Strong theories, burden, and weight

Hans Adler

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Abstract

We introduce the notion of the burden of a partial type in a complete first-order theory and call a theory strong if all types have almost finite burden. In a simple theory it is the supremum of the weights of all extensions of the type, and a simple theory is strong if and only if all types have finite weight. A theory without the independence property is strong if and only if it is strongly dependent. As a corollary, a stable theory is strongly dependent if and only if all types have finite weight. A strong theory does not have the tree property of the second kind.

Warning

This is a new and very unstable draft. I may change the title and some of the definitions for the final version.

1 Weight in simple theories

We begin by recalling the definitions of preweight and weight in simple theories. So in this section we fix a simple theory T.

Let us call a set $\mathcal{B}$ of tuples $\bar{b}$ a *free forking system* for $\mathrm{tp}(\bar{a}/C)$ if $\mathcal{B}$ is independent over C, and $\bar{a} \not\perp_C B$ for all $\bar{b} \in \mathcal{B}$. If $C' \supseteq C$ and $\bar{a} \perp_C C'$, $\mathcal{B} \perp_C C'$, then clearly $\mathcal{B}$ is a free forking system for $\mathrm{tp}(\bar{a}/C')$ if and only if $\mathcal{B}$ is a free forking system for $\mathrm{tp}(\bar{a}/C)$. Moreover, if we have chains $\mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \dots$ and $C_0 \subseteq C_1 \subseteq C_2 \subseteq \dots$ such that $\bar{a} \perp_{C_n} C_{n+1}$ and each $\mathcal{B}_n$ is a free forking system for $\mathrm{tp}(\bar{a}/C_n)$, then $\bigcup_{n<\omega} \mathcal{B}_n$ is a free forking system for $\mathrm{tp}(\bar{a}/\bigcup_{n<\omega} C_n)$.

The *weight* of a complete type p, in symbols $\mathrm{wt}(p)$, is the supremum of the cardinalities of all free forking systems for nonforking extensions of p. We will see that if $\mathrm{wt}(p) = \aleph_0$ then this supremum is attained. In other words: If a type p has *almost finite weight*, i.e., there is no infinite free forking system for a nonforking extension of p, then p has finite weight. The analogous problem for $\mathrm{wt}(p) = \aleph_\omega$, however, is open even for stable theories.

There are several ways to fix this problem, and none of them is perfect. Here we propose a solution that hides the problem whenever it does not matter. If Card is the class of all cardinals, then by Card* we denote the class that results if we add a new element ∞ and replace every limit cardinal κ by two new elements, κ_- and κ_+. If κ is a successor cardinal, we may also write $\kappa_- = \kappa_+ = \kappa$. Card* has an obvious linear order. The standard embedding of Card into Card* identifies κ with κ_+. Now whenever we take the supremum of a set of cardinals, we compute it in Card*. Now if a type p has almost finite weight we can express this by the formula $\mathrm{wt}(p) \leq (\aleph_0)_-$.

Question 1. Can it happen that $\mathrm{wt}(p) = \kappa_-$ for a limit cardinal κ?

The theorem saying that $\mathrm{wt}(p) = (\aleph_0)_-$ is impossible is well known [15]. In order to keep this paper relatively self-contained, we sketch a proof anyway.

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Lemma 2. *Let $\mathcal{B}$ be a free forking system for* $\operatorname{tp}(\bar{a}/C)$. *If* $\bar{b} \in \mathcal{B}$ *and* $\operatorname{wt}(\bar{b}/C) > 1$, *then there are a set* $C' \supseteq C$ *and tuples* $\bar{b}_1, \bar{b}_2$ *such that* $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_C C'$ *and* $(\mathcal{B} \setminus \{\bar{b}\}) \cup \{\bar{b}_1, \bar{b}_2\}$ *is a free forking system for* $\operatorname{tp}(\bar{a}/C')$.

Lemma 3. *Suppose* $\operatorname{tp}(\bar{a}/C)$ *has almost finite weight. Let $\mathcal{B}$ be a free forking system for* $\operatorname{tp}(\bar{a}/C)$. *For every* $\bar{b} \in \mathcal{B}$ *there is a set* $C' \supseteq C$ *and a tuple* $\bar{b}'$ *such that* $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_C C'$, $\operatorname{wt}(\bar{b}'/C') = 1$ *and* $(\mathcal{B} \setminus \{\bar{b}\}) \cup \{\bar{b}'\}$ *is a free forking system for* $\operatorname{tp}(\bar{a}/C')$.

Proof. Suppose there is no such $\bar{b}'$. Let $\mathcal{B}_0 = \mathcal{B} \setminus \{\bar{b}\}$ and $C_0 = C$. Applying the previous lemma repeatedly, we can build sequences $\mathcal{B}_0 \subsetneq \mathcal{B}_1 \subsetneq \dots$ and $C_0 \subseteq C_1 \subseteq \dots$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{C_n} C_{n+1}$ (hence $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_C C_{n+1}$) and each $\mathcal{B}_n$ is a free forking system for $\operatorname{tp}(\bar{a}/C_n)$. Hence $\bigcup_{n<\omega} \mathcal{B}_n$ is an infinite free forking system for $\operatorname{tp}(\bar{a}/\bigcup_{n<\omega} C_n)$. It follows that $\operatorname{wt}(\bar{a}/C) \geq (\aleph_0)_+$. □

Lemma 4. *Suppose* $\operatorname{tp}(\bar{a}/C)$ *has almost finite weight. Then there is a set* $C' \supseteq C$ *such that* $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_C C'$ *and a free forking system $\mathcal{B}$ for* $\operatorname{tp}(\bar{a}/C')$ *such that* $\operatorname{wt}(\bar{b}/C') = 1$ *for all* $\bar{b} \in \mathcal{B}$, $\bar{a}$ *dominates $\mathcal{B}$ over C', and $\mathcal{B}$ is maximal in the following sense. Whenever* $C'' \supseteq C'$ *is such that* $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{C'} C''$ *and* $\mathcal{B}' \supseteq \mathcal{B}$ *is a free forking system for* $\operatorname{tp}(\bar{b}/C'')$ *such that $\bar{a}$ dominates $\mathcal{B}'$ over C'', we have* $\mathcal{B}' = \mathcal{B}$.

