar{b}, \bar{c})$ such that $\models \beta(e, \bar{b}, \bar{c})$. Let $e_0, \ldots, e_{n-1}$ be all the realisations of $\beta(u, \bar{b}, \bar{c})$ that are in $\operatorname{acl} D$.

Let $\chi(u, \bar{d}^*)$ be an algebraic formula with parameters in D that is satisfied at least by $e_0, \ldots, e_{n-1}$. We may assume that $\bar{d} = \bar{d}^*$. Note that every element e' that satisfies $\beta(u, \bar{b}, \bar{c})$, either satisfies $\chi(u, \bar{d})$ or is not algebraic over $C\bar{d}$ at all.

Let $\delta(\bar{v}, \bar{b}, \bar{c})$ be an isolating formula in the algebraic type $\operatorname{tp}(\bar{d}/B \cup C)$. Note that for any $\bar{d}'$ satisfying $\delta(\bar{v}, \bar{b}, \bar{c})$, every element e' that satisfies $\beta(u, \bar{b}, \bar{c})$ either satisfies $\chi(u, \bar{d}')$ or is not algebraic over $C\bar{d}'$ at all.

Let $\varphi(\bar{x}, \bar{b}, \bar{c})$ be the formula defined as

$$\exists u \exists \bar{v} \big(\delta(\bar{v}, \bar{b}, \bar{c}) \wedge \alpha'(u, \bar{x}, \bar{v}) \wedge \beta(u, \bar{b}, \bar{c}) \wedge \neg\chi(u, \bar{v})\big).$$

$\varphi(\bar{x}, \bar{b}, \bar{c})$ has the property desired:

First note that e and $\bar{d}$ witness that $\models \varphi(\bar{a}, \bar{b}, \bar{c})$ holds. On the other hand, suppose $\models \varphi(\bar{a}', \bar{b}, \bar{c})$ holds and let e' and $\bar{d}'$ witness this, i.e.,

$$\models \delta(\bar{d}', \bar{b}, \bar{c}) \wedge \alpha'(e', \bar{a}', \bar{d}') \wedge \beta(e', \bar{b}, \bar{c}) \wedge \neg\chi(e', \bar{d}').$$

Let $D' = C\bar{d}'$. From $\delta(\bar{d}', \bar{b}, \bar{c})$ we get $C \subseteq D' \subseteq \operatorname{acl}(BC)$. From $\models \alpha'(e', \bar{a}', \bar{d}')$ we get $e' \in \operatorname{acl}(\bar{a}'\bar{d}') \subseteq \operatorname{acl}(D'\bar{a}')$. From $\models \beta(e', \bar{b}, \bar{c}) \wedge \neg\chi(e', \bar{d}')$ we get $e' \in \operatorname{acl}(BC)$ and $e' \notin \operatorname{acl}(C\bar{d}') = \operatorname{acl} D'$. Hence e' witnesses $\operatorname{acl}(D'\bar{a}) \cap \operatorname{acl}(BD') \supsetneq \operatorname{acl} D'$. $\square$

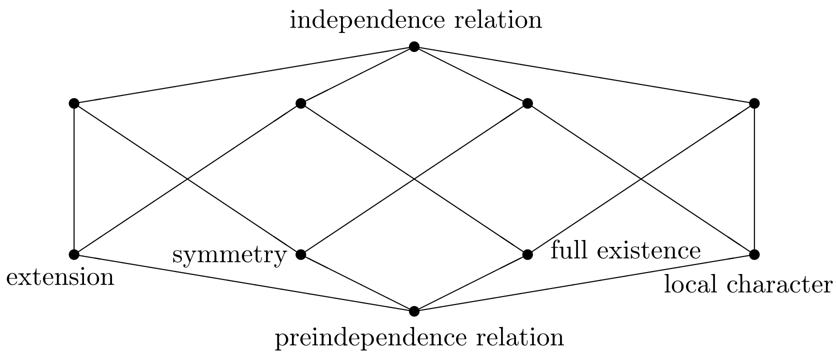

Figure 1: Classification of preindependence relations according to which of four properties hold. For each unlabelled node of this lattice diagram there is an example in the previous paper [2]. These can be used to assemble examples for the other nodes below.

The relations $\mathop{\smile\hskip-0.9em ^| \ }^{\mathrm{m}}$ and $\mathop{\smile\hskip-0.9em ^| \ }^{\text{þ-d}}$ which were also defined in the previous paper are, of course, also preindependence relations, and the same is true for relations derived from splitting, strong splitting (needs a harmless modification to fix transitivity) and weak dividing (left and right sides must be exchanged).

A type $p(\bar{x})$ is called *finitely satisfied* in a set C if for every formula $\varphi(\bar{x}) \in p$ (with parameters) there is a tuple $\bar{c} \in C$ such that $\models \varphi(\bar{c})$. By now the reader will not be surprised by my claim that this notion gives rise to a preindependence relation. But we are not going to use it, so we don't prove it.

Remark 1.4. Suppose $\mathop{\smile\hskip-0.9em ^| \ }$ satisfies monotonicity and strong finite character, and $\bar{a}, B, C$ are such that $C \mathop{\smile\hskip-0.9em ^| \ }_C B$ holds and $\operatorname{tp}(\bar{a}/BC)$ is finitely satisfied in C. Then $\bar{a} \mathop{\smile\hskip-0.9em ^| \ }_C B$.

Proof. Suppose $\bar{a} \not\mathop{\smile\hskip-0.9em ^| \ }_C B$. Let $\varphi(\bar{x}_0, \bar{y}, \bar{z})$ and $\bar{a}_0 \subseteq \bar{a}$, $\bar{b} \in B$, $\bar{c} \in C$ be as in the strong finite character condition. Since $\operatorname{tp}(\bar{a}/BC)$ is finitely satisfied in C there is $\bar{a}' \in C$ such that $\models \varphi(\bar{a}', \bar{b}, \bar{c})$ holds. Hence $\bar{a}' \not\mathop{\smile\hskip-0.9em ^| \ }_C B$, hence $C \not\mathop{\smile\hskip-0.9em ^| \ }_C B$ by monotonicity. □

This is quite useful because we also have the following well-known and easy fact.

Remark 1.5. For any $\bar{a}$, B there is a subset $C \subseteq \bar{a}$ of size $|C| \leq |T| + |B|$ such that $\operatorname{tp}(\bar{a}/BC)$ is finitely satisfied in C.

Putting the two remarks together it is easy to get the dual (left and right sides exchanged) of local character.

Theorem 1.6. *Suppose $\mathop{\smile\hskip-0.9em ^| \ }$ is a preindependence relation. Then $\mathop{\smile\hskip-0.9em ^| \ }$ is an independence relation if and only if $\mathop{\smile\hskip-0.9em ^| \ }$ satisfies full existence and symmetry.*

Proof. First note that strong finite character implies finite character. We already know the forward direction, so we only need to prove extension and local character from full existence and symmetry. Extension easily follows from transitivity, normality, full existence and symmetry. For local character we can take $\kappa(B) = (|T| + |B|)^+$: Given $\bar{a}$ and B there is $C \subseteq \bar{a}$ such that $|C| < \kappa(B)$ and $\operatorname{tp}(\bar{a}/BC)$ is finitely satisfied in C. Now $C \mathop{\smile\hskip-0.9em ^| \ }_C B$ holds by full existence, so $\bar{a} \mathop{\smile\hskip-0.9em ^| \ }_C B$ by monotonicity and strong finite character. □

This theorem allows us to 'improve' the last proposition of the previous paper.

Corollary 1.7. *Let $\mathop{\smile\hskip-0.9em ^| \ }$ be a preindependence relation.*

(1) If $\mathop{\smile\hskip-0.9em ^| \ }$ satisfies extension and local character, then $\mathop{\smile\hskip-0.9em ^| \ }$ also satisfies symmetry and full existence.

(2) If $\mathop{\smile\hskip-0.9em ^| \ }$ satisfies symmetry and full existence, then $\mathop{\smile\hskip-0.9em ^| \ }$ also satisfies extension and local character.

