Strict orders prohibit elimination of hyperimaginaries

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Abstract

A theory with the strict order property does not eliminate hyperimaginaries. Hence a theory without the independence property eliminates hyperimaginaries if and only if it is stable.

A *type definable equivalence relation* is an equivalence relation on tuples of a certain length (possibly infinite) which is defined by a partial type $E(\bar{x};\bar{y})$ over $\emptyset$. A *hyperimaginary* is an equivalence class $\bar{a}/E$. If $E(\bar{x};\bar{y})$, where $\bar{x}=(x_i)_{i\in I}$, $\bar{y}=(y_i)_{i\in I}$, defines an equivalence relation only on the realisations of a partial type $p(\bar{x})$ over $\emptyset$, then the partial type

$$E_p(\bar{x};\bar{y}) \quad = \quad E(\bar{x};\bar{y}) \cup \big\{(\varphi(\bar{x}) \wedge \varphi(\bar{y})) \vee x_i = y_i \bigm| \varphi(\bar{x}) \in p(\bar{x}),\ i \in I\big\}$$

defines an equivalence relation on all tuples of the appropriate length, which coincides with E on the realisations of p and is equality on the other tuples. Thus even in this more general case $\bar{a}/E$ is a hyperimaginary [2].

A hyperimaginary $\bar{a}/E$ is said to be *eliminable* if there is a set of imaginaries A such that an automorphism of the monster model fixes A pointwise if and only if it fixes $\bar{a}/E$ (i.e. maps $\bar{a}$ to a tuple $\bar{a}'$ such that $\models E(\bar{a};\bar{a}')$. It is a well-known and easy fact that a hyperimaginary $\bar{a}/E$ is eliminable if and only if the equivalence

$$E_p(\bar{x};\bar{y}) \quad \equiv \quad \big\{\epsilon(\bar{x};\bar{y}) \bigm| \epsilon(\bar{x};\bar{y}) \textit{ definable equivalence relation over } \emptyset\textit{, and } E(\bar{x};\bar{y}) \vdash \epsilon(\bar{x};\bar{y})\big\}$$

holds, where $p = \mathrm{tp}(\bar{a})$ [3].

Lemma 1. *Let $\bar{a}=(a_i)_{i\in\mathbb{Q}}$ be an indiscernible sequence which is ordered by a formula $\varphi(\bar{x};\bar{y})$ without parameters, i.e. $\models \varphi(\bar{a}_i;\bar{a}_j) \iff i<j$. Then the relation defined on the realisations of $p(\bar{x}) = \mathrm{tp}(\bar{a})$ by the partial type*

$$E(\bar{x};\bar{y}) \quad = \quad \big\{(\varphi(\bar{x}_i;\bar{y}_j) \wedge \varphi(\bar{y}_i;\bar{x}_j) \bigm| i,j \in \mathbb{Q},\ i<j\big\}$$

is clearly reflexive and symmetric. If it is also transitive, then the hyperimaginary $\bar{a}/E$ is not eliminable.

Proof. To simplify notation we will write the tuples $\bar{x},\bar{y}$ as elements and the formula φ as $<$. So there is an indiscernible strictly $<$-ascending chain $\bar{a}=(a_i)_{i\in\mathbb{Q}}$. Let $p(\bar{x}) = \mathrm{tp}(\bar{a})$. The partial type

$$E(\bar{x};\bar{y}) \quad = \quad \big\{(x_i<y_j) \wedge (y_i<x_j) \bigm| i,j\in\mathbb{Q}, i<j\big\}$$

is clearly reflexive, symmetric and transitive for realisations of p, so it defines a type-definable equivalence relation on p. Therefore $\bar{a}/E$ is a hyperimaginary. We will show that it is not eliminable.

