Theories controlled by formulas of Vapnik-Chervonenkis codimension 1

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Abstract

The notion of a VC-minimal theory is introduced, a slightly more general variant of C-minimality that also includes all strongly minimal or (weakly) o-minimal theories. The 1-dimensional definable sets in a VC-minimal theory have a good 'swiss cheese' representation similar to the C-minimal case. VC-minimal theories are dp-minimal; in particular they do not have the independence property, and if they are stable then every 1-type has weight 1.

1 VC-minimality

The underlying notion of dimension for this paper is essentially due independently to Vapnik and Chervonenkis and to Shelah [15, 14]. The connection between the VC dimension of a set system and the independence property was first observed by Laskowski [5]. For some basic facts about VC dimension see for example my paper on theories without the independence property [1].

The *VC codimension* of a formula $\varphi(\bar{x};\bar{y})$ is the maximal number $\mathrm{vc}^{\mathrm{opp}}(\varphi)$ of 'independent' instances $\varphi(\bar{x};\bar{b})$ of φ. More precisely, $\mathrm{vc}^{\mathrm{opp}}(\varphi)$ is the maximal natural number $N<\omega$ such that there are tuples $\bar{a}_I$ for each subset $I \subseteq N$ and $\bar{b}_j$ for each element $j \in N$ such that $\models \varphi(\bar{a}_I;\bar{b}_j)$ holds if and only if $j \in I$. For example in a linearly ordered theory both $x=y$ and $x<y$ have VC codimension 1. If no such maximum exists, the VC codimension of $\varphi(\bar{x};\bar{y})$ is ∞ and the formula is said to have the *independence property*.[1] A theory has the *independence property* if one of its formulas $\varphi(\bar{x};\bar{y})$ has it. Although the fact is not entirely trivial, this implies that a formula $\varphi(x;\bar{y})$, with x a single variable, has the independence property.

Remark 1. A formula $\varphi(\bar{x};\bar{y})$ has VC codimension ≤ 1 if and only if between any two instances $\varphi(\bar{x};\bar{b})$ and $\varphi(\bar{x};\bar{c})$ one of the following four relations holds:

- $M \models \forall x\big(\varphi(x;\bar{b}) \rightarrow \varphi(x;\bar{c})\big)$.
- $M \models \forall x\big(\varphi(x;\bar{c}) \rightarrow \varphi(x;\bar{b})\big)$.
- $M \models \neg\exists x\big(\varphi(x;\bar{b}) \wedge \varphi(x;\bar{c})\big)$.
- $M \models \forall x\big(\varphi(x;\bar{b}) \vee \varphi(x;\bar{c})\big)$.

Thus φ has VC codimension 1 if and only if φ has at least one non-trivial (not constantly true or constantly false) instance, and any two instances are comparable one with the other or the negation of the other.

Key words: Vapnik-Chervonenkis dimension, independence property, vc-minimal, o-minimal.
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[1] The VC codimension of a formula $\varphi(\bar{x};\bar{y})$ is dual to its VC dimension in the sense that it is the VC dimension of the dual formula $\varphi^{\mathrm{opp}}(\bar{y};\bar{x}) \equiv \varphi(\bar{x};\bar{y})$. $\varphi^{\mathrm{opp}}(\bar{y};\bar{x})$ is really exactly the same formula as $\varphi(\bar{x};\bar{y})$, only the order in which $\bar{x}$ and $\bar{y}$ are mentioned in the parentheses is changed. Laskowski defines a number that is greater by one than the VC codimension, and calls it the independence dimension. I do not use his definition because formulas of VC codimension 1 have independence dimension 2, which seems a bit odd. VC dimension and VC codimension are each bounded exponentially in the other. In particular, one is finite if and only if the other is finite.

Definition 2. A *VC-minimal formula* is a formula of VC codimension 1. A *finitely VC-minimal theory* is a complete theory T with a distinguished VC-minimal formula $\alpha(x;\bar{y})$, such that every formula $\varphi(x;\bar{c})$ with parameters in a model $M \models T$ is equivalent to a boolean combination of instances $\alpha(x;\bar{b})$, each with parameters $\bar{b} \in M$.

At this point we could check that every finitely VC-minimal theory is NIP. But we will see later that in fact every VC-minimal theory is even dp-minimal.

