Categoricity of countable first-order theories

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Abstract

Lecture notes for part of a model theory course given in Vienna. The proof of Morley's Categoricity Theorem is modelled on that of Ziegler's lecture notes [9], but has been restructured.

Contents

1 Preliminaries

For any L-structure M and any $B \subseteq M$, L_B denotes the language extended by a new constant $\dot{b}$ for every element $b \in B$. We do not usually have a need for the notational distinction between b and $\dot{b}$, so we will just write b for $\dot{b}$.

Let T be an L-theory and $\bar{x}$ a tuple of variables – possibly infinite, although it will usually be finite. Let $L^{\bar{x}}$ denote the set of all L-formulas with free variables contained in the tuple $\bar{x}$.

$$\mathrm{S}^{\bar{x}}(T) = \{p(\bar{x}) \subseteq L^{\bar{x}} \mid p \text{ maximal consistent}\}$$

denotes the set of complete types of T. If $M \models T$, then

$$\mathrm{S}^{\bar{x}}(M) = \{p(\bar{x}) \subseteq L_M^{\bar{x}} \mid p \text{ maximal consistent with the } L_M\text{-theory of } M\}$$

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denotes the set of complete types over M, and if $B \subseteq M$, then

$$\mathrm{S}_M^{\bar{x}}(B) = \{p(\bar{x}) \subseteq L_B^{\bar{x}} \mid p \text{ maximal consistent with the } L_B\text{-theory of } M\}$$

denotes the set of complete types over B (regarded as a subset of M). Normally we will only be dealing with extensions that are elementary, and no confusion can arise when we drop the index M from notation.

For any formula $\varphi(\bar{x}) \in L^{\bar{x}}$, we define the subset

$$[\varphi(\bar{x})] = \{p(\bar{x}) \in \mathrm{S}^{\bar{x}}(T) \mid \varphi(\bar{x}) \in p(\bar{x})\} \subseteq \mathrm{S}^{\bar{x}}(T).$$

We equip $\mathrm{S}^{\bar{x}}(T)$ with a topology by stipulating that the system of these sets $[\varphi(\bar{x})]$ (which is clearly closed under intersection) forms a basis of open sets. Since the complement $[\neg\varphi(\bar{x})]$ of every basic open set $[\varphi(\bar{x})]$ is again open, the *type space*[1] $\mathrm{S}^{\bar{x}}(T)$ has a basis of clopen sets, i.e. it is totally disconnected. We use the same (ambiguous) notation $[\varphi(\bar{x})]$ to define the basic open sets of the other type spaces $\mathrm{S}^{\bar{x}}(M)$ and $\mathrm{S}_M^{\bar{x}}(B)$.

We will not be doing anything non-trivial with this topology, but it is good to know about it because it motivates some of the terminology. E.g. the compactness theorem of first-order logic says precisely that every such type space is compact. Also a type $p(\bar{x}) \in \mathrm{S}_M^{\bar{x}}(B)$ is called *isolated* by a formula $\varphi(\bar{x}) \in p(\bar{x})$, if $\varphi(\bar{x}) \vdash p(\bar{x})$. This is because then $\{p(\bar{x})\} = [\varphi(\bar{x})]$ is an open set, i.e. $p(\bar{x})$ is an isolated point in the type space $\mathrm{S}_M^{\bar{x}}(B)$.

For a formula $\varphi(x)$ with one free variable and a structure M we write $\varphi^M = \{a \in M \mid M \models \varphi(a)\}$. Similarly for a family of formulas or a type $p(x)$. We will not need the natural generalisation to more variables.

2 Categoricity

Definition 1. A theory T is said to be *categorical* if any two models of T are isomorphic.

Remark 2. If T is a categorical first-order theory, then by the Löwenheim-Skolem Theorem the unique model of T must be finite.

Definition 3. Let κ be a cardinal. A theory T is said to be *κ-categorical* if any two models of T of size κ are isomorphic.

We will concentrate on the case when T is a countable first-order theory. A characterisation of $\aleph_0$-categoricity was published in 1959 indepently by Engeler, Ryll-Nardzewski and Svenonius [2, 7, 8]

Theorem 4 (Ryll-Nardzewski, Engeler, Svenonius)**.** *Let T be a countable complete first-order theory with infinite models. The following are equivalent:*

1. *Any two countable models of T are isomorphic.*
2. *T has a countable model which, for any $n < \omega$, realises only finitely many complete n-types (over $\emptyset$).*
3. *For any finite tuple $\bar{x}$ of variables the type space $\mathrm{S}^{\bar{x}}(T)$ is finite.*
4. *For any finite tuple $\bar{x}$ of variables there are only finitely many formulas with free variables contained in $\bar{x}$, up to equivalence modulo T.*
5. *For any finite tuple $\bar{x}$ of variables, all types $p(\bar{x}) \in \mathrm{S}^{\bar{x}}(T)$ are isolated.*

Proof. For now, see Hodges [4], Theorem 7.3.1, for the straightforward proof.[2] □

Uncountable categoricity, i.e. κ-categoricity for an uncountable cardinal κ, is a completely different matter, with a very different flavour. In his 1962 PhD thesis, Morley proved Łoś's Conjecture [5, 6]:

Theorem 5 (Morley)**.** *Let T be a countable complete first-order theory with infinite models. If T is κ-categorical for some uncountable κ, then T is κ-categorical for every uncountable κ.*

[1] Type spaces are also known as Stone spaces

[2] A later version will have a full proof and possibly a different formulation of the theorem.

We will have to do quite a bit of work to prove this theorem. Some examples of uncountably categorical theories:

- The theory of an infinite set in the empty signature.
- The theory of infinite-dimensional F-vector spaces for a fixed finite field F, and the theory of F-vector spaces for a fixed countable field F.
- The theory of algebraically closed fields of any fixed characteristic.[3]

A conjecture attributed to Zilber says that in some sense these are the only examples. Hrushovski developed a sophisticated generalisation of Fraïssé's Theorem and used it to contrive uncountably many counterexamples to Zilber's conjecture.

3 Matroids

In this section we are not actually dealing with model theory at all.