Suppose $\epsilon(\bar{x};\bar{y})$ is a definable equivalence relation such that $E_p(\bar{x};\bar{y}) \vdash \epsilon(\bar{x};\bar{y})$. Now let

$$\mathcal{A} = \big\{\bar{b}=(b_i)_{i\in\mathbb{Q}} \bigm| b_i = a_{f(i)}, \textit{ where } f:\mathbb{Q}\to\mathbb{Q} \textit{ is order-preserving}\big\}$$

be the set of all realisations of p that are actually subsequences of $\bar{a}$ (which have been re-ordered in an order-preserving way). Of course $\mathcal{A}$ is not type-definable, but we will examine ϵ on $\mathcal{A}$ anyway. In fact, we will show that ϵ is trivial on $\mathcal{A}$.

First note that $\epsilon(\bar{x};\bar{y}) = \epsilon(x_{i_0},\ldots,x_{i_k};y_{i_0},\ldots,y_{i_k})$ for some $i_0 < i_1 < \ldots i_k$ in $\mathbb{Q}$. We may assume that $i_0 = 0, i_1 = 1, \ldots, i_k = k$, so $\epsilon(\bar{x};\bar{y}) = \epsilon(x_0,\ldots,x_k;y_0,\ldots,y_k)$. The two obvious order-preserving maps $f : \mathbb{Q} \to \mathbb{Q} \setminus [0,1)$ and $g : \mathbb{Q} \to \mathbb{Q} \setminus (0,1]$ (which have $f(0) = 0$, $g(0) = 1$ and agree everywhere else) define two E_p-equivalent tuples $\bar{b} = f(\bar{a})$ and $\bar{c} = g(\bar{a})$ such that $b_0 = a_0 < a_1 = c_0$. Therefore $\epsilon(x_0,\ldots,x_k;y_0,\ldots,y_k) \not\vdash x_0 = y_0$, and similarly $\epsilon(x_0,\ldots,x_k;y_0,\ldots,y_k) \not\vdash x_i = y_i$ for all $i \leq k$. But by indiscernibility, ϵ must be equivalent on $\mathcal{A}$ to a quantifier-free formula in the language of order, hence to a quantifier-free equivalence relation in the language of order. Therefore ϵ is trivial on $\mathcal{A}$.

Now if E_p were equivalent to the set of all such formulas ϵ, then E_p would also have to be trivial on $\mathcal{A}$, which is clearly not the case. □

A formula $\varphi(\bar{x};\bar{y})$ is said to have the *strict order property* if there is an infinite sequence of tuples $(\bar{b}_i)_{i<\omega}$ such that $\varphi(\bar{x};\bar{b}_i)$ implies $\varphi(\bar{x};\bar{b}_{i+1})$ but $\varphi(\bar{x};\bar{b}_{i+1})$ does not imply $\varphi(\bar{x};\bar{b})$. For example if $\varphi(\bar{x};\bar{y})$ is a partial order with infinite chains, then φ has the strict order property. Conversely, if $\varphi(\bar{x};\bar{y})$ has the strict order property, then the formula

$$\psi(\bar{y};\bar{y}') \quad \equiv \quad \forall\bar{x}\big(\varphi(\bar{x};\bar{y}) \to \varphi(\bar{x};\bar{y}')\big) \quad \wedge \quad \neg\forall\bar{x}\big(\varphi(\bar{x};\bar{y}') \to \varphi(\bar{x};\bar{y})\big)$$

defines a partial order with infinite chains. A theory is said to have the strict order property if it has a formula which does [4].

Theorem 2. *Every theory with the strict order property has a hyperimaginary which is not eliminable.*

Proof. Let $\varphi(\bar{x};\bar{y})$ be a partial order with infinite chains. By standard arguments it follows that there are infinite indiscernible chains of arbitrary order type. By transitivity of the partial order and density of $\mathbb{Q}$ the partial type E in the lemma is transitive. □

Corollary 3. *The theory of dense linear orders, which is ω-categorical and therefore eliminates finitary hyperimaginaries, does not eliminate hyperimaginaries.*

Corollary 4. *A theory without the independence property is stable if and only if it eliminates hyperimaginaries.*

Proof. Stable theories are known to eliminate hyperimaginaries [3]. Unstable theories without the independence property are known to have the strict order property [4], so they don't eliminate hyperimaginaries. □

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References

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