Since the formula $x = y$ is VC-minimal for every theory (with infinite models), every strongly minimal theory is finitely VC-minimal when equipped with this formula. It is also easy to see that every o-minimal theory is finitely VC-minimal when equipped with the formula $(x \leq y_0 \wedge y_0 = y_1) \vee (x < y_0 \wedge y_0 \neq y_1)$. However, some weakly o-minimal theories are not finitely VC-minimal. For example consider the theory of a dense linear order together with a non-empty set of predicates, each of which defines an irrational cut. It is weakly o-minimal but not o-minimal. We can deal with the case of finitely many predicates by coding them in a single formula; yet here we have infinitely many irrational cuts and so this does not work and there is no way to regard the theory as finitely VC-minimal. To cover such theories as well, we need the notion of VC-minimality in its full generality.

The (stable) theory of two cross-cutting equivalence relations, which clearly does not deserve being called 'minimal' (since every 1-type has weight 2), shows that it is not enough to have a family of VC-minimal formulas. We must be a bit more careful, and will require that the family itself be VC-minimal.

We will call a family of formulas $\Phi(\bar{x}) = \{\varphi^s(\bar{x};\bar{y}^s) | s \in S\}$ an *instantiable family.* The *instances* of such a family are the formulas $\varphi^s(\bar{x};\bar{b})$, with parameters $\bar{b}$ taken from some model. An instantiable family is *instance-trivial* if all of its instances are trivial. By the coding trick which we already used to make o-minimal theories VC-minimal, every non-empty finite instantiable family can be replaced by a single formula which has exactly the same instances.

The VC codimension of an instantiable family $\Phi(\bar{x})$ is defined like that of a single formula: as the maximal natural number $N < \omega$ such that there are tuples $\bar{a}_I$ for each subset $I \subseteq N$ and instances $\varphi^{s_j}(\bar{x},\bar{b}^j)$ of Φ for each element $j \in N$ such that $\models \varphi^{s_j}(\bar{a}_I,\bar{b}^j)$ holds if and only if $j \in I$. This definition can be extended so that the VC codimension can also be an infinite cardinal, and for infinite families this would actually make sense; but there are several ways to do this, and we have no need to do it in this paper.

We say that a 1-sorted theory T has 1-*QE down to* instances of an instantiable family $\Phi(x)$, if every formula with parameters in a model $M \models T$ is equivalent to a Boolean combination of instances of $\Phi(x)$. Thus we can rephrase Definition 2 more elegantly, and say that a finitely VC-minimal theory is a theory which has 1-QE down to instances of a distinguished VC-minimal formula.

Definition 3. A *VC-minimal theory* is a complete 1-sorted theory which has 1-QE down to instances of a distinguished VC-minimal instantiable family.

Thus the finitely VC-minimal theories are those VC-minimal theories for which the instantiable family consists of a single formula or, up to a coding trick, those VC-minimal theories whose instantiable family is finite.

Remark 4. Every o-minimal theory is finitely VC-minimal, and every weakly o-minimal theory is VC-minimal.

Note that the theory of a dense cyclic order is not VC-minimal, although it becomes o-minimal after naming a parameter. This shows that the notion of VC-minimality, although arguably more natural than many other notions of 'minimality', is still not preserved under arbitrary reducts.

2 Directed systems and Swiss cheese

Definition 5. An instantiable family is a *directed family* if any two instances $\varphi(\bar{x};\bar{b})$ and $\psi(\bar{x};\bar{c})$ are either comparable (one of them implies the other) or inconsistent with each other; i.e. one of the following three relations holds between them:

- $M \models \forall x\big(\varphi(x;\bar{b}) \to \psi(x;\bar{c})\big)$.
- $M \models \forall x\big(\psi(x;\bar{c}) \to \varphi(x;\bar{b})\big)$.
- $M \models \neg\exists x\big(\varphi(x;\bar{b}) \wedge \psi(x;\bar{c})\big)$.

A *directed VC-minimal theory* is a VC-minimal theory whose distinguished VC-minimal family is directed.

Clearly every directed family is a VC-minimal family. The converse is false, but 'morally true':

Proposition 6. *Suppose $\Phi(x)$ is a VC-minimal instantiable family which has a non-trivial instance that is equivalent to a formula without parameters. Then there is a directed family $\Psi(x)$ of the same cardinality, such that every instance or negated instance of Ψ is equivalent to an instance or negated instance of Φ, and vice versa.*

Update proof notation!