Definition 6. A *closure operator* on a set M is a monotone, inflationary, idempotent function $\mathrm{cl}\colon \mathcal{P}(M) \to \mathcal{P}(M)$, i.e. it satisfies the following axioms:

- $X \subseteq Y \implies \mathrm{cl}\, X \subseteq \mathrm{cl}\, Y$.
- $X \subseteq \mathrm{cl}\, X$.
- $\mathrm{cl}(\mathrm{cl}\, X) = \mathrm{cl}\, X$.

It is called *finitary* if it also satisfies the *finite character axiom*:

- $\mathrm{cl}\, X = \bigcup_{X_0 \subseteq X,\ X_0 \text{ finite}} \mathrm{cl}\, X_0$.

A set is called closed with respect to a given closure operator cl if $X = \mathrm{cl}\, X$.

Remark 7. If cl is a closure operator on a set M, then its restriction cl_B to any subset $B \subseteq M$, defined by $\mathrm{cl}_B(X) = \mathrm{cl}(X) \cap B$, is a closure operator on the subset.

The abstract study of closure operators originated shortly after the First World War in Poland with Kuratowski and Tarski. A topological closure operator is one that satisfies $\mathrm{cl}(X \cup Y) = \mathrm{cl}\, X \cup \mathrm{cl}\, Y$ and corresponds exactly to a topology on the underlying set. Finitary closure operators, on the other hand, naturally arise in algebra and logic.

Definition 8. A *(finitary[4]) matroid* is a pair $\langle M, \mathrm{cl}\rangle$ such that cl is a finitary closure operator on M which also satisfies the *exchange principle*:

- $a \in \mathrm{cl}(X \cup \{b\}) \setminus \mathrm{cl}\, X$ implies $b \in \mathrm{cl}(X \cup \{a\})$.

Remark 9. If $\langle M, \mathrm{cl}\rangle$ is a matroid and $B \subseteq M$, then $\langle B, \mathrm{cl}_B\rangle$ is again a matroid.

The best known example of a matroid is a vector space with linear hull as the closure operator. By the remark, if we consider a matrix as a set of column vectors, we also get a natural matroid structure. This is the source of the term 'matroid'. Another example also related to uncountably categorical theories is an algebraically closed field with $\mathrm{cl}\, X$ defined as the algebraic closure of the field generated by X. Graphs give rise to matroids with a very different flavour: Let E be the set of edges of a graph. For any $X \subseteq E$ let $\mathrm{cl}\, X$ consist of those edges which connect two nodes that are connected by a path in X. Then $\langle E, \mathrm{cl}\rangle$ is a matroid.

Definition 10. In a matroid $\langle M, \mathrm{cl}\rangle$, an *independent system* is a system $\langle a_i\rangle_{i\in I}$ of elements $a_i \in M$ such that for all $i \in I$ we have $a_i \notin \mathrm{cl}\{a_j \mid i \neq j \in I\}$. A generating system is a system $\langle a_i\rangle_{i\in I}$ such that $\mathrm{cl}\{a_i \mid i \in I\} = M$. A *basis* of a matroid is an independent generating system.

[3] A later version will look at all these examples in detail.

[4] The modifier 'finitary' makes it explicit that the matroids that we are considering are neither finite nor of the most general infinite type. To date, most matroid theory literature silently assumes all matroids to be finite, and until recently there was no generally accepted notion of an infinite matroid. This has changed with a recent paper by Bruhn, Diestel, Kriesell and Wollan [1]. Finitary matroids are a special case of the general infinite matroids in the sense of that paper. The main advantage of what is likely to become a standard definition of infinite matroids is that it is closed under matroid duality. In model theory, matroids are usually known as pregeometries, but Rota's proposal to rename matroids in this way has ultimately failed and the word is considered obsolete in matroid theory.

Lemma 11. *In a matroid $\langle M, \mathrm{cl}\rangle$, if $\langle a_i\rangle_{j\neq i\in I}$ is an independent system and $a_j \notin \mathrm{cl}\{a_i \mid j \neq i \in I\}$, then $\langle a_i\rangle_{i\in I}$ is an independent system.*

Theorem 12. *In a matroid, a basis is the same thing as a maximal independent system, or a minimal generating system. Every matroid has a basis, and any two bases have the same size.*

4 Algebraicity and strong minimality

Definition 13. An L-formula $\varphi(x)$ is called *algebraic* if in every L-structure M it has only finitely many solutions. It is called algebraic modulo an L-theory T if in every model $M \models T$ it has only finitely many solutions. More generally, a possibly incomplete L-type $p(x)$ is called algebraic if in every L-structure it has only finitely many realisations, and similarly modulo a theory.

Note that by compactness an algebraic type over a subset $B \subseteq M$ of an L-structure implies an algebraic formula $\varphi(x, \bar{b})$ also with parameters in B, and an algebraic complete type contains an algebraic formula.

Usually one just says 'algebraic' for 'algebraic modulo T' if T is understood from the context. If T is complete, the distinction between absolute algebraicity and algebraicity often does not matter:

Remark 14. If $\varphi(x)$ has only finitely many solutions in M, then for some natural number $n < \omega$ the formula

$$\varphi(x) \wedge \exists^{\leq n} x' \varphi(x')$$

is equivalent to $\varphi(x)$ modulo the complete theory of M and is algebraic.

Remark 15. Let T be a complete first-order theory with infinite models. Any non-algebraic incomplete type has a non-algebraic completion.

Definition 16. A tuple $\bar{a} \in M$ is called *algebraic over* a set $B \subseteq M$ if its type $\mathrm{tp}_M(\bar{a}/B)$ is algebraic. The set of all elements of M that are algebraic over a set $B \subseteq M$ is denoted by $\mathrm{acl}(B)$.

Remark 17. $\langle M, \mathrm{acl}\rangle$ is a finitary closure operator.

The remainder of this section is not, strictly speaking, needed for the proof of Morley's Theorem. But it shows in the simplest case how we can get from a simple model theoretic condition (satisfied by vector spaces and algebraically closed fields) to uncountable categoricity. In the proof of Morley's Theorem we will need the slightly more general definition of a strongly minimal formula (see Definition 37).