No other relations between extension, local character, symmetry and full existence hold in general. More precisely, there are examples of complete theories and relations $\downarrow$ satisfying the basic axioms and showing that:

(3) $\downarrow$ may satisfy extension and symmetry, but neither full existence nor local character.
(4) $\downarrow$ may satisfy extension and full existence, but neither symmetry nor local character.
(5) $\downarrow$ may satisfy symmetry and local character, but neither extension nor full existence.
(6) $\downarrow$ may satisfy full existence and local character, but neither extension nor symmetry.

Proof. (1) and (2) are by Theorem 1.6. For (3)–(6) we can use the same examples as for Proposition 6.1 in the previous paper [2]. It suffices to check that these examples all have strong finite character. $\square$

2 Local Dividing

This section and the next one are very technical, and there is not much I can do to accomodate a reader who is not acquainted with Byunghan Kim's theory of dividing and forking in simple theories [7]. The experts will remember that Kim had to introduce the parameter k in k-dividing. It allowed him to use (implicitly) *the* negation of the formula $\exists\bar{x}(\varphi(\bar{x};\bar{y}_0)\wedge\cdots\wedge\varphi(\bar{x};\bar{y}_{k-1}))$. When Itay Ben-Yaacov wanted to generalise Kim's results to positive Robinson theories, the same problem came up again, because in positive model theory there is no canonical negation [8]. Ben-Yaacov used the same strategy, by working with *specific,* non-canonical, contradictions $\psi(\bar{y}_0,\ldots,\bar{y}_{k-1})$ of the formula $\exists\bar{x}(\varphi(\bar{x};\bar{y}_0)\wedge\cdots\wedge\varphi(\bar{x};\bar{y}_{k-1}))$. It turns out that this naturally leads to a way of expressing thorn-forking in terms of forking. Fortunately, the fact that we are working in the classical context allows us to take an important short cut: The only appearance of the scary word 'array-dividing' is in this very paragraph.

A formula $\psi(\bar{y}_{<k})$ is called a *k-inconsistency witness* for $\varphi(\bar{x};\bar{y})$ if the formula $\big(\bigwedge_{i<k}\varphi(\bar{x};\bar{y}_i)\big)\wedge\psi(\bar{y}_{<k})$ is inconsistent. When the precise value of k is immaterial we will omit it. We write

$$\Psi=\big\{\psi(\bar{y}_0,\ldots,\bar{y}_{k-1})\;\big|\;k<\omega;\ \bar{y}_0,\ldots,\bar{y}_{k-1}\text{ are compatible}\big\}$$

for the set of all potential inconsistency witnesses. A k-inconsistency witness $\psi(\bar{y}_{<k})$ for $\varphi(\bar{x};\bar{y})$ 'witnesses' k-inconsistency in the following way: Suppose $(\bar{b}_i)_{i<\omega}$ is a sequence such that $\models\psi(\bar{b}_{i_0},\ldots,\bar{b}_{i_{k-1}})$ for any $i_0<\cdots<i_{k-1}<\omega$. Then the set $\{\varphi(\bar{x};\bar{b}_i)\mid i<\omega\}$ is k-inconsistent, i. e., there is no tuple $\bar{a}$ satisfying k formulas from the set simultaneously.

A formula $\varphi(\bar{x};\bar{b})$ ψ-divides over a set C if $\psi\in\Psi$ is an inconsistency witness for $\varphi(\bar{x};\bar{y})$ and there is a sequence $\bar{b}_{<\omega}$ such that each $\bar{b}_i$ realises $\operatorname{tp}(\bar{b}/C)$, and $\models\psi(\bar{b}_{i_0},\ldots,\bar{b}_{i_{k-1}})$ holds for all $i_0<\ldots<i_{k-1}<\omega$. We say that $\bar{b}_{<\omega}$ *witnesses* that $\varphi(\bar{x};\bar{b})$ ψ-divides over C. A partial type $p(\bar{x})$ *ψ-divides* over a set C if it contains a formula $\varphi(\bar{x};\bar{b})\in p(\bar{x})$ which ψ-divides over C. Sometimes we need to make $\varphi(\bar{x};\bar{y})$ explicit, so we will say that the type *(φ,ψ)-divides* over C. Note that when $\varphi(\bar{x};\bar{b})$ ψ-divides over a set C, then there is a sequence $\bar{b}_{<\omega}$ witnessing this with $\bar{b}_0=\bar{b}$. Also note that $\varphi(\bar{x};\bar{b})$ also ψ-divides over every subset of C.

Now we will consider subsets $\Omega\subseteq\Psi$ that are closed under variable substitution in the following sense: If $\psi(\bar{y}_{<k})\in\Omega$ and $\bar{v}_{<k}$ is compatible with $\bar{y}_{<k}$, then $\psi(\bar{v}_{<k})\in\Omega$. We will call Ω *normal* if the following principle also holds. If $\psi(\bar{y}_0,\ldots,\bar{y}_{k-1})\in\Omega$, then also $\psi'(\bar{y}_0\bar{z}_0,\ldots,\bar{y}_{k-1}\bar{z}_{k-1})\in\Omega$, where ψ' is defined as $\psi(\bar{y}_{<k})\wedge(\bar{z}_0=\bar{z}_1=\cdots=\bar{z}_{k-1})$.

We say that a partial type $p(\bar{x})$ *Ω-divides* over a set C if it ψ-divides over C for some $\psi\in\Omega$. We define a relation $\downarrow^{\Omega}$ as follows:

$$A\mathop{\downarrow}^{\Omega}_{C}B\iff\textit{there is no }\bar{a}\in A\textit{ such that }\operatorname{tp}(\bar{a}/BC)\ \Omega\textit{-divides over }C.$$

Proposition 2.1. *If $\Omega\subseteq\Psi$ is normal, then $\downarrow^{\Omega}$ is a preindependence relation. Moreover, $A\downarrow^{\Omega}_{B}B$ and $A\downarrow^{\Omega}_{A}B$ for any sets A and B.*

Proof. *Invariance* and *monotonicity* are obvious. *Base monotonicity:* Suppose $A \mathop{\not\perp\!\!\!\perp}^{\Omega}_{C} B$ and $D \subseteq C \subseteq B$. It suffices to show that $A \mathop{\not\perp\!\!\!\perp}^{\Omega}_{D} B$. There is $\bar{a} \in A$, $\psi \in \Omega$ and a formula $\varphi(\bar{x}; \bar{b}) \in \mathrm{tp}(\bar{a}/B)$ which ψ-divides over C. It is immediate from the definition of ψ-dividing that $\varphi(\bar{x}; \bar{b})$ also ψ-divides over D. So $A \mathop{\not\perp\!\!\!\perp}^{\Omega}_{D} B$ does in fact hold. *Strong finite character:* Suppose $A \mathop{\not\perp\!\!\!\perp}^{\Omega}_{C} B$. Let $\bar{a} \in A$ be such that $\mathrm{tp}(\bar{a}/BC)$ Ω-divides over C. So there is a formula $\varphi(\bar{x}; \bar{b}) \in \mathrm{tp}(\bar{a}/BC)$ which ψ-divides over C for some $\psi \in \Omega$. Hence for every $\bar{a}'$ satisfying $\models \varphi(\bar{a}'; \bar{b})$, $\mathrm{tp}(\bar{a}'/\bar{b}C)$ also ψ-divides over C, so $\bar{a}' \mathop{\not\perp\!\!\!\perp}^{\Omega}_{C} \bar{b}$. *Transitivity:* Suppose $\bar{b} \mathop{\not\perp\!\!\!\perp}^{\Omega}_{D} A$ and $C \supseteq D$. We will show that $\bar{b} \mathop{\not\perp\!\!\!\perp}^{\Omega}_{C} A$ or $C \mathop{\not\perp\!\!\!\perp}^{\Omega}_{D} A$. There is a formula $\varphi(\bar{y}; \bar{a}) \in \mathrm{tp}(\bar{b}/AD)$ which ψ-divides over D for some $\psi \in \Omega$. Let $\bar{a}_{<\omega}$ witness this. Now let $\bar{c}$ enumerate C, let $p(\bar{z}, \bar{x}) = \mathrm{tp}(\bar{c}, \bar{a}/D)$, and consider the partial type

$$\bigcup_{i<\omega} p(\bar{z}, \bar{x}_i) \;\cup\; \bigl\{\psi(\bar{x}_{i_0}, \ldots, \bar{x}_{i_{k-1}}) \bigm| i_0 < \cdots < i_{k-1} < \omega\bigr\}.$$