Proof. Fix a model $M \models T$. Let $\mathcal{A}$ consist of the sets of realisations of all instances of Φ and their complements, except $\emptyset$ and M. We denote the complement of any $A \in \mathcal{A}$ by A'. By VC-minimality, for any two elements $A, B \in \mathcal{A}$, either A and B are comparable or A and B' are comparable.

Fix an element $C \in \mathcal{A}$ which represents an instance $\chi(\bar{x};\bar{c})$ of a formula $\chi(\bar{x};\bar{z}) \in \Phi(\bar{x})$. For every $A \in \mathcal{A}$ we see that A is comparable to either C or C', but not both. Therefore $\mathcal{A}$ is a disjoint union of the set $\{A \in \mathcal{A} \mid A \subseteq C\} \cup \{A \in \mathcal{A} \mid A \subsetneq C'\}$ and its dual $\{A \in \mathcal{A} \mid A \supseteq C'\} \cup \{A \in \mathcal{A} \mid A \supsetneq C\}$. As a consequence, the system consisting of the following formulas $\varphi'(\bar{x};\bar{y})$ (one for each $\varphi(\bar{x};\bar{y}) \in \Phi(\bar{x})$),

$$\big(\alpha(\bar{y},\bar{c}) \wedge \varphi(\bar{x};\bar{y})\big) \quad \vee \quad \big(\neg\alpha(\bar{y},\bar{c}) \wedge \neg\varphi(\bar{x};\bar{y})\big),$$

where $\alpha(\bar{y},\bar{c})$ is defined as

$$\big(\forall\bar{x}(\varphi(\bar{x};\bar{y}) \to \chi(\bar{x};\bar{c})\big) \quad \vee \quad \big([\forall\bar{x}(\varphi(\bar{x};\bar{y}) \to \neg\chi(\bar{x};\bar{c}))] \ \wedge\ \exists\bar{x}(\neg\varphi(\bar{x};\bar{y}) \wedge \neg\chi(\bar{x};\bar{c}))\big),$$

is a directed family and defines the same set $\mathcal{A}$ as does φ. □

Thus every VC-minimal theory can be made directed, although we may have to add the parameters of one non-trivial instance first. Obvious examples of directed families are $x = y$ in any theory, and $x < y$ in a linearly ordered theory. An interesting observation is that in any theory, if $\Phi(x)$ is a directed family, then $\Phi(x) \cup \{x = y\}$ is again a directed family.

Weakly o-minimal theories have 1-QE down to the infinite directed family consisting of all formulas $\varphi(x;\bar{y})$ with the property that every instance defines a downward closed set.

1-dimensional sets in a directed VC-minimal theory have a very nice representation similar to the C-minimal case. We will call the set of realisations of an instance of the distinguished directed VC-minimal family a *ball*. We will refer to any set of the form $A \setminus (B_0 \cup \cdots \cup B_{n-1})$, where A is a ball or the full domain, and $B_0, \ldots, B_{n-1}$ are disjoint balls and subsets of A, as a *swiss cheese*.

Proposition 7. *In a directed VC-minimal theory, the realisation set of every formula with parameters is a disjoint union of swiss cheeses.*

Proof. By induction on the boolean combinations: Show that the property of being a disjoint union of swiss cheeses is preserved under union and complementation. □

In the case of a strongly minimal theory, the balls are the singletons, and therefore the swiss cheeses are the singletons and the cofinite sets. For a (weakly) o-minimal theory, the balls are the definable downward-closed sets. In this case a ball cannot have two disjoint holes, and the swiss cheeses are precisely the definable convex sets.

Proposition 8. *The following are equivalent for a directed VC-minimal theory T:*