Definition 18. A complete first-order theory T with infinite models is *strongly minimal* if for all models $M \models T$ and all formulas $\varphi(x, \bar{m})$ with parameters $\bar{m} \in M$, either $\varphi(x, \bar{m})$ or $\neg\varphi(x, \bar{m})$ is algebraic.

Remark 19. Every strongly minimal theory is totally transcendental.

Proposition 20. *Let T be a complete first-order theory with infinite models. The following are equivalent:*

1. *T is strongly minimal.*
2. *For every model $M \models T$ and every single variable x, the type space $\mathrm{S}^x(M)$ contains precisely one non-algebraic type.*
3. *For every model $M \models T$, every subset $B \subseteq M$ of its domain and every finite tuple $\bar{x} = x_0 \dots x_{n-1}$ of distinct variables the type space $\mathrm{S}^{\bar{x}}_M(B)$ contains precisely one type $p(\bar{x})$ such that for any realisation $\bar{a} \models p$ in an elementary extension of M, a_i is not algebraic over $B \cup \{a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_{n-1}\}$, for all $i < n$.*

Moreover, if T is strongly minimal, then acl *satisfies the exchange property, so* (M, acl) *is a matroid for any model $M \models T$.*

Proof. 1 implies 2: Since M has proper elementary extensions and algebraic types are realised only in M itself, $\mathrm{S}^x(M)$ contains at least one non-algebraic type $p(x)$. If T is totally transcendental, it is uniquely determined as follows: For any formula $\varphi(x, \bar{m})$ with parameters from

M, either $\varphi(x, \bar{m})$ or $\neg\varphi(x, \bar{m})$ is algebraic. Since $p(x)$ is complete and not algebraic, the one that is not algebraic must be in p.

2 implies 1: Suppose for some model $M \models T$ there is a formula $\varphi(x, \bar{m})$ with parameters $\bar{m} \in M$ which witnesses that T is not strongly minimal, i.e. neither $\varphi(x, \bar{m})$ nor $\neg\varphi(x, \bar{m})$ is algebraic. By Remark 15, each of the two mutually inconsistent formulas can be extended to a complete type over M.

2 implies the 'moreover' part: We will use Remark 15 freely. Let $M \models T$. We need to show that $a \in \operatorname{acl}(C \cup \{b\}) \setminus \operatorname{acl} C$ implies $b \in \operatorname{acl}(C \cup \{a\})$ inside M. Equivalently, we can show that $b \notin \operatorname{acl}(C \cup \{a\})$ and $a \notin \operatorname{acl} C$ implies $a \notin \operatorname{acl}(C \cup \{b\})$. Note that by two applications of 2, the type $p(xy) = \operatorname{tp}_M(ab/C)$ is uniquely determined by the assumption. Replacing M by an elementary extension if necessary, we may assume that M contains an infinite set $A = \{a_0, a_1, \ldots\}$ of elements that are not algebraic over C. Similarly, we may assume that the unique non-algebraic type over $A \cup C$ is realised by an element $b' \in M$. Note that for any $a_i \in A$ we have $b' \notin \operatorname{acl}(C \cup \{a'\})$ and $a' \notin \operatorname{acl} C$, so $\operatorname{tp}_M(a_i b'/C) = p(xy)$. Hence all elements $a_i \in A$ have the same type $\operatorname{tp}(a_i/C \cup \{b'\}) = p(x, b')$ over $C \cup \{b'\}$, which is therefore not algebraic. It follows that $a_0 \notin \operatorname{acl}(C \cup \{b'\})$. Since $\operatorname{tp}(a_0, b'/C) = p(x, y) = \operatorname{tp}(a, b/C)$, we can conclude that $a \notin \operatorname{acl}(C \cup \{b\})$.

2 and the 'moreover' part imply 3: By 2 and a straightforward induction, for each $n < \omega$ there is a unique type $p(x_0 \ldots x_{n-1})$ such that for any realisation $a_0 \ldots a_{n-1} \models p$ in an elementary extension $N \succeq M$, a_i is not algebraic over $B \cup \{a_0, \ldots, a_{i-1}\}$, for all $i < n$. By repeated use of Lemma 11, $\langle a_0, \ldots, a_{n-1} \rangle$ is a basis of the matroid (N, acl), and so $a_i \notin B \cup \{a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n-1}\}$ for all $i < n$.

3 implies 2: Trivial. □

Theorem 21. *Let T be a countable complete first order theory. If T is strongly minimal, then T is κ-categorical for all $\kappa > \aleph_0$.*

Proof. Let $M \models T$ be a model of size $|M| = \kappa$. By Proposition 20, (M, acl) is a matroid, so let $B \subseteq M$ be a basis. Since $\kappa > |T|$, it is easy to check that $|B| = |M|$.

If $M' \models T$ is another model of the same size and $B' \subseteq M'$ is a basis of M', then also $|B'| = \kappa$. Let $f \colon B \to B'$ be a bijection. By 3 of Proposition 20, f is an elementary map. Since algebraic types are isolated, it can be extended to an elementary map from $M = \operatorname{acl} B$ to M', and it is easy to see that any such extended map must be surjective. Thus M and M' are isomorphic. □

5 Proof outline for Morley's Theorem

To get an idea of the proof of Morley's Theorem on a very high level, we need two fundamental definitions.

Definition 22. A complete first-order theory T with infinite models is called *totally transcendental* if in no model $M \models T$ can we find a binary tree $\langle \varphi_\eta(x) \rangle_{\eta \in 2^{<\omega}}$ of formulas with parameters such that for every tree node $\eta \in 2^{<\omega}$ the branching

$$\{\varphi_{\eta^\frown 0}(x), \varphi_{\eta^\frown 1}(x)\},$$

in η is inconsistent, but for every branch $\eta \in 2^\omega$ the corresponding set of formulas

$$\{\varphi_{\eta \restriction n}(x) \mid n < \omega\},$$

is consistent.

