Independence Relations

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January 26, 2004
Organisation

- Examples (3 slides)
- Preview (1 slide)
- Axioms of Independence (5 slides)
- $b$-Forking Independence $\downarrow^b$ (2 slides)
- Proof of Theorem on $\downarrow^b$ (5 slides)
Examples

Vector space

$A, B, C$ sets of vectors.
$\langle AB \rangle = \text{span of } A \cup B.$

\[ A \downarrow_{\text{C}} B \quad \Leftrightarrow \quad \langle AC \rangle \cap \langle BC \rangle = \langle C \rangle \]

\[ \Leftrightarrow \quad \begin{cases} \bar{a} \in A \text{ lin. indep.} / \langle C \rangle \\ \Rightarrow \quad \bar{a} \text{ lin. indep.} / \langle BC \rangle \end{cases} \]
Algebraically closed field

$A$, $B$, $C$ sets of field elements.

$\text{acl}(AB) = \text{smallest alg. closed field} \supseteq A \cup B$.

\[
\begin{array}{c}
A \downarrow B \\ C
\end{array} : \Leftrightarrow \begin{cases}
\bar{a} \in A \ \text{alg. indep.} / \ acl(C) \\
\Rightarrow \bar{a} \ \text{alg. indep.} / \ acl(BC)
\end{cases}
\]

\[
\Rightarrow \text{acl}(AC) \cap \text{acl}(BC) = \text{acl}(C)
\]

More complicated— not ‘modular’!
Examples

Everywhere infinite forest

Disjoint union of infinitely many unordered trees.
Each tree vertex has infinite degree.

$A$, $B$, $C$ sets of vertices.
d($a$, conv($B$)) = distance between $a$ and the convex hull of $B$. (May be ∞!)

\[ A \downarrow_{\forall C} B \implies \begin{cases} 
  d(a, \text{conv}(BC)) = d(a, \text{conv} C) \\
  \text{for all } a \in A 
\end{cases} \]

\[ \Rightarrow \text{conv}(AC) \cap \text{conv}(BC) = \text{conv}(C) \]

*There is no such thing as a basis!*
• 10 axioms for ‘independence relations’.

• Definition of $\downarrow^b$.

• **Theorem**
  $T$ complete consistent first-order theory.
  $\exists$ independence relation for $T$
  $\Rightarrow \downarrow^b$ is an independence relation.

• Proof outline.

• Proof details.
**Axioms of Independence**  

(1/5)

$T$ is a complete consistent first-order theory. $T$ has a big and homogeneous model.

**Independence relation** $A \perp_C B$:

(invariance)

\[
A \perp_C B \\
\{ (A', B', C') \equiv (A, B, C) \} \implies A' \perp_{C'} B'
\]

(symmetry)

\[
A \perp_C B \iff B \perp_C A
\]
Axioms of Independence (2/5)

(monotonicity)

\[ A \downarrow B \quad \text{if} \quad A_0 \subseteq A, \ B_0 \subseteq B \]

\[ \Rightarrow \quad A_0 \downarrow B_0 \]

(finite character)

\[ A \updownarrow B \quad \Rightarrow \quad \exists \text{finite } A_0 \subseteq A, \ B_0 \subseteq B : \]

\[ A_0 \updownarrow B_0 \]

(anti-reflexivity)

\[ a \downarrow B \quad \Rightarrow \quad a \in \text{acl} \ B. \]

\[ \ldots \]
Axioms of Independence (3/5)

(base monotonicity)

\[
\begin{align*}
A \downarrow C \quad \quad B_0 \subseteq B \\
\{ \implies \} \\
\Rightarrow \quad A \downarrow_{B_0C} B
\end{align*}
\]

(transitivity)

\[
\begin{align*}
B \downarrow_{CD} A \\
C \downarrow_D A \\
\{ \implies \} \\
\Rightarrow \quad BC \downarrow_D A
\end{align*}
\]

\ldots
Axioms of Independence (4/5)

(existence)
\[
\forall A, B, C \quad \exists A' \equiv_C A : \\
A' \downarrow_C B.
\]

(extension)
\[
A \downarrow_C B \\
\hat{B} \supseteq B \quad \Rightarrow \\
\exists A' \equiv_{BC} A : \\
A' \downarrow_C \hat{B}
\]

\[
\Leftrightarrow \\
\exists \hat{B}' \equiv_{BC} \hat{B} : \\
A \downarrow_C \hat{B}'
\]

\[
\ldots
\]
Axioms of Independence (5/5)