- *There is a bound on the lengths of chains of balls.*

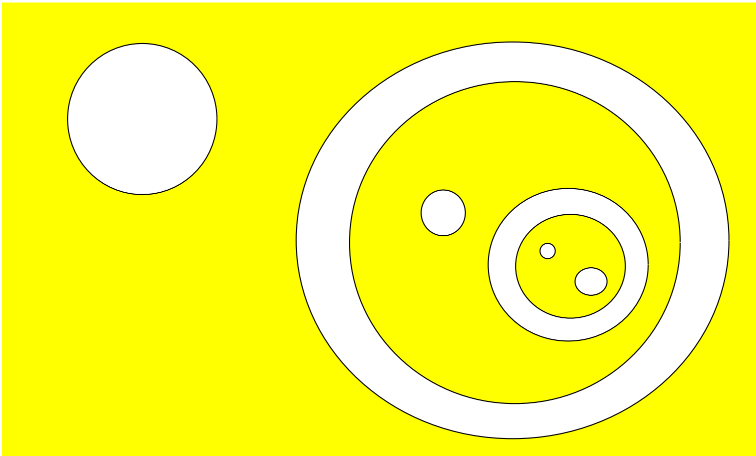

Figure 1: Disjoint union of three swiss cheeses, each with two holes.

- *For every formula in the distinguished family there is a finite upper bound on the lengths of chains of balls defined by the formula.*
- *T is equational.*
- *T is stable.*
- *T is simple.*
- *T does not have the strict order property.*

How about superstability?

Proposition 9. *Every VC-minimal theory is dp-minimal.*

Under what conditions are VC-minimal groups abelian-by-finite? Note that there are C-minimal counter-examples. (Is every o-minimal—not ordered—group abelian-by-finite?)

Under what conditions do VC-minimal theories have the exchange property?

Can we define a natural topology that specialises correctly to the o-minimal and C-minimal case? The strongly minimal case does not help: The topology will be either discrete or trivial. In the C-minimal case, those balls which are not points form the basic open sets. The following produces the discrete topology in the strongly minimal case, and also the correct o-minimal (and weakly o-minimal) and C-minimal topologies, provided that all maximal chains of balls are dense: For every ball or co-ball, take the intersection of all balls or co-balls that are strictly greater as a basic clopen set. In the o-minimal case, this definition glues together as inseparable any sequence of discrete points. While not the expected result, this does seem to make some kind of sense. There is a similar effect for valued fields with a discrete value group. I have no intuition for this case, and the resulting topology is coarser (has less open sets) than the usual C-minimal topology.

Can we prove a cell decomposition theorem?

3 D-relation of a finitely VC-minimal theory

Following Adeleke and Neumann, we say that a D-relation is a quarternary relation D satisfying the following axioms:

D1 $D(a, b; c, d) \Rightarrow D(b, a; c, d) \wedge D(c, d; a, b)$.

D2 $D(a, b; c, d) \Rightarrow \neg D(a, c; b, d)$.

D3 $D(a,b;c,d) \Rightarrow D(a,e;c,d) \vee D(a,b;c,e)$.

Axiom D1 says that we can think of D as a relation between the two sets $\{a,b\}$ and $\{c,d\}$, and that this relation is symmetric. Note that by D2, $D(a,b;a,d)$ can never hold. Therefore by D3, $D(a,b;c,d)$ implies $D(a,a;c,d)$.

Similarly, we can see that $D(a,b;c,d) \wedge D(a',b;c,d)$ implies $D(a,a';c,d)$. (Otherwise $D(a,b;c,a')$ and $D(a',b;c,a)$ by D3. But this contradicts D3.) This makes it expedient to think of D as a (symmetric) relation that can hold between two sets with an arbitrary number of elements, e.g. between $\{a,a',b\}$ and $\{c,d\}$.

We usually write $D(a,b;c,d)$ to stress the symmetries of the D-relation. But when talking about instances we think of the division $D(a;b,c,d)$, and so by the instances of D we mean the formulas of the form $D(x,b;c,d)$. The location of the semicolon is mostly a matter of clarity and style.

Remark 10. Every D-relation $D(x;y,z,u)$ is a VC-minimal formula.

Proof. Suppose D is not VC-minimal. Let $D(x,b;c,d)$ and $D(x,e;f,g)$ be two instances that are not comparable, and such that one is not comparable to the complement of the other. Let a_1,a_2,a_3,a_4 witness this:

$$\begin{array}{cccc} D(a_1,b;c,d) & D(a_2,b;c,d) & \neg D(a_3,b;c,d) & \neg D(a_4,b;c,d) \\ D(a_1,e;f,g) & \neg D(a_2,e;f,g) & D(a_3,e;f,g) & \neg D(a_4,e;f,g). \end{array}$$

From $D(a_1,b;c,d) \wedge \neg D(a_3,b;c,d)$ we get $D(a_1,b;a_3,c)$ using D3. From $D(a_2,b;c,d) \wedge \neg D(a_3,b;c,d)$ we get $D(a_2,b;a_3,c)$ using D3. Putting both results together we get $D(a_1,a_2;a_3,c)$. Exactly the same argument with a_4 in place of a_3 shows $D(a_1,a_2;a_4,c)$. From $D(a_1,a_2;a_3,c) \wedge D(a_1,a_2;a_4,c)$ we get $D(a_1,a_2;a_3,a_4)$.