Remark 23. We get an equivalent definition if we also require for such a tree that for each $\eta \in 2^{<\omega}$ the formula $\varphi_{\eta^\frown 0}(x)$ is of the form $\varphi_\eta(x) \wedge \neg\psi(x)$ and the formula $\varphi_{\eta^\frown 1}(x)$ is of the form $\varphi_\eta(x) \wedge \psi(x)$.

Definition 24. $M \prec N$ is said to be a *Vaught pair* for the formula $\varphi(x)$ (which may have parameters in M), if $M \neq N$ and $\varphi(x)$ is not algebraic (i.e. $\varphi(x)$ has infinitely many solutions), but nevertheless $\varphi^M = \varphi^N$.

Throughout the proof (i.e. from here to the end of Section 9) we will fix a complete first-order theory T with infinite models and a cardinal $\kappa > |T|$. Any further assumptions on T will be made explicit. Morley's Theorem is an obvious consequence of the following three lemmas.

Lemma A. *If T is countable and κ-categorical, then T is totally transcendental.*

Lemma B. *If T is countable, κ-categorical and totally transcendental, then T does not admit Vaught pairs.*

Lemma C. *If T is totally transcendental and does not admit Vaught pairs, then T is κ-categorical.*

6 Proof of Lemma A

Recall that we have fixed a complete first-order theory T with infinite models and a cardinal $\kappa \geq |T|$. Our aim in this section is to prove:

Lemma A. *If T is countable and κ-categorical, then T is totally transcendental.*

The following result says that there is always a model of size κ that is 'modest'. Afterwards we will show that if T is not totally transcendental, then there is also a model of size κ that is 'immodest', and so T cannot be κ-categorical.[5]

Theorem 25. *There is a model $M \models T$ of size $|M| = \kappa$ which, for every subset $B \subseteq M$ of size $|B| \leq |T|$, realises at most $|T|$ 1-types over B.*

Proof. Let T^* be a consistent expansion of T to a signature which has Skolem functions, i.e. for every formula $\varphi(x, \bar{y})$ over the extended signature there is a function symbol f_φ and an axiom in T^* which says that $\forall \bar{y}(\exists x \varphi(x, \bar{y}) \leftrightarrow \varphi(f_\varphi(\bar{y}), \bar{y}))$. Let $\langle c_i \rangle_{i<\kappa}$ be a sequence of new constants. We add to T^* axioms that say $c_i \neq c_j$ for $i \neq j$ and that for all $n < \omega$ and any two ascending n-tuples $i_0 < i_1 < \ldots < i_{n-1} < \kappa$ and $j_0 < j_1 < \ldots < j_{n-1} < \kappa$, we have

$$\mathrm{tp}(c_{i_0}, c_{i_1}, \ldots, c_{i_{n-1}}) = \mathrm{tp}(c_{i_0}, c_{j_1}, \ldots, c_{j_{n-1}}),$$

taking the types in the sense of L^*. This expanded theory T^{**} is consistent by Ramsey's Theorem. Take any model, then take the substructure M generated by its constants. Clearly $|M| = \kappa$. Due to the Skolem functions, M is an elementary substructure, so again a model of T^{**}.

Now for any subset $B \subset M$ of size $|B| \leq |T|$, we can find a subsequence $I \subseteq \kappa$ of size $|I| \leq |T|$ such that B is contained in the L^*-substructure generated by $C = \{c_i \mid i \in I\}$. It is sufficient to show that M realises at most $|T|$ 1-types over C in the sense of T^*.

Recalling the construction of M, we see that for any element $a \in M$ we can write $a = t(c_{i_0}, c_{i_1}, \ldots, c_{i_{n-1}})$ for an L^*-term t. Therefore $\mathrm{tp}(a/C)$ is determined by t and the type $\mathrm{tp}(c_{i_0} c_{i_1} \ldots c_{i_{n-1}}/C)$. But by indiscernibility of the sequence $\langle c_i \rangle_{i<\kappa}$ the latter only depends on the number n and the n cuts in the well-order I that are defined by the ordinals $j_0 < j_1 < \ldots < j_{n-1} < \kappa$. As there are at most $|T|$ such cuts, at most $|T|$ different types $\mathrm{tp}(a/C)$ can occur.[6] □

Proposition 26. *If the type spaces $\mathrm{S}^x(M)$ are countable for all countable models $M \models T$, then T is totally transcendental.*[7]

Proof. Take a tree in some model $M \models T$, witnessing that T is not totally transcendental. Since the tree has only countably many nodes, the parameters that appear in it fit into a countable set B. By Löwenheim-Skolem, M has an elementary substructure M' such that $B \subseteq M'$. Each node in the tree gives rise to an incomplete type over B, hence over M', which can be extended to a complete type in $\mathrm{S}^x(M')$. The types we get in this way are pairwise distinct because any two branches are inconsistent due to the inconsistent branching. There are $2^{\aleph_0}$ branches in the tree, so $|\mathrm{S}^x(M')| \geq 2^{\aleph_0} > \aleph_0$. □

[5] I have only seen this terminology in a recent book by Hinman [3], but I like it.

[6] Models constructed around indiscernibles as in this proof are known as Ehrenfeucht-Mostowski models.

[7] Theories with the property assumed in the proposition are called ω-stable. For countable theories the converse to the proposition is also true, but we will not need it.

Proof of Lemma A. Suppose T is not totally transcendental. Then there is a countable model $M \models T$ whose type space $\mathrm{S}^x(M)$ is uncountable. By compactness we can find an elementary extension N which realises at least $\aleph_1$ 1-types over M and such that $|N| \geq \kappa$. By Löwenheim-Skolem we can assume $|N| = \kappa$. The model N we have constructed is not 'modest', so it cannot be isomorphic to the model provided by the theorem. Hence T is not κ-categorical. □

7 Prime models

Recall that we have fixed a complete first-order theory T with infinite models and a cardinal $\kappa \geq |T|$. In this section we will prove a result that will be used in the proofs of both Lemma B and Lemma C.

Definition 27. $M \models T$ is said to be a *prime model* for T if for every model $N \models T$ there is an elementary embedding $f\colon M \xrightarrow{\preceq} N$, i.e. M is isomorphic to an elementary substructure of N.

