Independence Relations

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Organisation

- Examples (3 slides)
- Preview (1 slide)
- Axioms of Independence (5 slides)
- þ-Forking Independence $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{þ}}$ (2 slides)
- Proof of Theorem on $\mathop{\smile\!\!\!\!\!\!\;|}^{\text{þ}}$ (5 slides)

Examples (1/3)

Vector space

A, B, C sets of vectors.
$\langle AB \rangle =$ span of $A \cup B$.

$$A \underset{C}{\smile\!\!\!\!\!\!\mid}\; B \quad :\Leftrightarrow \quad \langle AC \rangle \cap \langle BC \rangle = \langle C \rangle$$

$$\Leftrightarrow \quad \begin{cases} \bar{a} \in A \text{ lin. indep. / } \langle C \rangle \\ \qquad \Rightarrow \quad \bar{a} \text{ lin. indep. / } \langle BC \rangle \end{cases}$$

Examples (2/3)

Algebraically closed field

A, B, C sets of field elements.
acl(AB) = smallest alg. closed field $\supseteq A \cup B$.

$$A \underset{C}{\smile\!\!\!\!|}\; B \quad :\Leftrightarrow \quad \begin{cases} \bar{a} \in A \text{ alg. indep. / acl}(C) \\ \quad \Rightarrow \quad \bar{a} \text{ alg. indep. / acl}(BC) \end{cases}$$

$$\Rightarrow \quad \text{acl}(AC) \cap \text{acl}(BC) = \text{acl}(C)$$

More complicated—not ‘modular’!

Examples (3/3)

Everywhere infinite forest

Disjoint union of infinitely many unordered trees.
Each tree vertex has infinite degree.

A, B, C sets of vertices.
$\mathsf{d}(a, \mathsf{conv}(B))$ = distance between a and the convex hull of B. (May be ∞!)

$$A \underset{C}{\downarrow} B \quad :\Leftrightarrow \quad \begin{cases} \mathsf{d}(a, \mathsf{conv}(BC)) = \mathsf{d}(a, \mathsf{conv}\, C) \\ \text{for all } a \in A \end{cases}$$

$$\Rightarrow \quad \mathsf{conv}(AC) \cap \mathsf{conv}(BC) = \mathsf{conv}(C)$$

There is no such thing as a basis!

Preview

- 10 axioms for 'independence relations'.
- Definition of $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$.
- **Theorem**
  T complete consistent first-order theory.
  $\exists$ independence relation for T
  $\Rightarrow \quad \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ is an independence relation.
- Proof outline.
- Proof details.

Axioms of Independence (1/5)

T is a complete consistent first-order theory.
T has a big and homogeneous model.

Independence relation $A \mathop{\smile\!\!\!\!\!\!|}_C B$:

(invariance)

$$\left. \begin{array}{l} A \underset{C}{\smile\!\!\!\!\!\!|} B \\ (A', B', C') \equiv (A, B, C) \end{array} \right\} \Rightarrow \quad A' \underset{C'}{\smile\!\!\!\!\!\!|} B'$$

(symmetry)

$$A \underset{C}{\smile\!\!\!\!\!\!|} B \quad \Leftrightarrow \quad B \underset{C}{\smile\!\!\!\!\!\!|} A$$

...

Axioms of Independence (2/5)

(monotonicity)

$$\left.\begin{array}{l} A \underset{C}{\downarrow} B \\[2ex] A_0 \subseteq A,\ B_0 \subseteq B \end{array}\right\} \quad \Rightarrow \quad A_0 \underset{C}{\downarrow} B_0$$

(finite character)

$$A \underset{C}{\not\downarrow} B \quad \Rightarrow \quad \begin{cases} \exists \text{ finite } A_0 \subseteq A,\ B_0 \subseteq B : \\ \qquad A_0 \underset{C}{\not\downarrow} B_0 \end{cases}$$

(anti-reflexivity)

$$a \underset{B}{\downarrow} a \quad \Rightarrow \quad a \in \operatorname{acl} B.$$

. . .

Axioms of Independence (3/5)

(base monotonicity)

$$\left.\begin{array}{c} A \underset{C}{\perp\!\!\!\smile} B \\ \\ B_0 \subseteq B \end{array}\right\} \Rightarrow \quad A \underset{B_0 C}{\perp\!\!\!\smile} B$$

(transitivity)

$$\left.\begin{array}{c} B \underset{CD}{\perp\!\!\!\smile} A \\ \\ C \underset{D}{\perp\!\!\!\smile} A \end{array}\right\} \Rightarrow \quad BC \underset{D}{\perp\!\!\!\smile} A$$

. . .

Axioms of Independence (4/5)

(existence)

$$\forall A, B, C \quad \left\{ \begin{array}{c} \exists A' \underset{C}{\equiv} A : \\ \\ A' \underset{C}{\downarrow} B. \end{array} \right.$$

(extension)

$$\left. \begin{array}{c} A \underset{C}{\downarrow} B \\ \\ \hat{B} \supseteq B \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} \exists A' \underset{BC}{\equiv} A : \\ \\ A' \underset{C}{\downarrow} \hat{B} \end{array} \right\}$$

$$\left(\Leftrightarrow \left\{ \begin{array}{c} \exists \hat{B}' \underset{BC}{\equiv} \hat{B} : \\ \\ A \underset{C}{\downarrow} \hat{B}' \end{array} \right\} \right)$$

...