By exactly the same series of arguments for $D(x,e;f,g)$, but with a_2 and a_3 exchanged, we can show $D(a_1,a_3;a_2,a_4)$. Taken together, our two final results contradict D2. □

Definition 11. A *D-minimal theory* is a complete 1-sorted theory which has 1-QE down to instances of a distinguished D-relation.

By the previous remark, every D-minimal theory is VC-minimal, with the same distinguished formula. Conversely, every VC-minimal theory is equipped with a natural D-relation. Given any formula $\varphi(x;\bar{y})$, let us define a quarternary relation D_φ by

$$D_\varphi(x,u;v,w) \equiv \\ \exists\bar{y}\big(\varphi(x;\bar{y}) \wedge \varphi(u;\bar{y}) \wedge \neg\varphi(v;\bar{y}) \wedge \neg\varphi(w;\bar{y})\big) \vee \exists\bar{y}\big(\neg\varphi(x;\bar{y}) \wedge \neg\varphi(u;\bar{y}) \wedge \varphi(v;\bar{y}) \wedge \varphi(w;\bar{y})\big).$$

Remark 12. If $\varphi(x;\bar{y})$ is a VC-minimal formula, then D_φ is a D-relation.

Proof. It is trivial to check axiom D1. Suppose axiom D2 fails, i.e. we have a,b,c,d such that $D(a,b;c,d) \wedge D(a,c;b,d)$. Without loss of generality, we have parameters $\bar{h}$ and $\bar{e}'$ such that $\varphi(a;\bar{h}) \wedge \varphi(b;\bar{h}) \wedge \neg\varphi(c;\bar{h}) \wedge \neg\varphi(d;\bar{h})$. and $\varphi(a;\bar{h}) \wedge \varphi(c;\bar{h}) \wedge \neg\varphi(b;\bar{h}) \wedge \neg\varphi(d;\bar{h})$. But this contradicts VC-minimality of φ. For axiom D3, we may assume $\varphi(a;\bar{h}) \wedge \varphi(b;\bar{h}) \wedge \neg\varphi(c;\bar{h}) \wedge \neg\varphi(d;\bar{h})$. Depending on whether $\varphi(e;\bar{h})$ holds or not, we get either $D(a,e;c,d)$ or $D(a,b;c,e)$. □

It would be natural to expect that if T is VC-minimal with distinguished formula $\varphi(x;\bar{y})$, then T is also D-minimal with distinguished formula D_φ. But this is not true in general. Consider the theory of a dense linear order without endpoints, together with a predicate P that defines an irrational cut. This theory is finitely VC-minimal with distinguished family $\{x<y, x\leq y, Px\}$ and weakly o-minimal, but not o-minimal. But it turns out that the instances of D_φ are exactly the instances of $\{x<y, x\leq y, x>y, x\geq y\}$, and so Px is not a Boolean combination of instances of D_φ. We will deal with this problem later, and also with the case of general (not finitely) VC-minimal theories. It is easy to check that strongly minimal and o-minimal theories are D-minimal. C-minimal theories become D-minimal if we suitably adjoin a new point ∞ such that $C(x;y,z)$ can be expressed as $D(\infty,x;y,z)$.

4 Topology of a finitely VC-minimal theory

Our motivating examples for VC-minimality are strongly minimal, weakly o-minimal and C-minimal theories. Sometimes similar methods and concepts can be used in these three disjoint classes of theories to prove similar results. Our hope is that some of these methods and concepts can be generalised to VC-minimal theories. One very notable concept in (weak) o-minimality and C-minimality is that of a topology on the unary sets. The absence of such a topology in the case of strong minimality need not worry us: It can easily be explained by the plausible assumption that the topology is trivial or discrete. In this section we will see that the D-relation of a finitely VC-minimal theory gives rise to a topology which in our applications is the familiar one.

The following definition essentially follows the terminology of Schoutens [13].