Expressed topologically, the following result says that in any type space for a totally transcendental theory the isolated points are dense.

Lemma 28. *Suppose T is totally transcendental, $B \subseteq M \models T$, and $\varphi(x)$ is a consistent formula with parameters in B. Then there is a complete type $p(x) \in \mathrm{S}^x_M(B)$ and a formula $\chi(x) \in p(x)$ which isolates $p(x)$.*

Proof. Let us temporarily call a formula $\varphi(x)$ perfect if it is such that the conclusion of the lemma does not hold. It is easy to see that for any perfect formula $\varphi(x)$ there is a formula $\psi(x)$, also with parameters in M, such that the two formulas $\varphi(x) \wedge \psi(x)$ and $\varphi(x) \wedge \neg\psi(x)$ are both consistent. Both must again be perfect. Using this, we can build a tree that contradicts total transcendentality. □

Proposition 29.

1. *If $\mathrm{tp}(\bar{a}\bar{b})$ is isolated, then $\mathrm{tp}(\bar{a})$ is isolated.*
2. *If $\mathrm{tp}(\bar{a}/\bar{b})$ and $\mathrm{tp}(\bar{b})$ are both isolated, then $\mathrm{tp}(\bar{a}\bar{b})$ is also isolated.*
3. *Suppose $\bar{a}, \bar{b}, \bar{a}', \bar{b}' \in M$ are such that $\mathrm{tp}(\bar{b}') = \mathrm{tp}(\bar{b})$, and $\mathrm{tp}(\bar{a}/\bar{b})$ is isolated by a formula $\chi(\bar{x}, \bar{b})$. If $M \models \chi(\bar{a}', \bar{b}')$, then $\mathrm{tp}(\bar{a}\bar{b}) = \mathrm{tp}(\bar{a}'\bar{b}')$.*

Proof. 1 is easy. For 2 suppose $\chi(\bar{x}, \bar{b})$ isolates $\mathrm{tp}(\bar{a}/\bar{b})$ and $\psi(\bar{y})$ isolates $\mathrm{tp}(\bar{b})$. Then $\mathrm{tp}(\bar{a}\bar{b})$ is isolated by $\chi(\bar{x}, \bar{y}) \wedge \psi(\bar{y})$. For 3 let $p(\bar{x}\bar{y}) = \mathrm{tp}(\bar{a}\bar{b})$ and $q(\bar{y}) = \mathrm{tp}(\bar{b}) = \mathrm{tp}(\bar{b}')$. Since $p(\bar{x}, \bar{b})$ is isolated by $\chi(\bar{x}, \bar{b})$, we have

$$p(\bar{x}\bar{y}) \quad \supseteq \quad \{\chi(\bar{x}, \bar{y})\} \cup q(\bar{y}) \quad \vdash \quad p(\bar{x}\bar{y}).$$

Since $a'b'$ realises $\{\chi(\bar{x}, \bar{y})\} \cup q(\bar{y})$, $a'b'$ also realises $p(\bar{x}, \bar{y})$. □

Theorem 30. *Suppose T is totally transcendental.*

1. *T has a prime model.*
2. *If $M \models T$ is a prime model of T and $\bar{a} \in M$, then $\mathrm{tp}(\bar{a})$ is isolated.*[8]

Proof. 1. Let T be totally transcendental and $M \models T$. Consider a sequence $\langle a_i \rangle_{i<\alpha}$ without repetitions and such that each type $\mathrm{tp}(a_i/a_{<i})$ is isolated. (We are using the abbreviation $a_{<i} = \{a_j \mid j < i\}$.) Since the empty sequence has these properties, such sequences do exist.

If $a_{<\alpha}$ is not an elementary substructure of M, then there is a formula $\varphi(x)$ with parameters in $a_{<\alpha}$ such that $M \models \exists x \varphi(x)$ but $M \models \neg\varphi(a_i)$ for all $i < \alpha$. By Lemma 28 there is a consistent formula $\chi_\alpha(x)$, also with parameters in $a_{<\alpha}$, such that $\chi_\alpha(x) \vdash \varphi(x)$ and moreover if we take any $a_\alpha \in M$ such that $M \models \varphi(a_\alpha)$, then $\mathrm{tp}(a_\alpha/a_{<\alpha})$ is isolated by $\chi_\alpha(x)$. Thus by fixing such an element a_α we can extend the original sequence to a new sequence $\langle a_i \rangle_{i<\alpha+1}$. Also if λ is a limit ordinal and we have a coherent system of such sequences $\langle a_i \rangle_{i<\alpha}$ for all $\alpha < \lambda$, then $\langle a_i \rangle_{i<\lambda}$ also has the required properties, allowing us extend the sequence transfinitely.

[8] Models with this property are called atomic. Prime models need not be atomic in general, and atomic models need not be prime.

Since the elements a_i are pairwise different, this process cannot continue forever and we must run into an elementary substructure $a_{<\alpha}$ of M at some point. It only remains to show that $a_{<\alpha}$ can be elementarily embedded into every model $N \models T$. But an easy induction shows that for every such sequence in M we can find one in N that has precisely the same type.

2. By induction on α we can show for sequences $\langle a_i \rangle_{i<\alpha}$ as in 1 that for any $i_0 < i_1 < \ldots < i_k < \alpha$ the type $\mathrm{tp}(a_{i_0} a_{i_1} \ldots a_{i_k})$ is isolated. For this we use Proposition 29. It follows that the prime model $M = a_{<\alpha}$ constructed in 1 has the property described. If M' is another prime model, then there is an elementary embedding of M' into M, so we can assume $M' \preceq M$ and the claim follows easily. □

8 Proof of Lemma B

Recall that we have fixed a complete first-order theory T with infinite models and a cardinal $\kappa \geq |T|$. Our aim in this section is to prove:

Lemma B. *If T is countable, κ-categorical and totally transcendental, then T does not admit Vaught pairs.*

Initially we will observe that there is always a model of size κ whose infinite definable sets all have size κ. Then we will have to work much harder to find a model of the same size for which this is not the case. We will first show (assuming countability) that if T admits any Vaught pairs at all, then T also admits Vaught pairs $M \prec N$ in which $|M| = \aleph_0$ and $|N| = \aleph_1$. Afterwards we will also assume total transcendence to inflate N to size $|N| = \kappa$ (using a theorem due to Lachlan).