It follows that $\mathcal{B}$ dominates $\bar{a}$ over C', and that $\operatorname{wt}(\bar{a}/C) = |\mathcal{B}| < \aleph_0$.

Theorem 5. *Suppose* $\operatorname{tp}(\bar{a}/C)$ *has almost finite weight. Then there is a set* $C' \supseteq C$ *such that* $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_C C'$ *and a free forking system $\mathcal{B}$ for* $\operatorname{tp}(\bar{a}/C')$ *such that* $\operatorname{wt}(\bar{b}/C') = 1$ *for all* $\bar{b} \in \mathcal{B}$, *$\bar{a}$ and $\mathcal{B}$ dominate each other over C', and* $\operatorname{wt}(\bar{a}/C) = |\mathcal{B}|$ *is in fact finite.*

Proof. Let C', $\mathcal{B}$ be as in the last lemma. First we observe that we may assume that $\bar{a}$ dominates $\bar{a}\mathcal{B}$ over C'. If not, this is witnessed by some $C'' \supseteq C'$ such that $C'' \mathop{\smile\!\!\!\!\!\mid}_{C'} \bar{a}$ but $C'' \not\mathop{\smile\!\!\!\!\!\mid}_{C'} \bar{a}\mathcal{B}$, and we can replace C' by C''. We may have to do this transfinitely, taking unions at limit steps. But since $\bar{a}\mathcal{B} \not\mathop{\smile\!\!\!\!\!\mid}_{C'} C''$ this cannot go on forever, by local character.

Now towards a contradiction, suppose $\mathcal{B}$ does not dominate $\bar{a}$ over C'. So there is $\bar{b}$ such that $\mathcal{B} \mathop{\smile\!\!\!\!\!\mid}_{C'} \bar{b}$ but $\bar{a} \not\mathop{\smile\!\!\!\!\!\mid}_{C'} \bar{b}$. So $\mathcal{B}' = \mathcal{B} \cup \{\bar{b}\}$ is a free forking system for $\operatorname{tp}(\bar{a}/C')$. By the previous lemma, we get $C'' \supseteq C'$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{C'} C''$ and some $\bar{b}'$ such that $\operatorname{wt}(\bar{b}'/C'') = 1$ and $\mathcal{B} \cup \{\bar{b}'\}$ is a free forking system over C''. Now we show that $\bar{a}$ dominates $\mathcal{B}'$ over C'', so $\mathcal{B}'$, C'' contradict maximality of $\mathcal{B}$.

If $X \mathop{\smile\!\!\!\!\!\mid}_{C''} \bar{a}$, then $XC'' \mathop{\smile\!\!\!\!\!\mid}_{C'} \bar{a}$, $XC'' \mathop{\smile\!\!\!\!\!\mid}_{C'} \bar{a}\mathcal{B}$, and $X \mathop{\smile\!\!\!\!\!\mid}_{C''} \bar{a}\mathcal{B}$.

Since $\bar{a} \not\mathop{\smile\!\!\!\!\!\mid}_{C''} \bar{b}$ and $\mathcal{B} \mathop{\smile\!\!\!\!\!\mid}_{C''} \bar{b}$, we have $\bar{a} \not\mathop{\smile\!\!\!\!\!\mid}_{\mathcal{B}C''} \bar{b}$ by transitivity. Since $\operatorname{wt}(\bar{b}/\mathcal{B}C'') = \operatorname{wt}(\bar{b}/C'') = 1$ and $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{\mathcal{B}C''} X$, we have $X \mathop{\smile\!\!\!\!\!\mid}_{\mathcal{B}C''} \bar{b}$. With $X \mathop{\smile\!\!\!\!\!\mid}_{C''} \mathcal{B}$ and transitivity we get $X \mathop{\smile\!\!\!\!\!\mid}_{C''} \mathcal{B}'$.

So $\bar{a}$ and $\mathcal{B}$ dominate each other over C'. Since $\operatorname{wt}(\bar{a}/C) \leq (\aleph_0)_-$, clearly $|\mathcal{B}|$ is finite. It is easy to check that $\operatorname{wt}(\bar{a}/C) = |\mathcal{B}|$. □

It is well known that the class of simple theories such that all (finitary) types have finite weight includes all supersimple theories; but in fact it is much wider, as it also includes all stable theories with no dense forking chains [3, 4, 7]. Finally, let us remark that the definitions and arguments in this section make perfect sense if we replace forking in a simple theory by thorn-forking in a rosy theory. But we will not make use of this.

2 Burden; its relation to weight

An *independent partition pattern* or *inp-pattern* for a partial type $p(\bar{x})$ is a sequence $((\varphi^\alpha(\bar{x};\bar{y}^\alpha), k^\alpha))_{\alpha<\kappa}$ such that there is an 'array' of tuples $\bar{b}_i^\alpha$ for $\alpha < \kappa$, $i < \omega$ which *witnesses* it: For all $\alpha < \kappa$ the 'row' $\{\varphi^\alpha(\bar{x};\bar{b}_i^\alpha) \mid i < \omega\}$ is k^α-inconsistent, but every 'path' $\{\varphi^\alpha(\bar{x};\bar{b}_{\eta(\alpha)}^\alpha) \mid \alpha < \kappa\}$, $\eta \in \omega^\kappa$, is consistent with $p(\bar{x})$. The *burden* of $p(\bar{x})$, in symbols $\operatorname{bdn}(p)$, is the supremum of the cardinalities κ of all inp-patterns for p. It is easy to see that the burden of a type is greater than zero if and only if it is non-algebraic. We call a theory *strong* if the burden of every finitary partial type $p(\bar{x})$ is bdn $\leq (\aleph_0)_-$.