Definition 13. A *definable topology* for a structure M is a formula $\varphi(x;\bar{y})$ such that the instances of φ define a basis of open sets of a topology on M. An *ind-definable topology* for M is an instantiable family $\Phi(x)$ such that for every elementary extension N of M, the instances of Φ define a basis of open sets of a topology on N.

By compactness, an instantiable family $\Phi(x)$ is an ind-definable topology if and only if for any two formulas $\varphi_1(x;\bar{y}_1), \varphi_2(x;\bar{y}_2) \in \Phi(x)$ there is another formula $\psi(x;\bar{z}) \in \Phi(x)$, such that

$$T \models \forall x\bar{y}_1\bar{y}_2\Big(\varphi_1(x;\bar{y}_1) \wedge \varphi_2(x;\bar{y}_2) \;\rightarrow\; \exists\bar{z}\Big(\psi(x;\bar{z}) \wedge \forall x'\big(\varphi_1(x';\bar{y}_1) \wedge \varphi_2(x';\bar{y}_2) \rightarrow \psi(x';\bar{z})\big)\Big)\Big).$$

Every definable topology is ind-definable, and it makes sense to call a formula (instantiable family) a definable (ind-definable) topology for a theory.

Pillay called a theory equipped with a definable topology a first-order topological theory [12]. First-order topological structures can be regarded as a special case of topological structures in the sense of Flum and Ziegler [3].

We define the topology of a finitely VC-minimal theory as follows. Let D be the D-relation associated with the distinguished VC-minimal formula. Then the *open cones* $D(x,b;c,d)^M$ form a subbasis of the topology, i.e. the open sets of the topology are the unions of finite intersections of open cones. In a strongly minimal theory the open cones are the singletons and their complements, and so the topology is discrete. In a weakly o-minimal theory the open cones are the sets defined by $x < b$ and $x > b$. The basis also contains the intersections of two open cones, i.e. the empty set and the rational intervals $b_1 < x < b_2$. Thus we get the order topology, as expected.

5 Weakly D-minimal theories, and the topology of a VC-minimal theory

The following is a relatively straightforward generalisation of D-minimality, analogous to weak o-minimality.

Definition 14. A *weakly D-minimal theory* is a complete 1-sorted theory T equipped with a formula $D(x,y;z,u)$ which is a D-relation, and with a distinguished instantiable family $\Phi(x)$ such that for every instance $\varphi(x;\bar{h})$ we have

$$\varphi(a;\bar{h}) \wedge \varphi(b;\bar{h}) \wedge \neg\varphi(c;\bar{h}) \wedge \neg\varphi(d;\bar{h}) \implies D(a,b;c,d)$$

and T has 1-QE down to instances of Φ.

In particular every finitely VC-minimal theory is weakly D-minimal by taking D to be its natural D-relation and keeping the same VC-minimal family.

Remark 15. If $\Phi(x)$ is the distinguished family of a weakly D-minimal theory, then the family $\Phi'(x) = \Phi(x) \cup \{D(x;y,z,u)\}$ is a VC-minimal family. In particular the theory when equipped with Φ or Φ' is a VC-minimal theory.

We can define the topology in the same way as for finitely VC-minimal theories.

If we start with a VC-minimal theory that is not finitely VC-minimal, then we may not be able to describe the D-relation by a type. In general it will be the negation of a partial type over $\emptyset$. We could allow this possibility in the definition of weak D-minimality; but this does not seem worthwhile, because it would be counterintuitive, and it would make weak D-minimality a synonym for VC-minimality. However, it does seem worthwhile to spell out the definition of the topology of a general VC-minimal theory.

So let T be a VC-minimal theory with distinguished VC-minimal family $\Phi(x)$. Let $D(a, b; c, d)$ hold if and only if there is an instance, or negated instance, $\varphi(x)$ of $\Phi(x)$ such that $\models \varphi(a) \wedge \varphi(b) \wedge \neg\varphi(c) \wedge \neg\varphi(d)$. It is easy to see that $\neg D$ can be described by a partial type $p(x, y; z, u)$ over $\emptyset$.

An open cone is a set defined by $D(x, b; c, d)$. It is more convenient to say that a closed cone is a set of the form $p(x, b; c, d)^M$. A basic closed set is then a finite union of closed cones, and a general open set is an intersection of basic closed sets.

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