Proposition 31. *There is a model $M \models T$ of size $|M| = \kappa$ such that for all non-algebraic formulas $\varphi(x)$ with parameters in M we have $\left|\varphi^M\right| = \kappa$. I.e. each φ^M is either finite or of size κ.*

Proof. Let $M_0 \models T$ be a model of size $|M_0| = |T|$. Given a model M_α, let $M_{\alpha+1}$ be an elementary extension such that for each of the $|M_\alpha|$-many non-algebraic formulas $\varphi(x)$ with parameters in M_α there is at least one new solution in $M_{\alpha+1}$. Moreover, using Löwenheim-Skolem we make sure that $|M_{\alpha+1}| = |M_\alpha|$. Taking units at limit steps, we get a transfinite sequence. M_κ is as required. □

Definition 32. A model $M \models T$ is called *ω-homogeneous* if every elementary bijection $f\colon A \to B$ between finite subsets of $A, B \subset M$ can be extended to an automorphism of M.

Proposition 33. *Suppose T is countable.*

1. *$M \models T$ is ω-homogeneous if and only if for every finite subset $B \subset M$, every elementary map $f\colon B \to M$ and every type $p(x) \in \mathrm{S}^x_M(B)$ that is realised in M, the image $f(p)$ of p is also realised in M.*
2. *The union of an elementary chain of ω-homogeneous models is ω-homogeneous.*
3. *Two ω-homogeneous countable models are isomorphic if and only if they realise the same n-types, for all $n < \omega$.*
4. *Every countable model of T has a countable elementary extension which is ω-homogeneous.*

Proof. 1, 2 and 3 are by simple back-and-forth arguments. For the proof of 4, let $M_0 \models T$ be countable. Given any countable M_i, choose $M_{i+1} \succ M_i$ such that for every finite subset $B \subset M_i$, every elementary map $f\colon B \to M_i$ and every type $p(x) \in \mathrm{S}^x_{M_i}(B)$ that is realised in M_i, the image $f(p)$ of p is realised in M_{i+1}. Since only countably many types need realising, we can choose M_{i+1} countable by Löwenheim-Skolem. $\bigcup_{i<\omega} M_i$ is ω-homogeneous by 1. □

Theorem 34 (Vaught's Two-Cardinal Theorem)**.** *If T is countable and admits a Vaught pair, then T also admits a Vaught pair $M \prec N$ for $\varphi(x)$ which has $|M| = \aleph_0$ and $|N| = \aleph_1$.*

Proof. We may assume that T admits a Vaught pair for a formula $\varphi(x)$ is without parameters. (Otherwise add the parameters to the language. Any Vaught pair for the extended theory will be a Vaught pair for the original theory.)

First we note that T admits a Vaught pair for $\varphi(x)$ in which *both* models are countable. To see this, expand the language by a predicate P, and consider the theory T' of Vaught pairs $P^N \prec N \models T$ for $\varphi(x)$. By Löwenheim-Skolem it has a countable model $N_0 \models T'$. Let $M_0 = P^{N_0}$. Then $M_0 \prec M_1$ is a Vaught pair for $\varphi(x)$ as claimed.

Next we want a countable Vaught pair $M \prec N$ for $\varphi(x)$ in which $M \cong N$. By 1 of Proposition 33 it suffices to ensure that M and N are ω-homogeneous, and M realises all n-types that are realised by N. Our method is a refinement of the proof of 4 of Proposition 33.

Given any countable model $N_i' \models T'$, corresponding to a countable Vaught pair $M_i \prec N_i$, we choose a countable elementary extension $N_{i+1}' \succ N_i'$ such that

- every n-type in the language of T that is realised in N_i is also realised in $M_{i+1} = P^{N_{i+1}}$; and
- for every finite subset $B \subset N_i'$, every elementary map $f\colon B \to N_i'$ and every type $p(x) \in \mathrm{S}^x_{N_i'}(B)$ that is realised in N_i', the image $f(p)$ of p is realised in N_{i+1}'.

Taking $N' = \bigcup_{i<\omega} N_i'$ and $M = P^{N'} = \bigcup_{i<\omega} M_i$, we see that M and N are both homogeneous and realise the same n-types for all n.

Now we start again with a countable Vaught pair $M_0 \prec M_1$ for $\varphi(x)$ in which $M_0 \cong M_1$. Using the isomorphism, we can find M_2 such that $M_1 \prec M_2$ is a Vaught pair for φ and $M_2 \cong M_1$, etc., taking unions at limit stages. Then $M_0 \prec M_{\aleph_1}$ is a Vaught pair for φ with $|M_0| = \aleph_0$ and $|M_{\aleph_1}| = \aleph_1$, as desired. □

Theorem 35 (Lachlan)**.** *Suppose T is countable and totally transcendental, and $M \models T$ is an uncountable model of T. Then there is a proper elementary extension $N \succ M$ such that every countable set of formulas with parameters in M that is realised in N is also realised in M.*

Proof. First note that there is a formula $\mu(y)$ with parameters in M such that μ^M is uncountable but for all formulas $\psi(y)$ with parameters in M either $(\mu \wedge \psi)^M$ or $(\mu \wedge \neg\psi)^M$ is (at most) countable. (If there were no such formula, then every uncountable formula could be split into two uncountable formulas, and we would be able to build a tree that contradicts total transcendentality.) Choose such a formula $\mu(x)$, and let

$$q(y) = \{\psi(y) \mid (\mu \wedge \psi)^M \text{ is uncountable}\}.$$

By choice of μ, $q(y)$ is a complete type over M. Since $(y \neq b) \in q(y)$ for all $b \in M$, this type is not realised in M. Therefore we can find $b \in N \succ M$ such that $b \models p$. By Theorem 30 we may assume that N is prime over Mb.