If this type is consistent, we can realise it by $\bar{c}, \bar{a}'_{<\omega}$, and so $\bar{a}'_{<\omega}$ witnesses that $\varphi(\bar{y}; \bar{a})$ ψ-divides over C. If it is inconsistent, then this is caused by a formula $\varphi'(\bar{z}; \bar{x}) \in p(\bar{z}, \bar{x})$ and by a formula $\psi'(\bar{x}_{<k'})$ which is a finite conjunction of formulas in $\{\psi(\bar{x}_{i_0}, \ldots, \bar{x}_{i_{k-1}}) \mid i_0 < \cdots < i_{k-1} < \omega\}$. But then ψ' is an inconsistency witness for φ', and the original sequence $\bar{a}_{<\omega}$ witnesses that $\varphi'(\bar{z}; \bar{a})$ ψ'-divides over D. *Normality:* Suppose $AC \mathop{\not\perp\!\!\!\perp}^{\Omega}_{C} B$. So there are $\bar{a} \in A$, $\bar{c} \in C$ and a formula $\varphi(\bar{x}\bar{z}; \bar{b}) \in \mathrm{tp}(\bar{a}\bar{c}/BC)$ which ψ-divides over C for some $\psi \in \Psi$, witnessed by a sequence $\bar{b}_{<\omega}$. Consider the formulas $\varphi'(\bar{x}; \bar{b}\bar{c}) \equiv \varphi(\bar{x}\bar{c}; \bar{b})$ and $\psi'(\bar{y}_0\bar{z}_0, \ldots \bar{y}_{k-1}\bar{z}_{k-1}) \equiv \psi(\bar{y}_{<k}) \wedge \bar{z}_0 = \cdots = \bar{z}_{k-1}$. Then clearly ψ' is a k-inconsistency witness for φ', and the sequence $\bar{b}_0\bar{c}, \bar{b}_1\bar{c}, \bar{b}_2\bar{c}, \ldots$ witnesses that $\varphi'(\bar{x}; \bar{b}\bar{c})$ ψ'-divides over C. Since Ω, $\psi' \in \Omega$ is normal, $\bar{a} \mathop{\not\perp\!\!\!\perp}^{\Omega}_{C} B$.

For the first 'moreover' statement, suppose $\models \varphi(\bar{a}; \bar{b})$ for some tuples $\bar{a} \in A$ and $\bar{b} \in B$, and there is $\psi \in \Omega$ such that the formula $\varphi(\bar{x}; \bar{b})$ ψ-divides over B. This would be witnessed by a sequence $\bar{b}_{<\omega}$ of tuples realising $\mathrm{tp}(\bar{b}/B)$, so $\bar{b}_i = \bar{b}$. But then $\models \bigl(\bigwedge_{i<k} \varphi(\bar{a}; \bar{b}_i)\bigr) \wedge \psi(\bar{b}_{<k})$, contradicting the assumption that ψ is a k-inconsistency witness for $\varphi(\bar{x}; \bar{y})$. For the second 'moreover' statement, suppose $\models \varphi(\bar{a}; \bar{b})$ for some tuples $\bar{a} \in A$ and $\bar{b} \in B$, and there is $\psi \in \Omega$ such that the formula and $\varphi(\bar{x}; \bar{b})$ ψ-divides over A. This would be witnessed by a sequence $\bar{b}_{<\omega}$ of tuples realising $\mathrm{tp}(\bar{b}/A)$. But then again $\models \bigl(\bigwedge_{i<k} \psi(\bar{a}; \bar{b}_i)\bigr) \wedge \psi(\bar{b}_{<k})$, contradicting the assumption that ψ is a k-inconsistency witness for $\varphi(\bar{x}; \bar{y})$. $\square$

In Example 4.6 below we will see that $\mathop{\perp\!\!\!\perp}^{\Omega}$ need not satisfy extension, local character, full existence or symmetry. Moreover, without the assumption that Ω is normal, $\mathop{\perp\!\!\!\perp}^{\Omega}$ need not be normal.

Example 2.2. Let Ω consist of all formulas of the form $\psi(y_{<k}) \equiv \bigwedge_{i<j<k}(y_i \neq y_j)$. It is easy to check that $A \mathop{\perp\!\!\!\perp}^{\Omega}_{C} B$ if and only if $(\mathrm{acl}\, A) \cap B \subseteq (\mathrm{acl}\, C) \cap B$. Therefore $\mathop{\perp\!\!\!\perp}^{\Omega}$ is normal if and only if algebraic closure is trivial in the sense that $\mathrm{acl}\, A = \bigcup_{a \in A} \mathrm{acl}\{a\}$.

Now we will localise further, so that we can introduce the local D-ranks. We consider finite sets $\Delta = \Delta(\bar{x})$ which consist of pairs $\bigl(\varphi(\bar{x}; \bar{y}^i), \psi(\bar{y}^i_{<k^i})\bigr)$. We call Δ a *finite set of pairs over* Ω if $\psi(\bar{y}^i_{<k^i}) \in \Omega$ for all $\bigl(\varphi(\bar{x}; \bar{y}^i), \psi(\bar{y}^i_{<k^i})\bigr) \in \Delta$. (The only purpose of the indices i is to make it clear that different pairs $(\varphi, \psi) \in \Delta$ may have different $\bar{y}$, while they all share the same $\bar{x}$.) We say that a partial type $p(\bar{x})$ Δ-*forks* over a set C if there are $n < \omega$, $\bigl(\varphi^i(\bar{x}; \bar{y}^i), \psi^i(\bar{y}^i_{<k_i})\bigr) \in \Delta$ for $i < n$, and tuples $\bar{b}^0, \ldots, \bar{b}^{n-1}$ such that $p(\bar{x}) \vdash \bigvee_{i<n} \varphi^i(\bar{x}; \bar{b}^i)$ and $\varphi^i(\bar{x}; \bar{b}^i)$ ψ^i-divides over C for each $i < n$.

Remark 2.3. (1) A formula $\varphi(\bar{x}; \bar{b})$ divides over C iff $\varphi(\bar{x}; \bar{b})$ ψ-divides over C for some formula $\psi(\bar{y}_{<k}) \in \Psi$.

(2) $\varphi(\bar{x}; \bar{b})$ forks over C iff $\varphi(\bar{x}; \bar{b})$ Δ-forks over C for some finite $\Delta(\bar{x})$.

(3) $\bar{a} \mathop{\perp\!\!\!\perp}^{\Omega *}_{C} B$ iff $\mathrm{tp}(\bar{a}/BC)$ does not Δ-fork over C for any finite Δ over Ω.

Remark 2.4. $\mathop{\perp\!\!\!\perp}^{\mathrm{d}} = \mathop{\perp\!\!\!\perp}^{\Psi}$.

Proof. We know that $\bar{a} \not\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}_C B$ iff there is a formula in $\varphi(\bar{x};\bar{b}) \in \mathrm{tp}(\bar{a}/BC)$ that divides over C. On the other hand, $\bar{a} \not\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}_C B$ iff there is a formula $\varphi(\bar{x};\bar{b}) \in \mathrm{tp}(\bar{a}/BC)$ which Δ-forks over C for some finite $\Delta(\bar{x})$. But a formula forks over C iff it Δ-forks over C for a finite Δ. □

Question 2.5. Is every independence relation for a complete theory of the form $\mathop{\smile\!\!\!\!\!\!\mid}^{\Omega*}$ for a suitable (normal) $\Omega \subseteq \Psi$?

3 Dividing Patterns and their Applications

Let I be a linearly ordered set. By an *I-sequence of pairs over Ω* we understand an I-indexed sequence $\xi = ((\varphi^i, \psi^i))_{i\in I}$ of pairs $\big(\varphi^i(\bar{x};\bar{y}^i), \psi^i(\bar{y}^i_{<k^i})\big)$ with $\psi^i \in \Omega$. Let $p(\bar{x})$ be a partial type over a set C. An I-sequence of pairs $\xi = ((\varphi^i, \psi^i))_{i\in I}$ is a *dividing pattern for $p(\bar{x})$ (over C)* if there is an I-sequence $(\bar{b}^i)_{i\in I}$ that *realises ξ over C*. By this we mean that firstly, $p(\bar{x}) \cup \{\varphi^i(\bar{x};\bar{b}^i) \mid i \in I\}$ is consistent, and secondly, each formula $\varphi^i(\bar{x};\bar{b}^i)$ ψ^i-divides over $C\bar{b}^{<i}$. If $(\varphi^i, \psi^i) \in \Delta$ for all $i \in I$, we may call ξ a Δ-dividing pattern.