$$
\begin{array}{cccccccc}
 & & p(\bar{x}) & & & & & \\
\varphi^0(\bar{x};\bar{b}^0_0) & \varphi^0(\bar{x};\bar{b}^0_1) & \varphi^0(\bar{x};\bar{b}^0_2) & \varphi^0(\bar{x};\bar{b}^0_3) & \varphi^0(\bar{x};\bar{b}^0_4) & \cdots & & k^0\text{-inconsistent} \\
\varphi^1(\bar{x};\bar{b}^1_0) & \varphi^1(\bar{x};\bar{b}^1_1) & \varphi^1(\bar{x};\bar{b}^1_2) & \varphi^1(\bar{x};\bar{b}^1_3) & \varphi^1(\bar{x};\bar{b}^1_4) & \cdots & & k^1\text{-inconsistent} \\
\varphi^2(\bar{x};\bar{b}^2_0) & \varphi^2(\bar{x};\bar{b}^2_1) & \varphi^2(\bar{x};\bar{b}^2_2) & \varphi^2(\bar{x};\bar{b}^2_3) & \varphi^2(\bar{x};\bar{b}^2_4) & \cdots & & k^2\text{-inconsistent} \\
\varphi^3(\bar{x};\bar{b}^3_0) & \varphi^3(\bar{x};\bar{b}^3_1) & \varphi^3(\bar{x};\bar{b}^3_2) & \varphi^3(\bar{x};\bar{b}^3_3) & \varphi^3(\bar{x};\bar{b}^3_4) & \cdots & & k^3\text{-inconsistent} \\
\varphi^4(\bar{x};\bar{b}^4_0) & \varphi^4(\bar{x};\bar{b}^4_1) & \varphi^4(\bar{x};\bar{b}^4_2) & \varphi^4(\bar{x};\bar{b}^4_3) & \varphi^4(\bar{x};\bar{b}^4_4) & \cdots & & k^4\text{-inconsistent} \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & \vdots
\end{array}
$$

Figure 1: Independent partitions for a type $p(\bar{x})$. Every row α is k^α-inconsistent. Every set containing only one formula from each row is consistent with the type.

Proposition 6. *The following conditions are equivalent for every partial type $p(\bar{x})$ and every cardinal κ.*

(1) $\mathrm{bdn}(p) \geq \kappa_+$.

(2) There are sequences $(\bar{b}^\alpha_i)_{i<\omega}$, where $\alpha < \kappa$, and formulas $\varphi^\alpha(\bar{x};\bar{y}^\alpha)$ such that each system $\{\varphi^\alpha(\bar{x};\bar{b}^\alpha_i) \mid i < \omega\}$ is k^α-inconsistent for some $k^\alpha < \omega$, but for all functions $\eta \in \omega^\kappa$ the partial type $p(\bar{x}) \cup \{\varphi^\alpha(\bar{x};\bar{b}^\alpha_{\eta(\alpha)}) \mid \alpha < \kappa\}$ is consistent.

(3) The same as (2), with the additional requirement that the sequences $(\bar{b}^\alpha_i)_{i<\omega}$ are mutually indiscernible, i.e., each $(\bar{b}^\alpha_i)_{i<\omega}$ is indiscernible over $\bar{b}^{\neq\alpha}_{<\omega}$.

Proof. The equivalence of (1) and (2) is immediate from the definition of burden, so we only need to show that (2) implies (3). We can extend each sequence $(\bar{b}^\alpha_i)_{i<\omega}$ to a very long sequence, extract a sequence which is indiscernible over the other sequences, and replace the original $(\bar{b}^\alpha_i)_{i<\omega}$ with the new one. □

Corollary 7. *Let $p(\bar{x})$ be a partial type defined over a set C of parameters. The burden of p is the supremum of the burdens of the completions of p over C.*

Proof. Clearly the burden of an extension of p is at most the burden of p. Now suppose $\mathrm{bdn}(p) \geq \kappa_+$. Then we have a configuration witnessing this as in (3) of the proposition. Let $\bar{a}$ be a realisation of p such that $\varphi^\alpha(\bar{x};\bar{b}^\alpha_0)$ holds for all $\alpha < \kappa$. Then clearly $\mathrm{bdn}(\bar{a}/C) \geq \kappa_+$. □

Proposition 8. *In a simple theory the burden of a partial type is the supremum of the weights of its complete extensions.*

Proof. Suppose that p has an extension $\mathrm{tp}(\bar{a}/C)$ which has weight $\mathrm{wt}(\bar{a}/C) \geq \kappa_+$ witnessed by a free forking system $(\bar{b}^\alpha)_{\alpha<\kappa}$ for (without restriction) $\mathrm{tp}(\bar{a}/C)$. We may assume that there are formulas $\varphi^\alpha(\bar{x};\bar{y}^\alpha)$ such that $\models \varphi^\alpha(\bar{a};\bar{b}^\alpha)$ and $\varphi(\bar{x};\bar{b}^\alpha)$ forks over C. (We may have to add some elements of C to $\bar{b}^\alpha$ to get this.) We build sequences $(\bar{b}^\alpha_i)_{i<\omega}$ by induction on α. Let $(\bar{b}^\alpha_i)_{i<\omega}$ be a Morley sequence over $C\bar{b}^{<\alpha}_{<\omega}\bar{b}^{>\alpha}$ starting with $\bar{b}^\alpha_0 = \bar{b}^\alpha$. Also by induction on α we can see that each of the sequences $(\bar{b}^\alpha_i)_{i<\omega}$ is actually independent over C. Using induction and standard arguments it is easy to see that $\bar{b}^{\leq\alpha}_{<\omega} \perp_C \bar{b}^{\geq\alpha}$, and $(\bar{b}^\alpha_i)_{i<\omega}$ is a Morley sequence over C, for all $\alpha < \kappa$. Therefore $\{\varphi^\alpha(\bar{x};\bar{b}^\alpha_i) \mid i < \omega\}$ is k^α-inconsistent for some $k^\alpha < \omega$. We can easily check that for all functions $\eta \in \omega^\kappa$ the partial type $p(\bar{x}) \cup \{\varphi^\alpha(\bar{x};\bar{b}^\alpha_{\eta(\alpha)}) \mid \alpha < \kappa\}$ is consistent, by doing an induction on the least $\beta < \kappa$ such that $\eta(\alpha) = 0$ for $\alpha \geq \beta$. Therefore $\mathrm{bdn}(p) \geq \kappa_+$.