Given any element $a \in N$ and countable $C \subset M$, we will find an element $a' \in M$ such that $\mathrm{tp}(a'/C) = \mathrm{tp}(a/C)$. This will conclude the proof.

Since $\mathrm{tp}(a/Mb)$ is isolated by Therem 30, there is a formula $\chi(x, y)$ with parameters in M, such that $\chi(x, b)$ isolates $\mathrm{tp}(a/Mb)$. Let C^* be C together with the parameters that appear in the formulas $\mu(y)$ and $\chi(x, y)$. Note that C^* is still countable, $C \subseteq C^* \subset M$, and

$$\chi(x, b) \text{ isolates } \mathrm{tp}(a/C^*b).$$

We next find an element $b' \in M$ such that

$$\mathrm{tp}(b'/C^*) = \mathrm{tp}(b/C^*).$$

Consider $q_0(y) = \mathrm{tp}(b/C^*) \subset q(y)$. By definition of $q(y)$, each of the countably many formulas $\psi(y) \in q_0(y)$ holds for all but countably many elements of M. It follows that $M \setminus q_0^M$ is a countable union of countable sets, hence countable. Therefore q_0^M is uncountable, hence not empty. Any $b' \in q_0^M$ will do.

Finally, we can find an element $a' \in M$ such that

$$M \models \chi(a', b').$$

Such an element exists because $N \models \exists x \chi(x, b)$, hence $N \models \exists x \chi(x, b')$, hence $M \models \exists x \chi(x, b')$ as the formula is over M.

Using Remark 29, applied to N and the language extended by constants for the elements of C^*, we see that $\mathrm{tp}(\bar{a}'\bar{b}'/C^*) = \mathrm{tp}(\bar{a}\bar{b}/C^*)$. It follows that $\mathrm{tp}(a'/C) = \mathrm{tp}(a/C)$, as required. □

Corollary 36. *Suppose T is countable, totally transcendental, and admits a Vaught pair for some formula $\varphi(x)$. Then there is a Vaught pair $M \prec N$ for $\varphi(x)$ such that $|M| = \aleph_0$ and $|N| = \kappa$.*

Proof. By Theorem 34 we can get a Vaught pair $M \prec N$ for $\varphi(x)$ such that $|M| = \aleph_0$ and $|N| = \aleph_1$. Consider the partial type $p(x) = \{\varphi(x)\} \cup \{x \neq a \mid a \in \varphi^M\}$, which is not realised in N. Using Theorem 35 we can find a proper elementary extension $N' \succ N$ such that N' also omits p. By transfinite induction, taking unions at limit steps, we get an arbitrarily long increasing sequence of elementary extensions with that property. At some point we get an elementary extension N^* of size $|N^*| \geq \kappa$ that still omits p. By a simple Löwenheim-Skolem argument we see that we can reduce N^* to size $|N^*| = \kappa$. □

Proof of Lemma B. Suppose T is countable and totally transcendental but admits a Vaught pair. By Corollary 36 there is a Vaught pair $M \prec N$ for $\varphi(x)$ such that $|M| = \aleph_0$ and $|N| = \kappa$. Thus we have $\left|\varphi^N\right| = \left|\varphi^M\right| = \aleph_0$. As N cannot be isomorphic to the model of Proposition 31, T is not κ-categorical. □

9 Proof of Lemma C

Recall that we have fixed a complete first-order theory T with infinite models and a cardinal $\kappa \geq |T|$. Our aim in this section is to prove:

Lemma C. *If T is totally transcendental and does not admit Vaught pairs, then T is κ-categorical.*

Definition 37. A formula $\mu(x)$ is called *minimal* in a model M, if for every formula $\varphi(x)$ with parameters in M either $\mu(x) \wedge \varphi(x)$ or $\mu(x) \wedge \neg\varphi(x)$ is algebraic. The formula is called *strongly minimal* for a theory T if it is minimal in every model of T.

Lemma 38. *Suppose T is totally transcendental. For every model $M \models T$ there is a minimal formula $\mu(x)$ with parameters in M.*

Proof. If there is no minimal formula, then every non-algebraic formula with parameters can be split into two non-algebraic formulas with parameters. Thus we can build a tree that contradicts total transcendence. □

Lemma 39. *Suppose T does not admit Vaught pairs. Every formula $\mu(x)$ without parameters that is minimal in a model $M \models T$ is strongly minimal for T.*

Proof. Let $\mu(x)$ be minimal in $M \models T$. If $\mu(x)$ is not strongly minimal for T, then there is a model $N \models T$ such that $\mu(x)$ is not minimal in N. Replacing N by a common elementary extension of M and N (such things always exist by compactness), we can assume that $M \prec N$. Having extended the language by a unary predicate P, let T^* be the theory of N with the subset M described by P, i.e. $N^P = M$. We denote the expanded model by N^*. Note that T^* 'knows' that the subset defined by P is a proper elementary substructure.

Let $\varphi(x, \bar{b})$ witness that $\mu(x)$ is not minimal in N, i.e. neither $\mu(x) \wedge \varphi(x, \bar{b})$ nor $\mu(x) \wedge \neg\varphi(x, \bar{b})$ is algebraic. Note that because M is elementary in N, for every natural number n we have $M \models \exists \bar{y}(\exists^{\geq n} x(\mu(x) \wedge \varphi(x, \bar{y})) \wedge \exists^{\geq n} x(\mu(x) \wedge \neg\varphi(x, \bar{y})))$. Keeping in mind that $\mu(x)$ is minimal in M it follows that, possibly after replacing φ by $\neg\varphi$, there are arbitrarily big definable finite sets of the form $\{a \in M \mid M \models \mu(a) \wedge \varphi(a, \bar{b})\}$ with $\bar{b} \in M$. Hence for every natural number n there is a tuple $\bar{b} = b_0 \ldots b_{k-1} \in M$ such that

$$N^* \quad \models \quad \exists^{\geq n} x(\mu(x) \wedge \varphi(x, \bar{b})) \quad \wedge \quad \forall x(\mu(x) \wedge \varphi(x, \bar{b}) \rightarrow Px).$$