Vaguely speaking, dividing patterns measure how many dividing extensions a type has. Under certain conditions an extension of a type that admits exactly the same dividing patterns will be shown not to divide. Similar notions already exist in the case that I is an ordinal with either the natural or the opposite order. Itay Ben-Yaacov works with a total rank $\mathrm{D}(p, \Xi)$, which has the property that for any ordinal α and any α^{opp}-sequence of pairs $\xi \in \Xi^{\alpha^{\mathrm{opp}}}$, ξ is a dividing pattern for p if and only if $\xi \in \mathrm{D}(p, \Xi)$. The idea that ξ being a dividing pattern can be expressed by a partial type is also from Ben-Yaacov [9]. On the other hand, a realisation of an α-indexed dividing pattern is precisely what Enrique Casanovas calls a dividing chain [11]. Admitting arbitrary linear orders in the definition is by no means harder, and it allows us to use one theorem for treating both D-ranks and the tree property.

If I is a linearly ordered set and $i \in I$ we will temporarily write $< i$ and $\leq i$ for the initial sequences $\{j \in I \mid j < i\}$ and $\{j \in I \mid j \leq i\}$, respectively.

Theorem 3.1. *Let $p(\bar{x})$ be a partial type, definable over a set C. An I-sequence of pairs $\xi = \big(\big(\varphi^i(\bar{x};\bar{y}^i), \psi^i(\bar{y}^i_{<k_i})\big)\big)_{i\in I}$ over Ψ is a dividing pattern for $p(\bar{x})$ over C iff the following partial type $\mathrm{divpat}^{\xi}_{p}\big((\bar{x}_\alpha)_{\alpha\in\omega^I}, (\bar{y}_\alpha)_{\alpha\in\omega^{\leq i}, i\in I}\big)$ is consistent:*

$$\bigcup_{\alpha\in\omega^I} p(\bar{x}_\alpha) \quad\cup\quad \big\{\varphi^i(\bar{x}_\alpha;\bar{y}_{\alpha\restriction\leq i}) \;\big|\; i \in I, \alpha \in \omega^I\big\}$$
$$\cup\quad \Big\{\ \psi^i(\bar{y}_{\alpha_0}, \ldots, \bar{y}_{\alpha_{k_i-1}}) \quad\Big|\quad i \in I,\ \alpha_0, \ldots, \alpha_{k_i-1} \in \omega^{\leq i},$$
$$(\alpha_0\restriction < i) = \cdots = (\alpha_{k_i-1}\restriction < i),\ \text{and}\ \alpha_0(i) < \cdots < \alpha_{k_i-1}(i)\ \Big\}.$$

Before proving this theorem let us try to understand what it says. Without understanding the structure of the type $\mathrm{divpat}^{\xi}_{p}$ it is at least easy to see that it does not mention the set C. That's why the qualification 'over C' is in parentheses in the definition—we can choose any set C we like, as long as p is defined over it. The next easy observation is that the surrounding theory is not involved in the definition of $\mathrm{divpat}^{\xi}_{p}$. Hence if p and ξ make sense in a reduct T' of T, then ξ is a dividing pattern for p in the context of T iff it is one in the context of T'. We will use this to prove that simplicity and rosiness are preserved in reducts.

For understanding the structure of $\mathrm{divpat}^{\xi}_{p}$ it is perhaps best to imagine this type partially realised by tuples $(\bar{b}_\alpha)_{\alpha\in\omega^{\leq i}, i\in I}$. These tuples form a non-standard tree, and the last part of the conjunction requires that the tuples $\bar{b}_\alpha$ of level i (i.e.: $\alpha \in \omega^{\leq i}$) that define the same (non-standard) path $\alpha\restriction < i$ through the tree are related by the inconsistency witness ψ^i. The type $\mathrm{divpat}^{\xi}_{p}\big((\bar{x}_\alpha)_{\alpha\in\omega^I}, (\bar{b}_\alpha)_{\alpha\in\omega^{\leq i}, i\in I}\big)$ then merely expresses that for every branch $\alpha \in \omega^I$ of this tree the set $\big\{\varphi^i(\bar{x}, \bar{b}_{\alpha\restriction\leq i}) \;\big|\; i \in I\big\}$ is consistent with $p(\bar{x})$.

With this tree structure in mind it is easy to see that, by compactness, the property of being a dividing pattern has finite character: $\mathrm{divpat}^{\xi}_{p}$ is consistent iff $\mathrm{divpat}^{\xi\restriction J}_{p}$ is consistent for every finite $J \subseteq I$. The tree structure already suggests a proof strategy.

Proof. We will prove the equivalence of the following statements.

(1) ξ is a dividing pattern for p over C.

(2) divpat_p^ξ is consistent.

(3) The type

$$\begin{aligned}\mathrm{divpat}'^{\,\xi}_p\big((\bar{x}_\alpha)_{\alpha\in\omega^I}, (\bar{y}_\alpha)_{\alpha\in\omega^{\le i}, i\in I}\big) \\ = \mathrm{divpat}_p^\xi\big((\bar{x}_\alpha)_{\alpha\in\omega^I}, (\bar{y}_\alpha)_{\alpha\in\omega^{\le i}, i\in I}\big) \\ \cup \big\{\bar{y}_\alpha \equiv_{C\{\bar{y}_{\alpha\restriction\le j} \mid j<i\}} \bar{y}_{\alpha'} \;\big|\; i\in I,\ \alpha,\alpha'\in\omega^{\le i},\ (\alpha\restriction<i)=(\alpha'\restriction<i)\big\}\end{aligned}$$

is consistent.

We first prove that (3) implies (1): Let the tuples $(\bar{b}_\alpha)_{\alpha\in\omega^{\le i}, i\in I}$ be a partial realisation of $\mathrm{divpat}'^{\,\xi}_p$. For $i\in I$ write ζ^i for the unique function $\zeta^i\in\{0\}^{<i}$, and for $m<\omega$ write $\zeta^{i\frown}(m)$ for the extension of ζ^i that maps i to m. Then for every $i\in I$ the sequence $(\bar{b}_{\zeta^{i\frown}(m)})_{m<\omega}$ witnesses that $\varphi(\bar{x};\bar{b}_{\zeta^{i\frown}(0)})$ ψ^i-divides over $C\{\bar{b}_{\zeta^{j\frown}(0)} \mid j<i\}$. Hence the I-sequence $(\bar{b}_{\zeta^{i\frown}(0)})_{i\in I}$ realises ξ over C.

Next we observe that we need only prove that (1) implies (2) and that (2) implies (3) in case I is finite. The general case then follows by compactness. Thus we can use induction on the size of I. The case $I=\emptyset$ is trivial: The 0-sequence () of pairs over Ψ is a dividing pattern for p over C iff p is consistent, and we have $\mathrm{divpat}_p^{()} = \mathrm{divpat}'^{\,()}_p = p(\bar{x}_{()})$.

Now suppose the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold for I, and we are given an $\{s\}\cup I$-sequence $((\varphi^s(\bar{x};\bar{y}),\psi^s(\bar{y}_{<k})))^\frown\xi$, where $s\notin I$ is less than every element of I. It is not hard to see that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) for $((\varphi^s,\psi^s))^\frown\xi$, using the following three easy facts:

(i) $((\varphi^s,\psi^s))^\frown\xi$ is a dividing pattern for p over C iff there is a tuple $\bar{b}$ such that $\varphi^s(\bar{x};\bar{b})$ ψ^s-divides over C and ξ is a dividing pattern for $p(\bar{x})\cup\varphi^s(\bar{x};\bar{b})$.