(local character)

\[ \forall A, B \quad \exists C \subseteq B : \left\{ \begin{array}{l}
A \downarrow B \\
|C| < \kappa(|A|)
\end{array} \right. \]
\textbf{b-Forking Independence} \( \perp^b \) \hspace{1cm} (1/2)

\textbf{Definition}

\( A \perp^a_C B \iff \text{acl}(AC) \cap \text{acl}(BC) = \text{acl}(C) \)

\( A \perp^m_C B \iff \begin{cases} C \subseteq D \subseteq \text{acl}(BC) \\ \Rightarrow A \perp^a_D B \end{cases} \)

\( A \perp^b_C B \iff \begin{cases} \hat{B} \supseteq B \\ \Rightarrow \exists A' \equiv_{BC} A : A' \perp^m_{BC} \hat{B} \end{cases} \)

\textit{Alf Onshuus:}

\textit{Thorn-forking in rosy theories (Berkeley 2002)}

\textbf{Theorem}

If there is any independence relation at all, then \( \perp^b \) is one.
Proof Outline

1. Delete existence and symmetry axioms—
   they are redundant!

2. $\downarrow^a$ satisfies all axioms except:
   base monotonicity.

3. $\downarrow^m$ satisfies all undeleted axioms except:
   extension, local character.

4. $\downarrow^b$ satisfies all undeleted axioms except:
   local character.

S’pose there is an independence relation $\downarrow$. Easy:

$$A \downarrow B \quad \Rightarrow \quad A \downarrow^b B.$$

So $\downarrow^b$ satisfies local character as well. □
Proof of Theorem (1/5)

1a. Existence is redundant

(existence)

\[ \forall A, B, C \exists A' \equiv A : \\
\quad A' \downarrow_C B. \]

Local character \(\Rightarrow A \downarrow_D C\) for some \(D \subseteq C\).

Base monotonicity \(\Rightarrow A \downarrow_C C\).

Extension \(\Rightarrow \exists A' \equiv_C A\) such that \(A' \downarrow_C BC\).

Monotonicity \(\Rightarrow A' \downarrow_C B\).  \qed
Proof of Theorem (2/5)

1b. Symmetry is redundant

\[ (\text{symmetry}) \]
\[
\begin{array}{ccc}
A & \downarrow & B \\
C & & \Rightarrow & B & \downarrow & A \\
\end{array}
\]

Enumerate \( A: \bar{a}_0 \).

Let \( \kappa \geq \kappa(|B|) \) be regular.

Find \( BC \)-indiscernible sequence \( (\bar{a}_i)_{i<\kappa} \)
such that \( \{ \bar{a}_i \mid i < \lambda \} \downarrow_C \bar{a}_\lambda \) for \( \lambda < \kappa \).

Local character
\[
\Rightarrow \exists D \subseteq C\{\bar{a}_i \mid i < \kappa \} \text{ s.t. } B \downarrow_D C\{\bar{a}_i \mid i < \kappa \}.
\]

\( \kappa \) regular and \( |D| < \kappa \)
\[
\Rightarrow D \subseteq C\{\bar{a}_i \mid i < \lambda \} \text{ for some } \lambda < \kappa
\]
\[
\Rightarrow B \downarrow_C \{\bar{a}_i \mid i < \lambda \} C\{\bar{a}_i \mid i < \kappa \} \text{ (base monot.)}
\]
\[
\Rightarrow B \downarrow_C \{\bar{a}_i \mid i < \lambda \} \bar{a}_\lambda \text{ (monotonicity)}
\]
\[
\Rightarrow B \downarrow_C \bar{a}_\lambda \text{ (transitivity)}
\]
\[
\Rightarrow B \downarrow_C \bar{a}_0 \text{ (invariance)} \]

\( \square \)
Proof of Theorem (3/5)

2. Properties of $\downarrow^a$

$\downarrow^a$ satisfies all axioms except:
base monotonicity.

Most axioms very straightforward.

Existence is existence of free amalgams over acl-closed sets (standard fact).

Extension follows from symmetry, existence and transitivity.
Proof of Theorem (4/5)

3. Properties of $\downarrow^m$

$\downarrow^m$ satisfies all undeleted axioms except:
extension, local character.

Base monotonicity was built into $\downarrow^m$.

Each of the other axioms is straightforward from the corresponding axiom for $\downarrow^a$. 
Proof of Theorem

4. Properties of $\downarrow^b$

$\downarrow^b$ satisfies all undeleted axioms except: local character.

Extension was built into $\downarrow^m$.

Finite character is a bit tricky.

Each of the other axioms is straightforward from the corresponding axiom for $\downarrow^a$. 