Conversely, suppose $\mathrm{bdn}(p) \geq \kappa_+$. This is witnessed by a configuration as in (3) of Proposition 6. If we extend the indiscernible sequence $(\bar{b}^{<\kappa}_i)_{i<\omega}$ to an indiscernible sequence $(\bar{b}^{<\kappa}_i)_{i<\omega+\omega}$ and set $C = \bar{b}^{<\kappa}_{\geq\omega}$, we find that the sequences $(\bar{b}^\alpha_i)_{i<\omega}$ are mutually indiscernible Morley sequences over C. It follows that $(\bar{b}^\alpha)_{\alpha<\kappa}$ is an independent sequence over C. Let $\bar{a}$ realise $p(\bar{x}) \cup \{\varphi(\bar{x};\bar{b}^\alpha_0)\}$. By the condition on k^α-inconsistency, clearly $\bar{a} \not\perp_C \bar{b}^\alpha_0$ for all $\alpha < \kappa$. Therefore $\mathrm{wt}(\bar{a}/C) \geq \kappa_+$. □

In Section III.7 of his book, Shelah defined the invariant $\kappa_{\mathrm{inp}}(T)$ of a theory. In our terminology, $\kappa_{\mathrm{inp}}(T)$ is the smallest infinite cardinal κ such that no finitary type has an inp-pattern of cardinality κ [10].

Corollary 9. *The following are equivalent for every simple theory T.*

(1) T is strong.

(2) $\kappa_{\mathrm{inp}} = \aleph_0$.

(3) There is no finite tuple $\bar{a}$ and infinite sequence $(\bar{b}_i)_{i<\omega}$ independent over a set C such that $\bar{a} \not\perp_C \bar{b}_i$ for all $i < \omega$.

(4) Every finitary type has p has weight $\mathrm{wt}(p) \leq (\aleph_0)_-$.

(5) Every finitary type has p has burden $\mathrm{bdn}(p) \leq (\aleph_0)_-$.

(6) Every finitary type has finite weight.

3 Strong theories without the independence property

We say that a theory is *NIP* if no formula has the independence property [1].

An *independent contradictory types pattern*, or *ict-pattern*, is a sequence $(\varphi^\alpha(\bar{x};\bar{y}^\alpha))_{\alpha<\kappa}$ such that there is an 'array' of tuples $(\bar{b}_i^\alpha)_{\alpha<\kappa,i<\omega}$ which *witnesses* it: For every 'path' $\eta \in \omega^\kappa$ through the array the following set of formulas is consistent.

$$\Gamma_\eta(\bar{x}) \quad = \quad \left\{\varphi^\alpha\left(\bar{x};\bar{b}_i^\alpha\right) \;\middle|\; \alpha<\kappa;\ i<\omega;\ \eta(\alpha)=i\right\} \;\cup\; \left\{\neg\varphi^\alpha\left(\bar{x};\bar{b}_i^\alpha\right) \;\middle|\; \alpha<\kappa;\ i<\omega;\ \eta(\alpha)\neq i\right\}$$

We say that a theory is *strongly NIP* (or *strongly dependent*) if there is no ict-pattern of infinite length. More generally, the least infinite cardinal κ such that there is no ict-pattern of length κ is denoted by κ_{ict}. So T is strongly NIP if and only if $\kappa_{\mathrm{ict}} = \aleph_0$ [12, 13]. κ_{ict} is a relatively recent definition. We only repeat it here so that we can state the next result, which says that it is not needed.

Proposition 10. *If T is NIP, then $\kappa_{\mathrm{ict}} = \kappa_{\mathrm{inp}}$. Otherwise $\kappa_{\mathrm{ict}} = \infty$.*

Proof. If $\varphi(\bar{x};\bar{y})$ has the independence property, then every constant sequence $(\varphi(\bar{x};\bar{y}))_{\alpha<\kappa}$ is easily seen to be an ict-pattern. Hence $\kappa_{\mathrm{ict}} = \infty$.

Now we show that $\kappa_{\mathrm{inp}} \leq \kappa_{\mathrm{ict}}$ always holds. So suppose $((\varphi^\alpha(\bar{x};\bar{y}^\alpha))_{\alpha<\kappa}$ is an inp-pattern, witnessed by an array $(\bar{b}_i^\alpha)_{\alpha<\kappa,i<\omega}$. By Proposition 6 we may assume that the sequences $(\bar{b}_i^\alpha)_{i<\omega}$ are mutually indiscernible. Let $\bar{a}$ be such that $\models \varphi(\bar{a};\bar{b}_0^\alpha)$ for all $\alpha<\kappa$. For every $\alpha<\kappa$ there are only finitely many other $i>0$ such that $\models \varphi(\bar{a};\bar{b}_i^\alpha)$. Therefore after removing finitely many $\bar{b}_i^\alpha$ for every α we have $\models \varphi(\bar{a};\bar{b}_i^\alpha)$ if and only if $i=0$. Now by mutual indiscernibility $(\bar{b}_i^\alpha)_{\alpha<\kappa,i<\omega}$ witnesses that $(\varphi^\alpha(\bar{x};\bar{y}^\alpha))_{\alpha<\kappa}$ is an ict-pattern.