(ii) $\mathrm{divpat}_p^{((\varphi^s,\psi^s))^\frown\xi}$ is consistent iff there is a sequence $(\bar{b}_m)_{m<\omega}$ such that $\models\psi^s(\bar{b}_{m_0},\dots,\bar{b}_{m_{k-1}})$ for any $m_0<\dots<m_{k-1}<\omega$ and the type $\mathrm{divpat}^\xi_{p(\bar{x})\cup\varphi^s(\bar{x};\bar{b}_m)}$ is consistent for every $m<\omega$.

(iii) $\mathrm{divpat}'^{\,((\varphi^s,\psi^s))^\frown\xi}_p$ is consistent iff there is a sequence $(\bar{b}_m)_{m<\omega}$ such that $\models\psi^s(\bar{b}_{m_0},\dots,\bar{b}_{m_{k-1}})$ for any $m_0<\dots<m_{k-1}<\omega$, $\bar{b}_m\equiv_C\bar{b}_0$ for all $m<\omega$, and the type $\mathrm{divpat}'^{\,\xi}_{p(\bar{x})\cup\varphi^s(\bar{x};\bar{b}_m)}$ is consistent for every $m<\omega$. (Thus the sequence $(\bar{b}_m)_{m<\omega}$ witnesses that $\varphi(\bar{x};\bar{b}_0)$ ψ^s-divides over C.) $\square$

D-rank and the tree property

Fix a finite set of pairs $\Delta(\bar{x})$. If there are arbitrarily long finite Δ-dividing patterns for p, then there is a pair $(\varphi,\psi)\in\Delta$ such that there are arbitrarily long finite (φ,ψ)-dividing patterns for p. It follows that $(\varphi,\psi)^I$ is a Δ-dividing pattern for p for every linearly ordered set I. Therefore the following definition makes sense.

Let $p(\bar{x})$ be a partial type and $\Delta(\bar{x})$ a finite set of pairs. Then $\mathrm{D}_\Delta(p)\in\omega\cup\{\infty\}$ is ∞ if p has Δ-dividing patterns of arbitrary order type, or otherwise the greatest number $n<\omega$ such that Δ-dividing patterns of length n exist for p.

A formula $\varphi(\bar{x};\bar{y})$ *has the tree-property (of order k)* if there is a tree of tuples $(\bar{b}_\alpha)_{\alpha\in\omega^{<\omega}}$ such that for every limit point $\alpha\in\omega^\omega$ the branch $\{\varphi(\bar{x};\bar{b}_{\alpha\restriction n}) \mid n<\omega\}$, is consistent, and at every node $\alpha\in\omega^{<\omega}$ the set of successors $\{\varphi(\bar{x};\bar{b}_{\alpha^\frown(i)}) \mid i<\omega\}$ is k-inconsistent (i. e., every subset with k elements is inconsistent).

Remark 3.2. A formula $\varphi(\bar{x};\bar{y})$ has the tree-property of order k if and only if there is a k-inconsistency witness $\psi(\bar{y}_{<k})$ for φ such that $\mathrm{D}_{\varphi,\psi}(\emptyset)=\infty$.

Proof. First observe that given any formula $\varphi(\bar{x};\bar{y})$ and $k<\omega$, the formula $\psi(\bar{y}_{<k})\equiv\neg\exists\bar{x}\bigwedge_{i<k}\varphi(\bar{x};\bar{y}_i)$ is the most general k-inconsistency witness for φ in the sense that whenever $\psi'(\bar{y}_{<k})$ is a k-inconsistency witness for φ and $(\bar{b}_i)_{i<\omega}$ is a sequence such that we have $\models\varphi'(\bar{b}_{i_0},\dots,\bar{b}_{i_{k-1}})$ for all $i_0<\dots<i_{k-1}<\omega$, then also $\models\varphi(\bar{b}_{i_0},\dots,\bar{b}_{i_{k-1}})$ for all $i_0<\dots<i_{k-1}<\omega$. Now note that $\mathrm{D}_{\varphi,\psi}(\emptyset)=\infty$ iff the unique element $\xi\in\{(\varphi,\psi)\}^\omega$ is a dividing pattern, iff the type $\mathrm{divpat}_\emptyset^\xi$ is consistent. But the tree that appears in the tree property is just a partial realisation of $\mathrm{divpat}_\emptyset^\xi$. $\square$

D-rank and forking symmetry

At last we have everything it takes for another beautiful symmetry proof. Let $\Omega \subseteq \Psi$ be closed under variable substitution. We will show a connection between $\mathop{\smile\hskip-0.9em ^|\ }^{\Omega *}$ and the D_Δ-ranks under a combinatorial condition (finite ranks) and a skew converse under a geometric condition (normality). If both conditions are satisfied, $\mathop{\smile\hskip-0.9em ^|\ }^{\Omega *}$ is symmetric.

Remark 3.3. For any $\bar{a}$, B, C and finite $\Delta(\bar{x})$ we have $\mathrm{D}_\Delta(\bar{a}/BC) \leq \mathrm{D}_\Delta(\bar{a}/C)$.

Proof. Let $p(\bar{}) = \mathrm{tp}(\bar{a}/BC)$. If $\xi \in \Delta(\bar{x})^n$ is a dividing pattern for p, then divpat_p^ξ is consistent. Hence $\mathrm{divpat}_{p\restriction C}^\xi$ is consistent, so ξ is a dividing pattern for $p\restriction C$. $\square$

Lemma 3.4. *Suppose* $\mathrm{D}_\Delta(\bar{a}/BC) = \mathrm{D}_\Delta(\bar{a}/C) < \infty$ *for all finite* $\Delta(\bar{x})$ *over* Ω*. Then* $\bar{a} \mathop{\smile\hskip-0.9em ^|\ }^{\Omega *}_C B$*.*

Proof. Towards a contradiction, suppose $\bar{a} \not\mathop{\smile\hskip-0.9em ^|\ }^{\Omega *}_C B$. Then there is a set $\hat{B} \supset B$ such that $\bar{a}' \not\mathop{\smile\hskip-0.9em ^|\ }^{\Omega}_C \hat{B}$ holds for every $\bar{a}'$ realising $p(\bar{x}) = \mathrm{tp}(\bar{a}/BC)$. Hence the set

$$p(\bar{x}) \cup \big\{\neg\varphi(\bar{x};\bar{b}) \bigm| (\varphi,\psi) \in \Delta, \textit{ and } \varphi(\bar{x};\bar{b}) \ \psi\textit{-divides over } C\big\}$$

is inconsistent. By compactness there are pairs $(\varphi^i,\psi^i) \in \Delta$ and tuples $\bar{b}^i$ such that $p(\bar{x}) \vdash \bigvee_{i<n} \varphi^i(\bar{x};\bar{b}^i)$, and $\varphi^i(\bar{x};\bar{b}^i)$ ψ^i-divides over C.

Let ξ be a Δ-dividing pattern for p of maximal length $|\xi| = \mathrm{D}_\Delta(p)$, realised over $C\bar{b}^0\bar{b}^1\ldots\bar{b}^{n-1}$ by, say, $(\bar{b}_j)_{j<|\xi|}$. Let $\bar{a}'$ realise $p(\bar{x}) \cup \{\varphi_j(\bar{x};\bar{b}_j) \mid j < |\xi|\}$. Then $\models \varphi^i(\bar{a}';\bar{b}^i)$ for an index $i < n$. Hence $(\varphi^i,\psi^i)^\frown\xi$ is a Δ-dividing pattern for p, realised by $\bar{b}^{i\frown}(\bar{b}_j)_{j<|\xi|}$. This contradicts maximality of $|\xi|$. $\square$

Lemma 3.5. *Suppose* Ω *is normal and* $B \mathop{\smile\hskip-0.9em ^|\ }^{\Omega *}_C \bar{a}$*.*
Then for every finite $\Delta(\bar{x})$ *over* Ω *we have* $\mathrm{D}_\Delta(\bar{a}/BC) = \mathrm{D}_\Delta(\bar{a}/C)$*.*