Therefore the partial T^*-type

$$q(\bar{y}) \quad = \quad \{ \quad \exists^{\geq n} x(\mu(x) \wedge \varphi(x, \bar{y})) \quad | \quad n < \omega \quad \} \\ \cup \quad \{\forall x(\mu(x) \wedge \varphi(x, \bar{y}) \to Px)\} \quad \cup \quad \{Py_0 \wedge \ldots \wedge Py_{k-1}\} \quad \cup \quad T^*$$

is consistent by compactness. Let $N_V^* \models T^*$ be a model of T^* in which this type is realised by a tuple $\bar{b} \models q(\bar{y})$. Denote the reduct of N_V^* to the original theory by N_V and the subset $P^{N_V^*} \subset N_V$ of N_V by M_V. Using the definition of T^*, we see that $M \prec N$ is a Vaught pair for $\mu(x) \wedge \varphi(x, \bar{b})$.[9] □

Remark 40. Let $q(\bar{y}) \in \mathrm{S}^{\bar{y}}(T)$ be an isolated type and let $\bar{c}$ be a tuple of new constants. If $T^* = p(\bar{c})$ is κ-categorical, then T is κ-categorical.

Proof. Suppose $M_1, M_2 \models T$ and $|M_1| = |M_2| = \kappa$. Since the type $q(\bar{y})$ is isolated, there are tuples $\bar{b}_1 \in M_1$ and $\bar{b}_2 \in M_2$ realising it. Thus by interpreting $\bar{c}$ as $\bar{b}$ we can expand M_1 to a model $M_1^* \models T^*$ of the κ-categorical theory, and similarly for M_2. By κ-categoricity M_1^* and M_2^* are isomorphic, i.e. there is an isomorphism from M_1 to M_2 which moreover maps $\bar{b}_1$ to $\bar{b}_2$. In particular T is κ-categorical. □

Proof of Lemma C. By Theorem 30, T has a prime model M. By Lemma 38, M has a minimal formula $\mu(x, \bar{b})$ with parameters $\bar{b} \in M$. By Theorem 30, $\mathrm{tp}(\bar{b})$ is isolated. By Remark 40 it is enough to prove that the original theory expanded by constants for $\bar{b}$ is κ-categorical. Since total transcendence and non-existence of Vaught pairs are clearly preserved under extension by constants, it all boils down to this: We may assume that there is a minimal formula $\mu(x)$ *without* parameters for (the expansion of) M. By Lemma 39 it then follows that $\mu(x)$ is a strongly minimal formula for T.

Let $M_1 \models T$ be a model of size $|M_1| = \kappa$. We first verify that $\left|\mu^{M_1}\right| = \kappa$. Let T' be T together with a new constant each for every element of μ^{M_1}. Let $M_1' \models T'$ be the obvious expansion of M_1. The theory T' is again totally transcendental, so it has a prime model N'. By the definition of prime models we can find N_1' such that $N' \cong N_1' \preceq M_1'$. Note that $\mu^{N_1} = \mu^{M_1}$. Like T, T' does not have a Vaught pair for $\mu(x)$, so we must have $N_1 = M_1$. By the Löwenheim-Skolem Theorem, the size of a prime model for T' can be at most $|T'|$, hence we have $\left|\mu^{M_1}\right| \leq |N_1| \leq |T| + \left|\mu^{M_1}\right|$. Since $|N_1| = |M_1| = \kappa > |T|$ it follows that $\left|\mu^{M_1}\right| = \kappa$.

If $M_2 \models T$ is another model of size $|M_2| = \kappa$, the by the same argument $\left|\mu^{M_2}\right| = \kappa$. At this point we engage in some hand-waving:[10] The material from Section 40 generalises to strongly minimal formulas as smoothly as one can hope. In particular, acl when restricted to the realisations of $\mu(x)$ describes a matroid, and any two bases of this matroid have the same type, provided they have the same size.

Let $B_i \subseteq M_i$ be a basis of the matroid $\langle \mu^{M_i}, \mathrm{acl}_{\mu^{M_i}} \rangle$, for $i \in \{1, 2\}$. As in Theorem 21, we can take any bijection $f \colon B_1 \to B_2$, note that it is an elementary bijection, and then extend it to the algebraic closures, noting that the result is again an elementary bijection. Since $\mu(M_i) \subseteq \mathrm{acl}\, B_i$ it follows in particular that there is an elementary bijection $f' \colon \mu(M_1) \to \mu(M_2)$.

Via f' we can define an expansion $M_2' \models T'$ of M_2 such that every element of μ^{M_2} is named by one of the new constants; f' is an elementary map also in the language of T'. As above for M_1 we can now find an isomorphic copy $N_2 \cong N$ such that $N_2 \preceq M_2$ and show that $N_2 = M_2$. It follows that $M_1 = N_1 \cong N \cong N_2 = M_2$. □

This concludes the proof of Morley's Theorem.

Corollary 41 (to the proof of Lemma C)**.** *If T is totally transcendental and does not admit Vaught pairs, then for any cardinal $\kappa \leq |T|$, T has at most κ non-isomorphic models of size κ.*

Proof. There is only one place in the proof of Lemma C where the assumption $\kappa > |T|$ was used: when we concluded $\left|\mu^{M_1}\right| = \kappa$ from $\left|\mu^{M_1}\right| \leq \kappa \leq |T| + \left|\mu^{M_1}\right|$. If we have instead $\kappa \leq |T|$, we only know that $\left|\mu^{M_1}\right| \leq \kappa$.

[9] Embedded in this proof is a proof that if T does not admit Vaught pairs, then T eliminates the quantifier $\exists^\infty$, i.e. every formula in first order extended by this quantifier (meaning: there exist infinitely many) is equivalent modulo T to a regular first-order formula.

[10] A later version will have a rigorous proof here.

Having added the parameters $\bar{b}$ of the minimal formula $\mu(x, \bar{b})$ to the language as in the proof of Lemma C, we see that a model $M \models T$ (of any cardinality) is characterised up to isomorphism by the cardinal $\left|\mu^M\right| \leq \kappa$. As before, any two models that are isomorphic in the expanded language are also isomorphic in the original language. $\square$

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