Finally, assuming NIP we prove $\kappa_{\mathrm{ict}} \leq \kappa_{\mathrm{inp}}$. So suppose $(\varphi^\alpha(\bar{x};\bar{y}^\alpha))_{\alpha<\kappa}$ is an ict-pattern, and let an array $(\bar{b}_i^\alpha)_{\alpha<\kappa,i<\omega}$ witness this. Again (by a similar argument as for inp witnesses) we may assume that the sequences $(\bar{b}_i^\alpha)_{i<\omega}$ are mutually indiscernible. Now consider the formulas $\psi^i(\bar{x};\bar{y}_1^\alpha\bar{y}_2^\alpha) \equiv \varphi^\alpha(\bar{x};\bar{y}_1^\alpha) \wedge \varphi^\alpha(\bar{x};\bar{y}_2^\alpha)$ and sequences $\bar{c}_i^\alpha = \bar{b}_{2i}^\alpha\bar{b}_{2i+1}^\alpha$. Since T is NIP, each formula ψ^α has finite alternation rank [1]. Therefore $(\psi^\alpha(\bar{x};\bar{c}_i^\alpha) \mid i<\omega)$ is k^α-inconsistent for some $k^\alpha<\omega$. Hence $((\psi^i(\bar{x};\bar{y}_1^\alpha\bar{y}_2^\alpha),k^\alpha))_{\alpha<\kappa}$ is an icp-pattern. □

Corollary 11. *A theory is strongly NIP if and only if it is strong and NIP.*

The following result shows that strongness is in fact a very strong and natural condition for NIP theories [13].

Fact 12. *The following conditions are equivalent for every complete theory T.*

(1) T is strongly NIP.

(2) For every indiscernible sequence $(\bar{b}_i)_{i\in\kappa}$, with tuples $\bar{b}_i$ that are at most countable, and every element a there is an equivalence relation $\sim$ on κ with finitely many convex classes and such that $\mathrm{tp}(\bar{b}_i/a)$ only depends on the $\sim$-class of i.

(3) For every indiscernible sequence $(\bar{b}_i)_{i\in I}$, with tuples $\bar{b}_i$ of arbitrary length, and every finite set A there is an equivalence relation $\sim$ on I with finitely many convex classes and such that each subsequence $(\bar{b}_i)_{i\in j/\sim}$ is indiscernible over C.

4 Unbounded burden and the two tree properties

Recall that a formula $\varphi(\bar{x};\bar{y})$ has the *tree property* (with respect to $k<\omega$) if there is a tree of tuples $(\bar{b}_\nu)_{\nu\in\omega^{<\omega}}$ such that for every $\eta\in\omega^\omega$ the path $\{\varphi(\bar{x};\bar{b}_{\eta\restriction\ell}) \mid \ell<\omega\}$ is consistent, but at every node $\nu\in\omega^{<\omega}$ the branching is k-inconsistent, i.e., the system $\{\varphi(\bar{x};\bar{b}_{\nu i}) \mid i<\omega\}$ is k-inconsistent [11].

A formula $\varphi(\bar{x};\bar{y})$ has the *tree property of the second kind (TP$_2$)* if there is an array of tuples $(\bar{b}_i^\alpha)_{\alpha,i<\omega}$ such that the system $\{\varphi(\bar{x};\bar{b}_i^\alpha) \mid i<\omega\}$ is 2-inconsistent for all $\alpha<\omega$, but for every function $\eta\in\omega^\omega$ the set $\{\varphi(\bar{x};\bar{b}^\alpha_{\eta(\alpha)} \mid \alpha<\omega\}$ is consistent [11]. (In other words, $\varphi(\bar{x};\bar{y})$ has the tree property of the second kind if there is a tree $(\bar{b}_\nu)_{\nu\in\omega^{<\omega}}$ witnessing that it has the tree property with respect to 2 and such that every non-root node $\bar{b}_\nu$ only depends on the length $\alpha=|\nu|$ and the last element $i=\nu(|\nu|-1)$.) One could say that strongness is related to not having the tree property of the second kind as supersimplicity is to simplicity. It is easy to see that the tree property of the second kind implies the independence property as well as the tree property.

Proposition 13. *The following conditions are equivalent for every complete theory.*

(1) There is a type of burden ∞.

(2) There is a finitary type of burden ∞.

(3) There are a formula $\varphi(\bar{x};\bar{y})$, tuples $(\bar{b}_i^\alpha)_{\alpha,i<\omega}$ and a number $k<\omega$ such that each system $\{\varphi(\bar{x};\bar{b}_i^\alpha) \mid i<\omega\}$ is k-inconsistent, but for every function $\eta\in\omega^\omega$ the set $\{\varphi(\bar{x};\bar{b}^\alpha_{\eta(\alpha)}) \mid \alpha<\omega\}$ is consistent.

(4) There is a formula $\varphi(\bar{x};\bar{y})$ which has the tree property of the second kind.

(5) There are a formula $\varphi(\bar{x};\bar{y})$ and tuples $(\bar{b}_i^\alpha)_{\alpha,i<\omega}$ such that each system $\{\varphi(\bar{x};\bar{b}_i^\alpha) \mid i<\omega\}$ is 2-inconsistent, for every function $\eta\in\omega^\omega$ the set $\{\varphi(\bar{x};\bar{b}^\alpha)_{\eta(\alpha)} \mid \alpha<\omega\}$ is consistent, the sequences $(\bar{b}_i^\alpha)_{i<\omega}$ are mutually indiscernible, and the sequence $(\bar{b}^\alpha_{<\omega})_{\alpha<\omega}$ is indiscernible.