Proof. Since $\mathrm{D}_\Delta(\bar{a}/BC) \leq \mathrm{D}_\Delta(\bar{a}/C)$ holds anyway we need only prove that $\mathrm{D}_\Delta(\bar{a}/C) \geq n$ implies $\mathrm{D}_\Delta(\bar{a}/BC) \geq n$. By definition of D_Δ there is a Δ-dividing pattern $\xi = ((\varphi^i,\psi^i))_{i<n} \in \Delta^n$ for $\mathrm{tp}(\bar{a}/C)$, and this is witnessed by tuples $(\bar{b}^i)_{i<n}$ such that $\models \varphi^i(\bar{a};\bar{b}^i)$ and $\varphi^i(\bar{x};\bar{b}^i)$ ψ^i-divides over $C\bar{b}^{<i}$ for all $i < n$. Since $B \mathop{\smile\hskip-0.9em ^|\ }^{\Omega *}_C \bar{a}$ we may assume that $B \mathop{\smile\hskip-0.9em ^|\ }^{\Omega}_C \bar{a}\bar{b}^{<n}$. Hence $BC\bar{b}^{<i} \mathop{\smile\hskip-0.9em ^|\ }^{\Omega}_{C\bar{b}^{<i}} \bar{b}_i$ by base monotonicity and normality for all $i < n$. Now since $\varphi^i(\bar{x};\bar{b}^i)$ ψ^i-divides over $C\bar{b}^{<i}$ we get by transitivity that $\varphi^i(\bar{x};\bar{b}^i)$ ψ^i-divides over $BC\bar{b}^{<i}$ as well. Therefore $\bar{b}^{<n}$ also witnesses that ξ is a dividing pattern for $\mathrm{tp}(\bar{a}/BC)$, so $\mathrm{D}(\bar{a}/BC) \geq n$. $\square$

Theorem 3.6. *Suppose* Ω *is normal, and* $\mathrm{D}_\Delta(\emptyset) < \infty$ *for all finite* $\Delta(\bar{x})$ *over* Ω*. Then the following conditions are equivalent:*

(1) $\bar{a} \mathop{\smile\hskip-0.9em ^|\ }^{\Omega *}_C B$*.*
(2) $\mathrm{D}_\Delta(\bar{a}/BC) = \mathrm{D}_\Delta(\bar{a}/C)$ *for all finite* $\Delta(\bar{x})$ *over* Ω*.*
(3) $B \mathop{\smile\hskip-0.9em ^|\ }^{\Omega *}_C \bar{a}$*.*

Proof. (3) implies (2) by Lemma 3.5, and (2) implies (1) by Lemma 3.4. Hence $\mathop{\smile\hskip-0.9em ^|\ }^{\Omega *}$ is symmetric, so (1) implies (3). $\square$

As it happens, we will not need this theorem to get symmetry (which we can get under slightly weaker conditions using Theorem 3.3 in the previous paper), but only to see that non-forking extensions are characterised by preservation of D-rank.

4 Sharper Results on Forking and Dividing

Again we fix a set $\Omega \subseteq \Psi$ which is closed under variable substitution.

Lemma 4.1. *The following statements are equivalent:*

(1) $\mathop{\smile\hskip-0.9em ^|\ }^{\Omega *}$ *satisfies the local character axiom.*
(2) $\mathop{\smile\hskip-0.9em ^|\ }^{\Omega}$ *satisfies the local character axiom.*
(3) $\mathrm{D}_{\varphi,\psi}(\emptyset) < \infty$ *for every* $(\varphi,\psi) \in \Omega$*.*

Proof. (1) implies (2): This follows from $A \mathop{\smile\hskip-0.9em^{|}\ }^{\Omega*}_{C} B \Rightarrow A \mathop{\smile\hskip-0.9em^{|}\ }^{\Omega}_{C} B$.

(2) implies (3): Suppose $\mathop{\smile\hskip-0.9em^{|}\ }^{\Omega}$ has local character with a constant κ, but $\mathrm{D}_{\varphi,\psi}(\emptyset) = \infty$. We may assume that κ is regular. $(\varphi,\psi)^\kappa$ is a dividing pattern for the empty type, so it has a realisation $(\bar{b}_i)_{i<\kappa}$ over $\emptyset$. Let $\bar{a}$ realise $\{\varphi(\bar{x};\bar{b}_i) \mid i<\kappa\}$. By local character there is a subset $C \subseteq \bar{b}_{<\kappa}$ such that $\bar{a} \mathop{\smile\hskip-0.9em^{|}\ }^{\Omega}_{C} \bar{b}_{<\kappa}$ and $|C| < \kappa$. Since κ is regular, there is $\alpha < \kappa$ such that $C \subseteq \bar{b}_{<\alpha}$. Hence $\bar{a} \mathop{\smile\hskip-0.9em^{|}\ }^{\Omega}_{\bar{b}_{<\alpha}} \bar{b}_{<\kappa}$, a contradiction to the fact that $\models \varphi(\bar{a};\bar{b}_\alpha)$ holds and $\varphi(\bar{x};\bar{b}_\alpha)$ ψ-divides over $\bar{b}_{<\alpha}$.

(3) implies (1): Note that $\mathrm{D}_\Delta(p(\bar{x})) < \omega$ for all finite $\Delta(\bar{x})$ over Ω and partial types $p(\bar{x})$. We will prove local character for $\mathop{\smile\hskip-0.9em^{|}\ }^{\Omega*}$ with $\kappa = |T|^+$. So suppose we have a type $p(\bar{x}) = \mathrm{tp}(\bar{a}/B)$ with finite $\bar{a}$.

For every finite $\Delta(\bar{x})$ over Ω we can find a finite subset $C_\Delta \subseteq B$ such that $\mathrm{D}_\Delta(p{\restriction}C_\Delta) = \mathrm{D}_\Delta(p)$: For each Δ-dividing pattern ξ of length $|\xi| = \mathrm{D}_\Delta(p)+1$ (there are only finitely many) the type divpat^ξ_p is inconsistent, so there is a finite subset $C_\xi \subseteq B$ such that $\mathrm{divpat}^\xi_{p{\restriction}C_\xi}$ is still inconsistent. If C_Δ is the union of these sets C_ξ, then clearly C_Δ is a finite set such that $\mathrm{D}_\Delta(p{\restriction}C_\Delta) = \mathrm{D}_\Delta(p)$.

Now let C be the union of these sets C_Δ for all finite $\Delta(\bar{x})$ over Ω. Then $|C| \le |T|$, so $|C| < \kappa$. Moreover, $\mathrm{D}_\Delta(p{\restriction}C) = \mathrm{D}_\Delta(p{\restriction}C_\Delta) = \mathrm{D}_\Delta(p)$ for all finite $\Delta(\bar{x})$ over Ω. Hence $\bar{a} \mathop{\smile\hskip-0.9em^{|}\ }^{\Omega*}_{C} B$ by Lemma 3.4. □

At last we can improve Theorem 3.3 of the previous paper in the case $\mathop{\smile\hskip-0.9em^{|}\ } = \mathop{\smile\hskip-0.9em^{|}\ }^{\Omega}$.

Theorem 4.2. *Let $\Omega \subseteq \Psi$ be normal.*
Then $\mathop{\smile\hskip-0.9em^{|}\ }^{\Omega*}$ is an independence relation if and only if the following, equivalent, conditions are satisfied.

(1) $\mathop{\smile\hskip-0.9em^{|}\ }^{\Omega*}$ satisfies the local character axiom.
(2) $\mathop{\smile\hskip-0.9em^{|}\ }^{\Omega}$ satisfies the local character axiom.
(3) $\mathrm{D}_{\varphi,\psi}(\emptyset) < \infty$ for every $(\varphi,\psi) \in \Omega$.
(4) $A \mathop{\smile\hskip-0.9em^{|}\ }^{\Omega*}_{C} B$ implies $B \mathop{\smile\hskip-0.9em^{|}\ }^{\Omega*}_{C} A$.
(5) $A \mathop{\smile\hskip-0.9em^{|}\ }^{\Omega*}_{C} B$ implies $B \mathop{\smile\hskip-0.9em^{|}\ }^{\Omega}_{C} A$.

Moreover, under these conditions $\bar{a} \mathop{\smile\hskip-0.9em^{|}\ }^{\Omega*}_{C} B$ if and only if $\mathrm{D}_\Delta(\bar{a}/BC) = \mathrm{D}_\Delta(\bar{a}/C)$ for all finite $\Delta(\bar{x})$ over Ω.