Axioms of Independence (5/5)

(local character)

$$\forall A, B \quad \exists C \subseteq B : \quad \begin{cases} A \underset{C}{\downarrow} B \\ |C| < \kappa(|A|) \end{cases}$$

þ-Forking Independence $\mathop{\smile\!\!\!\!\!\!|}\,^{þ}$ (1/2)

Definition

$$A \mathop{\smile\!\!\!\!\!\!|}_{C}^{\mathsf{a}} B \quad :\Leftrightarrow \quad \mathrm{acl}(AC) \cap \mathrm{acl}(BC) = \mathrm{acl}(C)$$

$$A \mathop{\smile\!\!\!\!\!\!|}_{C}^{\mathsf{m}} B \quad :\Leftrightarrow \quad \begin{cases} C \subseteq D \subseteq \mathrm{acl}(BC) \\ \quad \Rightarrow \quad A \mathop{\smile\!\!\!\!\!\!|}_{D}^{\mathsf{a}} B \end{cases}$$

$$A \mathop{\smile\!\!\!\!\!\!|}_{C}^{þ} B \quad :\Leftrightarrow \quad \begin{cases} \hat{B} \supseteq B \\ \quad \Rightarrow \quad \exists\, A' \equiv_{BC} A : \; A' \mathop{\smile\!\!\!\!\!\!|}_{C}^{\mathsf{m}} \hat{B} \end{cases}$$

Alf Onshuus:
Thorn-forking in rosy theories (Berkeley 2002)

Theorem

If there is any independence relation at all, then $\mathop{\smile\!\!\!\!\!\!|}\,^{þ}$ is one.

þ-Forking Independence $\mathop{\smile\!\!\!\!\!\!|}\,^{\text{þ}}$ (2/2)

Proof Outline

1. Delete existence and symmetry axioms—they are redundant!

2. $\mathop{\smile\!\!\!\!\!\!|}\,^{\text{a}}$ satisfies all axioms except: base monotonicity.

3. $\mathop{\smile\!\!\!\!\!\!|}\,^{\text{m}}$ satisfies all undeleted axioms except: extension, local character.

4. $\mathop{\smile\!\!\!\!\!\!|}\,^{\text{þ}}$ satisfies all undeleted axioms except: local character.

S'pose there is an independence relation $\mathop{\smile\!\!\!\!\!\!|}$. Easy:

$$A \mathop{\smile\!\!\!\!\!\!|}_{C} B \quad \Rightarrow \quad A \mathop{\smile\!\!\!\!\!\!|}_{C}^{\text{þ}} B.$$

So $\mathop{\smile\!\!\!\!\!\!|}\,^{\text{þ}}$ satisfies local character as well. □

Proof of Theorem (1/5)

1a. Existence is redundant

(existence)

$$\forall A, B, C \quad \begin{cases} \exists A' \underset{C}{\equiv} A : \\ \\ \quad A' \underset{C}{\perp} B. \end{cases}$$

Local character $\Rightarrow A \perp_D C$ for some $D \subseteq C$.

Base monotonicity $\Rightarrow A \perp_C C$.

Extension $\Rightarrow \exists\, A' \equiv_C A$ such that $A' \perp_C BC$.

Monotonicity $\Rightarrow A' \perp_C B$. $\square$

Proof of Theorem (2/5)

1b. Symmetry is redundant

(symmetry)

$$A \underset{C}{\perp} B \quad \Rightarrow \quad B \underset{C}{\perp} A$$

Enumerate A: $\bar{a}_0$.
Let $\kappa \geq \kappa(|B|)$ be regular.

Find BC-indiscernible sequence $(\bar{a}_i)_{i<\kappa}$
such that $\{\bar{a}_i \mid i < \lambda\} \perp_C \bar{a}_\lambda$ for $\lambda < \kappa$.

Local character
$\Rightarrow \exists\, D \subseteq C\{\bar{a}_i \mid i < \kappa\}$ s.t. $B \perp_D C\{\bar{a}_i \mid i < \kappa\}$.

κ regular and $|D| < \kappa$
$\Rightarrow D \subseteq C\{\bar{a}_i \mid i < \lambda\}$ for some $\lambda < \kappa$

$\Rightarrow B \perp_{C\{\bar{a}_i | i<\lambda\}} C\{\bar{a}_i \mid i < \kappa\}$ (base monot.)

$\Rightarrow B \perp_{C\{\bar{a}_i | i<\lambda\}} \bar{a}_\lambda$ (monotonicity)

$\Rightarrow B \perp_C \bar{a}_\lambda$ (transitivity)

$\Rightarrow B \perp_C \bar{a}_0$ (invariance) □

Proof of Theorem **(3/5)**

2. Properties of $\mathop{\smile\!\!\!\!\!\!\,|}^{a}$

$\mathop{\smile\!\!\!\!\!\!\,|}^{a}$ satisfies all axioms except:
base monotonicity.

Most axioms very straightforward.

Existence is existence of free amalgams over acl-closed sets (standard fact).

Extension follows from symmetry, existence and transitivity.

Proof of Theorem (4/5)

3. Properties of $\mathop{\smile\!\!\!\!\!\!|}^{m}$

$\mathop{\smile\!\!\!\!\!\!|}^{m}$ satisfies all undeleted axioms except:
extension, local character.

Base monotonicity was built into $\mathop{\smile\!\!\!\!\!\!|}^{m}$.

Each of the other axioms is straightforward from the corresponding axiom for $\mathop{\smile\!\!\!\!\!\!|}^{a}$.

Proof of Theorem **(5/5)**

4. Properties of $\smile^{b}$

$\smile^{b}$ satisfies all undeleted axioms except:
local character.

Extension was built into $\smile^{m}$.

Finite character is a bit tricky.

Each of the other axioms is straightforward from the corresponding axiom for $\smile^{a}$.