Proof. Clearly (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1), so it suffices to show that (1) implies (5). Let $p(\bar{x})$ be a type of burden ∞, let κ be a very large cardinal. Among the pairs $(\varphi^\alpha,k^\alpha)$ in an inp-pattern $((\varphi^\alpha,k^\alpha))_{\alpha<\kappa}$ for p, at least one must occur infinitely often. Let us choose such a pair (φ,k). Therefore there are tuples $(\bar{b}_i^\alpha)_{\alpha,i<\omega}$ such that each system $\{\varphi(\bar{x};\bar{b}_i^\alpha) \mid i<\omega\}$ is k-inconsistent and for every function $\eta\in\omega^\omega$ the set $\{\varphi(\bar{x};\bar{b}^\alpha)_{\eta(\alpha)} \mid \alpha<\omega\}$ is consistent. By compactness we can find such tuples $(\bar{b}_i^\alpha)_{\alpha<\kappa,i<\omega}$ for every cardinal κ. We already know how to make the sequences $(\bar{b}_i^\alpha)_{i<\omega}$ mutually indiscernible. If κ is big enough, we can, moreover, extract from $(\bar{b}^\alpha_{<\omega})_{\alpha<\kappa}$ a sequence of indiscernibles. The only remaining problem is to get from k-inconsistency to 2-inconsistency.

Suppose k is minimal such that a formula and a configuration of this kind exist. We need to show that $k=2$. If the system $\{\varphi(\bar{x};\bar{b}_0^\alpha),\varphi(\bar{x};\bar{b}_1^\alpha) \mid \alpha<\omega\}$ is consistent, then we can replace $\varphi(\bar{x};\bar{y})$ by the formula $\varphi(\bar{x};\bar{y}_0)\wedge\varphi(\bar{x};\bar{y}_1)$ and the tuples $\bar{b}_i^\alpha$ by $\bar{b}_{2i}^\alpha\bar{b}_{2i+1}^\alpha$. Otherwise already $\{\varphi(\bar{x};\bar{b}_0^\alpha),\varphi(\bar{x};\bar{b}_1^\alpha) \mid \alpha<n\}$ is inconsistent for some $n<\omega$. In this case we can replace $\varphi(\bar{x};\bar{y})$ by the formula $\varphi(\bar{x};\bar{y}^0)\wedge\cdots\wedge\varphi(\bar{x};\bar{y}^{n-1})$ and the tuples $\bar{b}_i^\alpha$ by $\bar{b}_i^{n\alpha}\bar{b}_i^{n\alpha+1}\ldots\bar{b}_i^{n\alpha+n-1}$. In the first case we get $[k/2]$-inconsistency, in the second we get 2-inconsistency. □

A formula $\varphi(\bar{x};\bar{y})$ has the *tree property of the first kind (TP$_1$)* if there is a tree of parameters $(\bar{b}_\eta)_{\eta\in\omega^{<\omega}}$ such that each of the sets $\{\varphi(\bar{x};\bar{b}_{\eta\restriction n}) \mid n<\omega\}$ representing a path $\eta\in\omega^\omega$ through the tree is consistent, while for any two incomparable nodes $\eta,\eta'\in\omega^{<\omega}$ the formulas $\varphi(\bar{x};\bar{b}_\eta)$ and $\varphi(\bar{x};\bar{b}_{\eta'})$ are inconsistent [11]. $\varphi(\bar{x};\bar{y})$ has the *strong order property SOP$_2$* if there is a tree of parameters $(\bar{b}_\eta)_{\eta\in 2^{<\omega}}$ such that each of the sets $\{\varphi(\bar{x};\bar{b}_{\eta\restriction n}) \mid n<\omega\}$ representing a path $\eta\in 2^\omega$ through the tree is consistent, while for any two incomparable nodes $\eta,\eta'\in 2^{<\omega}$ the formulas $\varphi(\bar{x};\bar{b}_\eta)$ and $\varphi(\bar{x};\bar{b}_{\eta'})$ are inconsistent [2, 14]. It is easy to see that a formula has the TP$_1$ if and only if it has the SOP$_2$, and that the strict order property implies SOP$_2$.

The following result is Theorem III.7.11 in Shelah's book [11, 10].

Theorem 14. *A theory has the tree property (i.e. is not simple) if and only if it has the tree property of the first kind or the tree property of the second kind.*

Proof. If a formula has TP$_1$ or TP$_2$, then it clearly has the tree property. For the converse, suppose that there is no formula with TP$_2$, but we do have a formula $\varphi(\bar{x};\bar{y})$ which has the tree property with respect to $k<\omega$, witnessed by a tree of tuples $(\bar{b}_\nu)_{\nu\in\omega^{<\omega}}$ with the following properties.

(1) Every path $\{\varphi(\bar{x};\bar{b}_{\eta\restriction\ell}) \mid \ell < \omega\}$, for $\eta \in \omega^\omega$, is consistent.

(2) Every branching $\{\varphi(\bar{x};\bar{b}_{\nu i}) \mid i < \omega\}$, for $\nu \in \omega^{<\omega}$, is k-inconsistent.

We will prove that there is a formula which has TP_1. We first give a brief outline. It can be shown that we can find $(\bar{b}_\nu)_{\nu\in\omega^\omega}$ which satisfy, moreover, the following indiscernibility condition.

(3) The type $\mathrm{tp}(\bar{b}_{\nu_1}\ldots\bar{b}_{\nu_n})$ only depends on the quantifier-free type of $\nu_1\ldots\nu_n$ in the structure of all $\nu \in \omega^{<\omega}$ which has the binary operation $\nu\cap\mu = \nu\restriction\ell$, where ℓ is maximal such that $\nu\restriction\ell = \mu\restriction\ell$, the relation $\nu \lhd \mu$ which says that ν is an initial sequence of μ, the relation $\nu \prec \mu$ which says that ν comes before μ in the lexicographic order, and the relation $\nu < \mu$ which says that $|\nu| < |\mu|$.

Now if T does not have the tree property of the second kind, then there are other φ, $(\bar{b}_\nu)$ which satisfy (1)–(3) and also the following condition.