Proof. By Proposition 2.1 (and since we have assumed transitivity and normality), $\mathop{\smile\hskip-0.9em^{|}\ }^{\Omega}$ is a preindependence relation. Hence by Lemma 1.2, the relation $\mathop{\smile\hskip-0.9em^{|}\ }^{\Omega*}$ satisfies all axioms for independence relations other than local character. Therefore it is an independence relation if and only if (1) holds. Conditions (1) to (3) are equivalent by Lemma 4.1. If $\mathop{\smile\hskip-0.9em^{|}\ }^{\Omega*}$ is an independence relation, then (4) holds by Theorem 3.6. (4) implies (5) because $B \mathop{\smile\hskip-0.9em^{|}\ }^{\Omega*}_{C} A$ implies $B \mathop{\smile\hskip-0.9em^{|}\ }^{\Omega}_{C} A$. (5) implies (2): We choose $\kappa(A) = (|T| + |A|)^+$ for every set A. Given sets A and B, let $\bar{b}$ be an enumeration of B. By Remark 1.5 there is a subset $C \subseteq B$ such that $|C| < \kappa(A)$ and $\mathrm{tp}(\bar{b}/AC)$ is finitely realised in C. Hence by Remark 1.4 we have $B \mathop{\smile\hskip-0.9em^{|}\ }^{\mathrm{f}}_{C} A$, so $A \mathop{\smile\hskip-0.9em^{|}\ }^{\mathrm{d}}_{C} B$ by (5). The 'moreover' statement is by Theorem 3.6. □

Corollary 4.3. *Suppose $\mathop{\smile\hskip-0.9em^{|}\ }$ is a* symmetric *preindependence relation such that $A \mathop{\smile\hskip-0.9em^{|}\ }^{\Omega*}_{C} B \implies A \mathop{\smile\hskip-0.9em^{|}\ }_{C} B \implies A \mathop{\smile\hskip-0.9em^{|}\ }^{\Omega}_{C} B$. Then $\mathop{\smile\hskip-0.9em^{|}\ }$ is an independence relation (and $\mathop{\smile\hskip-0.9em^{|}\ } = \mathop{\smile\hskip-0.9em^{|}\ }^{\Omega*}$). In particular, if $\mathop{\smile\hskip-0.9em^{|}\ }^{\Omega}$ is symmetric, then $\mathop{\smile\hskip-0.9em^{|}\ }^{\Omega}$ is an independence relation.*

Re-proving some well-known facts on global forking

Theorem 4.4. *A complete theory T is simple if and only if the following, equivalent, conditions are satisfied.*

(1) $\mathop{\smile\hskip-0.9em^{|}\ }^{\mathrm{f}}$ satisfies the local character axiom.
(2) $\mathop{\smile\hskip-0.9em^{|}\ }^{\mathrm{d}}$ satisfies the local character axiom.
(3) $\mathrm{D}_{\varphi,\psi}(\emptyset) < \omega$ for each $(\varphi,\psi) \in \Psi$.
(4) $\mathop{\smile\hskip-0.9em^{|}\ }^{\mathrm{f}}$ is symmetric.
(5) $A \mathop{\smile\hskip-0.9em^{|}\ }^{\mathrm{f}}_{C} B$ implies $B \mathop{\smile\hskip-0.9em^{|}\ }^{\mathrm{d}}_{C} A$.

(6) $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}$ *is symmetric.*
Moreover, in a simple theory $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}$ *is the strongest independence relation.*

Proof. Theorem 5.3 of the previous paper already taught us the 'moreover' statement, and that simplicity is equivalent to (1). It also implies that for a simple theory (6) holds. (6) clearly implies (5). For the equivalence of (1)–(5) recall that $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}*} = \mathop{\smile\!\!\!\!\!\!\mid}^{\Psi*}$ and apply Theorem 4.2. □

Corollary 4.5. *Every reduct of a simple theory is simple.*
T is simple if and only if T^{eq} is simple.

Proof. We first show that a reduct of a simple theory is simple. We already know that a theory T is simple iff $\mathrm{D}_{\varphi,\psi}(\emptyset) < \infty$ holds for every inconsistency pair (φ, ψ). Let T' be a reduct of T. For formulas φ and ψ in the signature of T', (φ, ψ) is an inconsistency pair for T' if and only if it is an inconsistency pair for T. Moreover, $\mathrm{D}_{\varphi,\psi}(\emptyset)$ is the maximal $n < \omega$ such that for the unique $\xi \in \{(\varphi, \psi)\}^n$ the type divpat_p^ξ from Theorem 3.1 is consistent. Since this type is independent of the ambient theory, it does not matter whether we evaluate $\mathrm{D}_{\varphi,\psi}$ in T or in T'. Thus if T is simple then so is T'.

One consequence is that if T^{eq} is simple, then so is its reduct T. We now show the converse. So suppose T is simples and (φ, ψ) is a k-inconsistency pair for T^{eq}. We may assume that as much as possible is coded in a single imaginary variable, so $\varphi \equiv \varphi(x; y)$ and $\psi \equiv \psi(y_{<k})$. The sorts of x and y correspond to definable equivalence relations ϵ_x and ϵ_y. Now consider $\varphi'(\bar{x}; \bar{y}) \equiv \varphi(\bar{x}/\epsilon_x; \bar{y}/\epsilon_y)$ and $\psi'(\bar{y}_{<k}) \equiv \psi(\bar{y}_0/\epsilon_y, \ldots, \bar{y}_{k-1}/\epsilon_y)$. φ' and ψ' can be expressed in T, and (φ', ψ') is a k-inconsistency pair for T. Clearly $\mathrm{D}_{\varphi,\psi}(\emptyset) = \mathrm{D}_{\varphi',\psi'}(\emptyset)$, so T^{eq} also satisfies condition (3) of Theorem 4.4. □

Example 4.6. Let T be a theory in which there is a type that forks over its domain.. Two examples of this phenomenon were given by Saharon Shelah as Exercise III.1.3 in his book [10]. For such a theory it easily follows that $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}$ does not satisfy extension or full existence. Moreover, it follows from Theorem 4.4 that $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}$ does not satisfy local character or symmetry, either.

5 Thorn-Forking, Local Forking and M-Symmetry

Let Ψ_{m} be the subset of Ψ which consists of all formulas $\psi((u\bar{v})_{<k}$ of the form $\bigwedge_{i<j<k}(u_i \neq u_j \wedge \bar{v}_i = \bar{v}_j)$. Of course Ψ_{m} is normal. We observe that if $\psi((u\bar{v})_{<k})$ is a k-inconsistency witness for $\varphi(\bar{x}; u\bar{v})$, then whenever $\varphi(\bar{a}; g\bar{h})$ holds, g must be algebraic over $\bar{a}\bar{h}$.

Proposition 5.1. *Some properties of* $\mathop{\smile\!\!\!\!\!\!\mid}^{\Psi_{\mathrm{m}}}$*:*
(1) $\mathop{\smile\!\!\!\!\!\!\mid}^{\Psi_{\mathrm{m}}}$ *has the following characterisation:*

$$A \mathop{\smile\!\!\!\!\!\!\mid}^{\Psi_{\mathrm{m}}}_{C} B \iff \left(\begin{array}{l} \operatorname{acl}(AD) \cap B \subseteq \operatorname{acl} D \\ \quad \textit{for every set } D \textit{ such that } C \subseteq D \subseteq BC \end{array} \right).$$

(2) $A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{M}}_{C} B$ *implies* $A \mathop{\smile\!\!\!\!\!\!\mid}^{\Psi_{\mathrm{m}}}_{C} B$*.*
(3) $\mathop{\smile\!\!\!\!\!\!\mid}^{þ} = \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{M}*} = \mathop{\smile\!\!\!\!\!\!\mid}^{\Psi_{\mathrm{m}}*}$*.*

Proof. (1) Suppose there is a set D such that $C \subseteq D \subseteq BC$ and $\operatorname{acl}(\bar{a}D) \cap B \not\subseteq \operatorname{acl} D$. So there is an element $e \in \operatorname{acl}(\bar{a}D) \cap B \setminus \operatorname{acl} D$. Let $\alpha(u, \bar{a}, \bar{d})$ with $\bar{d} \in D$ be an algebraic formula realised by e. Then for some $k < \omega$, $\models \varphi(\bar{a}; e\bar{d})$ holds, where $\varphi(\bar{x}; u\bar{v}) \equiv \alpha(u, \bar{x}, \bar{v}) \wedge \exists_{<k} u' \alpha(u', \bar{x}, \bar{v})$. We set $\psi((u\bar{v})_{<k}) \equiv \bigwedge_{i<j<k}(u_i \neq u_j \wedge \bar{v}_i = \bar{v}_j)$. Let $e_{<\omega}$ be a sequence of distinct realisations of the (non-algebraic) type $\operatorname{tp}(e/D)$. Then the sequence $(e_i\bar{d})_{i<\omega}$ witnesses that $\varphi(\bar{x}; e\bar{d})$ ψ-divides over C. Hence $\bar{a} \not\mathop{\smile\!\!\!\!\!\!\mid}^{\Psi_{\mathrm{m}}}_{C} B$.