(4) If $\nu \in \omega^{<\omega}$, $\eta \in \omega^\omega$ and $\nu \not\lhd \eta$, then $\{\varphi(\bar{x};\bar{b}_\nu)\} \cup \{\varphi(\bar{x};\bar{b}_{\nu i} \mid i < \omega\}$ is inconsistent.

By another change of φ, $(\bar{b}_\nu)$, we can make sure that in addition to (1)–(4) we also have the following condition.

(5) If $\nu,\mu \in \omega^{<\omega}$, $\nu \not\lhd \mu$ and $\mu \not\lhd \nu$, then $\varphi(\bar{x};\bar{b}_\nu)$ and $\varphi(\bar{x};\bar{b}_\mu)$ are inconsistent.

At this stage we have found a formula which, by (1) and (5), satisfies the tree property of the first kind. Now we explain how to get (3), (4) and (5).

How to get (3): LOOK IT UP!

How to get (4): First we consider the set $\{\varphi(\bar{x};\bar{b}_1),\varphi(\bar{x};\bar{b}_{01}),\varphi(\bar{x};\bar{b}_{001}),\varphi(\bar{x};\bar{b}_{0001}),\ldots\}$. If it is consistent, then by (3) for all functions $\eta\in\omega^\omega$ the set $\{\varphi(\bar{x};\bar{b}_{0^\ell(\eta(\ell)+1)}) \mid \ell<\omega\}$ is consistent, and so $(\bar{c}_i^\ell)_{\ell,i<\omega}$ defined as $\bar{c}_i^\ell = \bar{b}_{0^\ell(i+1)}$ witnesses that $\varphi(\bar{x};\bar{y})$ satisfies condition (3) of Proposition 13. Since T does not have TP_2 this is impossible.

Thus $\{\varphi(\bar{x};\bar{b}_1),\varphi(\bar{x};\bar{b}_{01}),\varphi(\bar{x};\bar{b}_{001}),\varphi(\bar{x};\bar{b}_{0001}),\ldots\}$ is inconsistent; on the other hand, by (1) the set $\{\varphi(\bar{x};\bar{b}_0),\varphi(\bar{x};\bar{b}_{00}),\varphi(\bar{x};\bar{b}_{000}),\varphi(\bar{x};\bar{b}_{0000}),\ldots\}$ is consistent. Therefore there is a maximal $n<\omega$ such that the set

$$\{\varphi(\bar{x};\bar{b}_1),\varphi(\bar{x};\bar{b}_{01}),\varphi(\bar{x};\bar{b}_{001}),\ldots,\varphi(\bar{x};\bar{b}_{0^{n-1}1})\}\cup\{\varphi(\bar{x};\bar{b}_{0^n0}),\varphi(\bar{x};\bar{b}_{0^{n+1}0}),\varphi(\bar{x};\bar{b}_{0^{n+2}0}),\ldots\}$$

is consistent. Let $\psi(\bar{x};\bar{c}) \equiv \varphi(\bar{x};\bar{b}_1)\wedge\varphi(\bar{x};\bar{b}_{01})\wedge\varphi(\bar{x};\bar{b}_{001})\wedge\cdots\wedge\varphi(\bar{x};\bar{b}_{0^{n-1}1})$. We replace $\varphi(\bar{x};\bar{y})$ by $\varphi'(\bar{x};\bar{y}\bar{z}) \equiv \varphi(\bar{x};\bar{y})\wedge\psi(\bar{x};\bar{z})$ and $(\bar{b}_\nu)_{\nu\in\omega^{<\omega}}$ by $(\bar{b}'_\nu)_{\nu\in\omega^{<\omega}}$, where $\bar{b}'_\nu = \bar{b}_{0^{n+1}\nu}\bar{c}$. Clearly φ', $(\bar{b}'_\nu)$ still satisfy (2) and (3), where in (3) we have indiscernibility even over $\bar{c}$. $\{\varphi'(\bar{x};\bar{b}'_{0^\ell}) \mid \ell<\omega\}$ is consistent by choice of n. Therefore (1) holds by (3).

Moreover, $\{\varphi'(\bar{x};\bar{b}'_1)\}\cup\{\varphi'(\bar{x};\bar{b}'_{00^\ell}) \mid \ell<\omega\}$ is inconsistent by the maximal choice of n. By (3), for any $\nu\in\omega^{<\omega}$, $\eta\in\omega^\omega$ such that $\nu\not\lhd\eta$ and $\nu\prec\eta$ the set $\{\varphi(\bar{x};\bar{b}_\nu)\}\cup\{\varphi(\bar{x};\bar{b}_{\eta\restriction\ell}) \mid |\nu|<\ell<\omega\}$ is inconsistent. Because of the condition $\nu\prec\eta$ we have only 'half' of (4).

To get the other half, 'invert' the tree as follows. Let ω^* be ω with the opposite order. Build a tree $(\bar{b}'_\nu)_{\nu\in(\omega^*)^{<\omega}}$, the finite subtrees of which have the same type as those of $(\bar{b}_\nu)_{\nu\in\omega^{<\omega}}$. Then this new tree also satisfies (1)–(3), and it satisfies the 'second half' of (4), i.e. (4) under the condition $\nu\succ\eta$. Repeat the above process to get the 'first half' again. Since the 'second half' is preserved, the result satisfies (4).

How to get (5): It follows from (4) that for a number m the following to formulas are inconsistent.

$$\begin{array}{lcl}\varphi(\bar{x};\bar{b}_0) & \wedge & \varphi(\bar{x};\bar{b}_{10})\wedge\varphi(\bar{x};\bar{b}_{100})\wedge\cdots\wedge\varphi(\bar{x};\bar{b}_{10^m})\\ \varphi(\bar{x};\bar{b}_1) & \wedge & \varphi(\bar{x};\bar{b}_{00})\wedge\varphi(\bar{x};\bar{b}_{000})\wedge\cdots\wedge\varphi(\bar{x};\bar{b}_{00^m})\end{array}$$

(*) Hence for $\nu,\mu\in\omega^{<\omega}$, if $\nu\not\lhd\mu$ and $\mu\not\lhd\nu$, then the formula

$$\varphi(\bar{x};\bar{b}_\nu)\wedge\varphi(\bar{x};\bar{b}_{\nu0})\wedge\cdots\wedge\varphi(\bar{x};\bar{b}_{\nu0^{2m-1}}) \quad\wedge\quad \varphi(\bar{x};\bar{b}_\mu)\wedge\varphi(\bar{x};\bar{b}_{\mu0})\wedge\cdots\wedge\varphi(\bar{x};\bar{b}_{\mu0^{2m-1}})$$

is inconsistent.