Conversely, suppose $\bar{a} \not\mathop{\smile\!\!\!\!\!\!\mid}^{\Psi_{\mathrm{m}}}_{C} B$. So there is a formula $\varphi(\bar{x}; e\bar{d}) \in \operatorname{tp}(\bar{a}/BC)$ which ψ-divides over C for some $\psi \in \Psi_{\mathrm{m}}$. Let this be witnessed by $(e_i\bar{d})_{i<\omega}$. We may assume that $e_0\bar{d} \in BC$. Since $e_i\bar{d} \equiv_C e_j\bar{d}$ for $i < j < \omega$, $e_i \equiv_{C\bar{d}} e_j$ holds as well, so the sequence $e_{<\omega}$ witnesses that

$e_0 \notin \operatorname{acl}(C\bar{d})$. In particular, $e_0 \in BC \setminus C$, so $e_0 \in B$. Moreover, $\models \varphi(\bar{a}; e_0\bar{d})$ implies that $e_0 \in \operatorname{acl}(\bar{a}\bar{d}) \subseteq \operatorname{acl}(\bar{a}C\bar{d})$. So choosing $D = C\bar{d}$ we get $e_0 \in \operatorname{acl}(\bar{a}D) \cap B \setminus \operatorname{acl} D$.

(2) Suppose $A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}_C B$. So $\operatorname{acl}(AD) \cap \operatorname{acl}(BD) \subseteq \operatorname{acl} D$ for every set D such that $C \subseteq D \subseteq \operatorname{acl}(BC)$. Hence $\operatorname{acl}(AD) \cap B \subseteq \operatorname{acl} D$ for every set D such that $C \subseteq D \subseteq BC$.

(3) By (1) we have $\mathop{\smile\!\!\!\!\!\mid}^{\Psi_{\mathrm{m}}} = \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{m}}$, where $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{m}}$ is as in Corollary 4.8 of the previous paper. $\square$

It is not true in general that $\mathop{\smile\!\!\!\!\!\mid}^{\Psi_{\mathrm{m}}} = \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$: Let T be the theory of an everywhere infinite forest, as in Example 4.4 of the previous paper. Let a, b_0 and b_1 be nodes such that $\models Rab_0$, $\models Rab_1$ and $\models b_0 \neq b_1$. Then $a \not\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}_{\emptyset} b_0b_1$ because $a \in \operatorname{acl}(a) \cap \operatorname{acl}(b_0b_1) \setminus \operatorname{acl}\emptyset$. But $a \mathop{\smile\!\!\!\!\!\mid}^{\Psi_{\mathrm{m}}}_{\emptyset} b_0b_1$ holds. This can be verified by checking $\operatorname{acl}(aD) \cap \{b_0, b_1\} \subseteq \operatorname{acl} D$ for the four possible values of D such that $\emptyset \subseteq D \subseteq \{b_0, b_1\}$.

Theorem 5.2. *$\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}$ is an independence relation for T if and only if the following, equivalent, conditions are satisfied:*

(1) $\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}$ satisfies the local character axiom.
(2) $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$ satisfies the local character axiom.
(3) $\mathrm{D}_{\varphi,\psi}(\emptyset) < \infty$ for every $(\varphi, \psi) \in \Psi_{\mathrm{m}}$.
(4) $A \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_C B$ implies $B \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_C A$.
(5) $A \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_C B$ implies $B \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}_C A$.
(6) T admits a strict independence relation

Moreover, in a theory T satisfying these conditions, $\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}$ is the weakest strict independence relation.

In particular, T is rosy iff T^{eq} satisfies the equivalent conditions above.

Proof. First note that $\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}} = \mathop{\smile\!\!\!\!\!\mid}^{\Psi_{\mathrm{m}}*}$ by Proposition 5.1. Therefore we can apply Theorem 4.2: (1), (3) and (4) are equivalent, and they hold if and only if $\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}$ is an independence relation. Moreover, they are equivalent to (2') $\mathop{\smile\!\!\!\!\!\mid}^{\Psi_{\mathrm{m}}}$ satisfies the local character axiom, and to (5') $A \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_C B$ implies $B \mathop{\smile\!\!\!\!\!\mid}^{\Psi_{\mathrm{m}}}_C A$. The 'moreover' statement and the equivalence of (6) with the other conditions are by Theorem 4.3 of the previous paper [2]. Finally, (1) $\Rightarrow$ (2) $\Rightarrow$ (2') and (4) $\Rightarrow$ (5) $\Rightarrow$ (5') since $A \mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}_C B \Rightarrow A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}_C B \Rightarrow A \mathop{\smile\!\!\!\!\!\mid}^{\Psi_{\mathrm{m}}}_C B$ by Proposition 5.1 (3) and (4). $\square$

Moreover, by Proposition 5.1 (2), if $\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}$ is an independence relation then we also have a characterisation of $\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}$ in terms of D-ranks by Theorem 3.6. Thus D-ranks can be used in place of the thorn-ranks defined by Alf Onshuus [12].

Corollary 5.3. *Every reduct of a rosy theory is rosy.*

Proof. Use condition (3) as in Corollary 4.5. $\square$

M-Symmetry

From the way we have reached it, the following result looks almost trivial. Yet it was what I considered the most important open problem in my diploma thesis [13]. I had only been able to show it under additional hypotheses such as existence of a strict independence relation (as in simple theories) or strong atomicity of the lattice of algebraically closed sets (as in pregeometric theories).

Corollary 5.4. *The relation $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$ is a (strict) independence relation iff it is symmetric.*

Proof. By Proposition 5.1, $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$ lies between $\mathop{\smile\!\!\!\!\!\mid}^{\Psi_{\mathrm{m}}}$ and $\mathop{\smile\!\!\!\!\!\mid}^{\text{þ}}$. So the statement follows from Corollary 4.3. $\square$

Two algebraically closed sets A and B form a *modular pair* in the lattice of algebraically closed sets, written $\mathrm{M}(A, B)$, if the following rule holds: For any algebraically closed set $C \subseteq B$, $\operatorname{acl}(AC) \cap B = \operatorname{acl}(C(A \cap B))$. We call T *M-symmetric* if the lattice of algebraically closed sets is M-symmetric, i.e., $\mathrm{M}(A, B) \implies \mathrm{M}(B, A)$. The relation to $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$ is as follows.

Remark 5.5. $\mathrm{M}(A,B) \iff A \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{M}}_{A\cap B} B$. $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{M}}$ is symmetric iff the lattice of algebraically closed sets is M-symmetric.

A well-known example due to Wilfrid Hodges for a 1-based stable theory with a reduct which is not 1-based shows that M-symmetry is not preserved under taking reducts either[14].

Example 5.6. (Symmetry of $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{M}}$ is not preserved under reducts)
Let T_0 be the theory, in the signature of one unary function f, which states the following: there is at least one element; for every element b there are infinitely many elements a such that $f(a) = b$; and f has no periodic points. Note that T is complete.

Let T be the theory extending T_0, in the signature consisting of f and a binary relation E, stating that $\forall xy(Exy \leftrightarrow f(x) = y \vee f(y) = x)$. Then $A \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{M}}_{C} B \iff \mathrm{acl}(AC) \cap \mathrm{acl}(BC) = \mathrm{acl}\, C$, so $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{M}}$ is clearly symmetric. (Indeed, the lattice of algebraically closed sets is even modular.) Yet the theory of an everywhere infinite forest (Example 4.4 in the previous paper), for which $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{M}}$ is not symmetric, is a reduct of T.

Question 5.7. Is every stable theory the reduct of a stable M-symmetric theory?

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