For any $\nu\in\omega^{<\omega}$ let ν' be ν with $2m$ padding zeros after every entry of ν, or formally $\nu' = \nu(0)0^{2m}\nu(1)0^{2m}\ldots\nu(|\nu|-1)0^{2m}$. We replace $\varphi(\bar{x};\bar{y})$ by the formula $\varphi'(\bar{x};\bar{y}_0\ldots\bar{y}_{2m-1})$ and $\bar{b}_\nu$ by $\bar{b}'_\nu = \bar{b}_{\nu'}\bar{b}_{\nu'0}\ldots\bar{b}_{\nu'0^{2m-1}}$. We still have (3), and due to the padding also (1). (5) easily follows from (*). (2) and (4) follow from (5). □

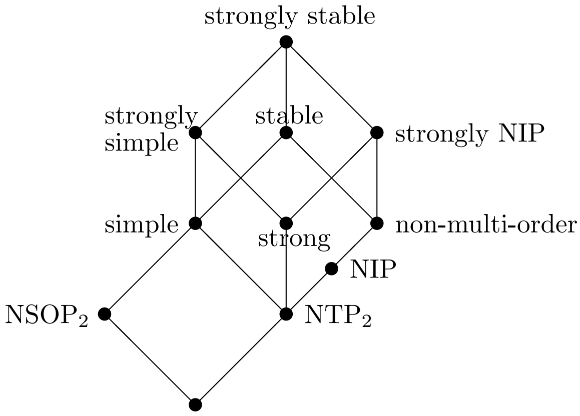

Figure 2: This lattice diagram visualises the classification of complete theories according to which of the properties covered in this paper they satisfy.

5 Conclusion

An *independent orders pattern* or *ird-pattern* is a sequence $(\varphi^\alpha(\bar{x};\bar{y}^\alpha))_{\alpha<\kappa}$ of formulas such that there is an array of tuples $(\bar{b}_i^\alpha)_{\alpha<\kappa,i<\omega}$ which witnesses it: For every path $\eta \in \omega^\kappa$ the following set of formulas is consistent.

$$\Gamma_\eta(\bar{x}) \quad = \quad \left\{\varphi^\alpha\left(\bar{x};\bar{b}_i^\alpha\right) \;\middle|\; \alpha<\kappa;\ i<\omega;\ i<\eta(\alpha)\right\} \cup \left\{\neg\varphi^\alpha\left(\bar{x};\bar{b}_i^\alpha\right) \;\middle|\; \alpha<\kappa;\ i<\omega;\ i\geq\eta(\alpha)\right\}$$

The smallest infinite cardinal κ such that there is no ird-pattern of cardinality κ is denoted by $\kappa_{\mathrm{ird}}(T)$. It is easy to see that $\kappa_{\mathrm{ird}}(T)=\infty$ if and only if T has the independence property. Also in stable theories there is not even an ird-pattern of length one, so $\kappa_{\mathrm{ird}}(T)=\aleph_0$. Moreover, if $(\varphi^\alpha(\bar{x};\bar{y}^\alpha))_{\alpha<\kappa}$ is an ird-pattern, then clearly $(\varphi^\alpha(\bar{x};\bar{y}_1^\alpha)\wedge\neg\varphi^\alpha(\bar{x};\bar{y}_2^\alpha))_{\alpha<\kappa}$ is an ict-pattern. Therefore $\kappa_{\mathrm{ird}}\leq\kappa_{\mathrm{ict}}$. A theory is called *multi-order* if $\kappa_{\mathrm{ird}}>\aleph_0$ and *non-multi-order* if $\kappa_{\mathrm{ird}}=\aleph_0$.

Non-multi-order theories are interesting because under certain set-theoretic assumptions (negating the continuum hypothesis) countable NIP theories that are non-multi-order have strictly less types than those which are multi-order [10]. Since $\kappa_{\mathrm{ird}}\leq\kappa_{\mathrm{ict}}$, strongly dependent theories are non-multi-order.

In closing, I should mention the draft paper by Alf Onshuus and Alex Usvyatsov which inspired me to writing the present paper [6]. In Section 3 of his paper on strongly dependent theories, Shelah explores possibilities to define a rank for strongly NIP theories. In fact, he defines ranks dp-rk$_{\bar{\Delta},\ell}$ for '$\ell=1,2,3,4,5,6,8,9,11,12$ (but not 7,10)', and a few pages later also for '$\ell\in\{14,15,17,18\}$' [13]. Fortunately, the question whether these ranks are always ≤ 1 in the case of 1-types seems to be related to a very simple property, and so Onshuus and Usvyatsov can start from the following straightforward definition.

A theory is *dp-minimal* if there is no ict-pattern $(\varphi^\alpha(x,\bar{y}^\alpha))_{\alpha<2}$ of length 2, where x is a single variable. In other words, dp-minimality means that T is NIP and every 1-type $p(x)$ has burden $\mathrm{bdn}(p)\leq 1$. It is not hard to check that in a simple theory $\mathrm{bdn}(p)=1$ implies that p is regular [PROVE]. Superstable theories of U-rank 1 and o-minimal theories are dp-minimal, but also an artificial example of a stable non-superstable theory. [SHOW: dp-minimality is preserved under 1-minimal expansions.]

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